begin to get 100 percent depletion. While our experimental accuracy is such that this agreement is somewhat fortuitous we may, however, conclude that the back diffusion of He^3 in the liquid during the heat flush is small.

It is also of interest to estimate the possible heat flush in a closed system produced by a creeping film. We will consider the simple case of a bulb filled with a sample of liquid helium, connected to a filling tube, the bulb being immersed in a bath of liquid helium, the whole forming an isolated system. The film creeps up the filling tube and evaporates[3] and, in the steady state condition, a mass of helium gas recondenses at the liquid surface equal to the mass flowing in the film. If we assume that all of the heat of condensation is transmitted to the liquid sample, and take Daunt and Mendelssohn's[6] value for the creep rate at 1.8°K and a tube diameter of 2 mm we obtain a power input to the surface of the liquid of 0.1 milliwatt. From Fig. 2 it may be seen that at 0.1 milliwatt power there is a finite heat flush. It is possible that the use of the Daunt and Mendelssohn creep rate may not be justified here. Later work[7] has given different results, indicating in some cases a much higher creep rate and this would, of course, lead to a larger heat flush. We believe in the particular apparatus used in the present experiment, the effect of creeping film heat flush is small. The surface of the liquid sample is in the Kovar section, and hence most of the creeping film heat would proceed directly out into the bath, passing through only that amount of the sample which is in the Kovar section.

The method appears to be a rapid and efficient way of concentrating He^3 possibly up to 100 percent purity. The process may, however, break down for concentrations much higher than our present record of 4 percent He^3 although we have, as yet, no indication that this is so.

Finally we are indebted to Mr. Ernest Lynton for his assistance with the low temperature phases of this work.

[6] J. G. Daunt and K. Mendelssohn, Proc. Roy. Soc. **A170**, 423 (1939); **170**, 439 (1939).

[7] K. R. Atkins, Nature **161**, 925 (1948).

PHYSICAL REVIEW VOLUME 76, NUMBER 1 JULY 1, 1949

The Interaction Representation of the Proca Field

FREDERIK J. BELINFANTE
Department of Physics, Purdue University, Lafayette, Indiana

(Received December 16, 1948)

The methods used by Schwinger in quantum electrodynamics can be generalized in such a way that they become applicable to meson theory. This is shown by an example. The method used seems slightly simpler than the method proposed by the Japanese school. It turns out that the covariant field variables in interaction representation are not simply the transformed of the covariant variables used in Heisenberg representation. Also it turns out to be necessary to confine the space-like surfaces used in many applications to flat surfaces perpendicular to the time direction. The direct interaction between two particles through the meson field is obtained by a canonical transformation similar to the first approximation Schwinger transformation in quantum electrodynamics.

The example of a neutral vector meson field discussed in the present paper has been chosen in such a way as to show the analogy to quantum electrodynamics. The interaction energy between particles obtained by Schwinger's relativistic treatment in meson theory (and also obtainable by the other usual perturbation methods) goes over into the Møller interaction for vanishing meson mass.

INTRODUCTION

In recent developments of quantum electrodynamics much use has been made of the so-called interaction representation, in which the q-numbers describing various fields of particles or quanta satisfy field equations of a form as if no interactions between these fields would exist, while the interaction is described by a generalized Schroedinger equation for the situation functional (Schroedinger state vector) Ψ. The theory of this interaction representation and its use in quantum electrodynamics have been developed here in America by Schwinger.[1] The basic ideas of this theory of the interaction representation had been developed independently and published earlier by Tomonaga[2] in Japan. As the theory may be considered as a generalization of the many-times theory of Dirac, Fock, and Podolsky,[3] the new theory was called by him the "super-many-time theory."[4]

If one tries to apply the super-many-time forma-

[1] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

[2] S. Tomonaga, Bull. I.P.C.R. (Riken-iho) 22, 545 (1943) (*in Japanese*); Prog. Theor. Phys. 1, 27 (1946) and 2, 101 (1947).

[3] Dirac, Fock and Podolsky, Physik. Zeits. Sowjetunion 2, 468 (1932). See also Chapter 18 of G. Wentzel, *Einführung in die Quantentheorie der Wellenfelder* (F. Deuticke, Vienna, 1943; reprinted by Edwards Brothers, Inc., Ann Arbor, 1946).

[4] Compare for instance T. Miyazima, Prog. Theor. Phys. 2, 94(A) (1947).

lism to meson fields, one encounters the following complications:

1. The interaction operator is no longer a scalar.

2. The commutator of the interaction operators at two different points x and x' separated by a space-like four-vector involves derivatives of three-dimensional delta-functions.

3. As a consequence, the q-numbers transformed from Heisenberg representation to interaction representation will sometimes depend on the local slope of the super-many-time "surface" used for the transformation.

4. Consequently, the Lorentz transformations of q-numbers in Heisenberg representation and in interaction representation are then different. Therefore, the tensors describing the field in interaction representation need not necessarily be the transformed of the tensors used in Heisenberg representation.

5. Also the integrability of the generalized Schroedinger equation presents a problem more delicate than in quantum electrodynamics.

6. In the derivation of the field equations for the transformed q-numbers, more care has to be exerted about the order of sequence of limits than is necessary for the usual q-numbers of quantum electrodynamics.[5]

7. *Some of the field equations may have the form of so-called "identities"* (not involving $\partial/\partial t$), *but yet contain interactions in Heisenberg representation.*

In the present paper we shall deal with all these points[6] and shall develop a satisfactory theory of the interaction representation for the particular case of a theory of neutral vector mesons interacting only with the current-density four-vector of Dirac particles. It should be expected that the same methods as used here for this particular example can also be applied to other types of meson fields. The particular interest we take in this one case is based on certain applications of this theory, which can be made in quantum electrodynamics.[7]

1. THE NEUTRAL PROCA FIELD IN HEISENBERG REPRESENTATION

We shall discuss here the theory of neutral vector mesons (case b of Kemmer[8] with real field variables), interacting only with the charge current-density four-vector of a field of Dirac particles (say, electrons), and omitting the tensor interaction. Apart from the vector interaction, this meson field constitutes a Proca field[9] with real field components. We shall use a notation that brings out the analogies between this field and an ordinary electromagnetic field. Indeed, the electromagnetic interactions between electrons could be described as interactions through a field of mesons with negligible mass.[7]

If μ is the mass of these mesons, we shall write κ_0 for $\mu c/\hbar$, while κ stands for $mc/\hbar$, m being the mass of the Dirac particles. The interaction constant is called $(-e)$ instead of g, in order to emphasize the similarity of this theory to a theory of electromagnetism. For this same purpose, we do not give the meson field equations in the form used by Kemmer[8] and later by Yukawa and others,[10] in which the mass factor κ_0 occurs in the equation expressing the field strengths in terms of the four-dimensional rotation of the potentials, as well as in the equations giving these potentials in terms of the four-dimensional divergences of the field strengths. This form of the field equations may have the advantage of making the field strengths and the potentials to quantities of the same dimensions, so that they can then be collected into one single 10-component symmetric undor of the second rank.[11] But obviously the field equations in this form would not have Maxwell's equations as their limit for $\kappa_0\to 0$. Therefore, we prefer here the equations in the form originally used by Proca himself[9] and later by Bhabha,[12] Fröhlich, and others.[13]

As our starting point, we shall therefore take the following first order[14] Lagrangian function:[15]

$$L = \frac{1}{4\pi}\{\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - F^{\mu\nu}\partial_\mu A_\nu - \tfrac{1}{2}\kappa_0{}^2 A_\nu A^\nu\} + A_\nu j^\nu - \hbar c\bar{\psi}(\kappa + \gamma^\nu\partial_\nu)\psi. \quad (1)$$

[5] For less usual q-numbers like the gradients of the electric field strengths, however, even in quantum electrodynamics the method used by Schwinger (reference 1, Eq. (2.9)) leads to wrong results. Compare footnote 23.

[6] Just after completion of this work I received Progress of Theoretical Physics, Vol. 3, Nos. 1 and 2, in which S. Kanesawa and S. Tomonaga (pp. 1 and 101) and Y. Miyamoto (p. 124) deal with the difficulties Nos. 1, 2, and 5 listed above, by a method which seems different from the one used here and apparently more complicated. I could not find a discussion of the other four points listed above.

[7] If the limit to vanishing meson mass is taken at the end of the calculation, the formulas for mesonic interaction between electrons give the usual electromagnetic interactions, if the meson field discussed in this paper is used. See also F. J. Belinfante, Phys. Rev. **75**, 1321(A), (1949); Prog. Theor. Phys. **4**, 2 (1949).

[8] N. Kemmer, Proc. Roy. Soc. **A166**, 127 (1938).

[9] A. Proca, J. de Phys. et rad. **7**, 347 (1936); **8**, 23 (1937).

[10] Yukawa, Sakata and Taketani, Proc. Phys. Math. Soc. Japan **20**, 319 (1938).

[11] F. J. Belinfante, Physica **6**, 849, 870 (1939). See also Kramers, Belinfante and Lubański, Physica **8**, 597 (1941).

[12] H. J. Bhabha, Proc. Roy. Soc. **A166**, 501 (1938).

[13] Fröhlich, Heitler and Kemmer, Proc. Roy. Soc. **A166**, 154 (1938).

[14] Some advantages of the use of *first-order* Lagrangians (linear in the gradient operators ∂_μ) have been discussed by me in Physica **7**, 449 (1940). As far as I know, the proof of the covariance of the canonical commutation relations used in meson theories in Heisenberg representation has as yet been published only for the theory using first-order Lagrangians (see Physica **7**, 765 (1940)). For applications of a first-order Lagrangian in quantum electrodynamics see, for instance, Physica **6**, 887 (1939); **7**, 449 (1940); or **12**, 17 (1946). For an application in classical mechanics (electrodynamics), see Phys. Rev. **74**, 779 (1948).—For the rest, the formalism in this paper could be developed just as easily from a Lagrangian of the second order as far as the Proca field is concerned.

[15] Here and in the following, products of various factors like, for instance, $F^{\mu\nu}\partial_\mu A_\nu$ or $\bar{\psi}\gamma^\nu\partial_\nu\psi$ or $\kappa_0\bar{\psi}\psi$ are *tacitly* assumed

Here $\partial_0 = -\partial^0 = \partial/c\partial t$, $\partial_1 = \partial^1 = \nabla_x = \partial/\partial x$, etc. The γ^ν are Dirac matrices satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \{\gamma^\mu; \gamma^\nu\} = 2g^{\mu\nu}$, with $g^{00} = -1$, $g^{11} = g^{22} = g^{33} = +1$. They are related to the Dirac matrices β and α^ν (with $\alpha^0 = 1$) by $\gamma^\nu = -i\beta\alpha^\nu$, while $\bar{\psi} = \psi^\dagger\beta$, if $\psi^\dagger$ is the one-row four-column hermitian conjugate of the four-row one-column matrix of the q-number ψ. The charge current-density four-vector j^ν (with $j^0 = \rho$ in e.s.u.) is given by

$$j^\nu = -ie\bar{\psi}\gamma^\nu\psi, \tag{2}$$

where $(-e)$ stands for the "mesonic charge" of the Dirac particles. (For $\kappa_0 \to 0$ this formalism will then describe the electromagnetic interactions of Dirac electrons with an electric charge $-e$.)

Independent variation of the six-component antisymmetric tensor $F^{\mu\nu}$, of the four-vector A_ν and of the undors (four-spinors) ψ and $\bar{\psi}$ in $\delta\int L d\omega = 0$ (with $d\omega = dxdydzdct$), yields in the usual way the field equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3}$$

$$\kappa_0^2 A^\nu = 4\pi j^\nu + \partial_\mu F^{\mu\nu}, \tag{4}$$

$$\{\kappa + \gamma^\nu\partial_\nu + (ie/\hbar c)A_\nu\gamma^\nu\}\psi = 0, \text{ and conj.} \tag{5}$$

From (2) with (5) follows

$$\partial_\nu j^\nu = 0. \tag{6}$$

This with (4) gives

$$\partial_\nu A^\nu = 0, \tag{7}$$

so that, by (3), $\partial_\mu F^{\mu\nu} = \Box A^\nu$ (with $\Box = \partial_\mu\partial^\mu$), or, by (4),

$$(\Box - \kappa_0^2)A^\nu = -4\pi j^\nu. \tag{8}$$

A solution of this equation is given by the four-dimensional integral

$$A^\nu{}_{\text{direct}} = -2\pi \int d\omega' \Delta_0(x - x') \operatorname{sgn}(t - t') j^\nu(x'), \tag{9}$$

where x stands for $\mathbf{x}$ and x^0 (that is, for x, y, z and ct), while $\operatorname{sgn} t = |t|^{-1}t = \operatorname{sgn} x^0$, so that

$$\partial_0 \operatorname{sgn} t = 2\delta(ct). \tag{10}$$

Finally, $\Delta_0(x)$ stands for

$$\Delta_0(x) = \frac{-i}{8\pi^3}\int (dk)^4 \cdot \delta(k_\mu k^\mu + \kappa_0^2) \cdot \operatorname{sgn} k^0 \cdot \exp(ik_\nu x^\nu), \tag{11}$$

where $(dk)^4 = \mathbf{dk}dk^0 = dk_x dk_y dk_z dk^0$, and where $\delta(\ )$ is the ordinary delta-function. Integration over k^0 in (11) gives

$$\Delta_0(x) = -\int \frac{\mathbf{dk}}{8\pi^3} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{\sin\{(\kappa_0^2 + k^2)^{\frac{1}{2}} ct\}}{(\kappa_0^2 + k^2)^{\frac{1}{2}}}, \tag{12}$$

with $k = |\mathbf{k}|$; so that obviously Δ_0 is an improper function of r and t, odd in t, and satisfying

$$\{\Box - \kappa_0^2\}\Delta_0(x) = 0, \quad \Delta_0(x) = -\Delta_0(-x), \tag{13a–b}$$

$$\Delta_0(t = 0) = 0, \tag{14a}$$

$$\{\partial_0\Delta_0(x)\}_{t=0} = -\delta(\mathbf{x}) = -\delta(x)\delta(y)\delta(z), \tag{14b}$$

so that

$$\partial_0^2\{\Delta_0(x - x') \operatorname{sgn}(t - t')\} = \partial_0^2\Delta_0 . \operatorname{sgn}(t - t') - 2\delta(x_0 - x_0')\delta(\mathbf{x} - \mathbf{x}') - \partial_0'\{2\Delta_0(x - x')\delta(x_0 - x_0')\}.$$

From this, with (13a) and (14a), it follows easily that (9) is a solution of (8). For $\kappa_0 \to 0$ this solution would correspond to the average of the advanced and the retarded potential, as seen from

$$\lim_{\kappa_0 \to 0} \Delta_0(x) = D(x) = (4\pi r)^{-1}\{\delta(r + ct) - \delta(r - ct)\}, \tag{15}$$

which follows from (11) by integration over angles after $\kappa_0 \to 0$. The solution of (8) corresponding to the retarded potential could easily be obtained by replacing $\operatorname{sgn}(t - t')$ in (9) by $\{1 + \operatorname{sgn}(t - t')\}$.

The general solution of (8) can be written as the sum of (9) and some arbitrary superposition of plane wave solutions of the homogeneous wave equation with a factor $\exp(ik_\mu x^\mu)$, with $k_\mu k^\mu = k^2 - k_0^2 = -\kappa_0^2$.

Before we bring our formalism in canonical form, it is convenient to introduce three-dimensional notation throughout by $F_{l0} = F^{0l} = \mathbf{E}_l$, $F_{12} = \mathbf{H}_z$, etc., by $A^0 = \Phi$, $j^0 = \rho = -e\psi^\dagger\psi$, $\mathbf{j} = -e\psi^\dagger\boldsymbol{\alpha}\psi$, etc. Thus, the first-order Lagrangian function (1) becomes

$$L = (1/4\pi)\{\tfrac{1}{2}\mathbf{H}^2 - \mathbf{H}\cdot\operatorname{curl}\mathbf{A} - \tfrac{1}{2}\mathbf{E}^2 - \mathbf{E}\cdot\nabla\Phi - \mathbf{E}\cdot\partial_0\mathbf{A} + \tfrac{1}{2}\kappa_0^2(\Phi^2 - \mathbf{A}^2)\} + \mathbf{A}\cdot\mathbf{j} - \Phi\rho - \hbar c\psi^\dagger(\kappa\beta - i\boldsymbol{\alpha}\cdot\nabla - i\partial_0)\psi. \tag{16}$$

The field Eqs. (3)–(5) partially yield equations of motion

$$\partial\mathbf{A}/c\partial t = -\mathbf{E} - \nabla\Phi, \tag{17}$$

$$\partial\mathbf{E}/c\partial t = \operatorname{curl}\mathbf{H} + \kappa_0^2\mathbf{A} - 4\pi\mathbf{j}, \tag{18}$$

$$i\hbar\partial\psi/\partial t = (mc^2\beta - i\hbar c\boldsymbol{\alpha}\cdot\nabla - e\Phi + e\mathbf{A}\cdot\boldsymbol{\alpha})\psi; \tag{19}$$

partially they yield so-called "identities" defining $\mathbf{H}$ and Φ as *derived variables:*[16]

$$\mathbf{H} = \operatorname{curl}\mathbf{A}, \tag{20}$$

$$\kappa_0^2\Phi = 4\pi\rho - \operatorname{div}\mathbf{E}. \tag{21}$$

From (16) it is readily seen that the canonical conjugates to $\mathbf{A}$ and ψ are given by $(-\mathbf{E}/4\pi c)$ and

to have been "symmetrized" as $\frac{1}{2}\{F^{\mu\nu}\partial_\mu A_\nu + (\partial_\mu A_\nu)F^{\mu\nu}\}$ or $\frac{1}{2}\{\bar{\psi}(\gamma^\nu\partial_\nu\psi) - (\gamma^\nu\partial_\nu\psi)\bar{\psi}\}$ or $\frac{1}{2}\kappa_0\{\bar{\psi}\psi - \psi\bar{\psi}\}$, according to the rules of the general quantum theory of wave fields discussed in Physica **7**, 765 (1940), even if such "symmetrization" is not written down here explicitly.—A minus sign in such symmetrizations occurs only if two field variables are involved, *both* describing a field of Fermi-Dirac particles (obeying the Pauli exclusion principle).

[16] Compare F. J. Belinfante, Physica **7**, 765 (1940).

($i\hbar\psi^\dagger$), so that the commutation relations read

$$[\mathbf{A}_i(\mathbf{x})\,;\,\mathbf{E}_j(\mathbf{x}')] = -4\pi i\hbar c\delta_{ij}\delta(\mathbf{x}-\mathbf{x}'), \quad (22a)$$

$$\{\psi_\alpha(\mathbf{x})\,;\,\psi_\beta^\dagger(\mathbf{x}')\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'), \quad (22b)$$

$$\{\psi_\alpha\,;\,\psi_\beta\} = \{\psi_\alpha^\dagger\,;\,\psi_\beta^\dagger\} = 0, \text{ etc.}$$

The commutation relations between $\mathbf{H}$ and $\mathbf{E}$ or between Φ and $\mathbf{A}$ or Φ and ψ follow in accordance with the general rules of the theory of quantized fields[16] from the canonical commutation relations (22) by the identities (20)–(21); for instance:

$$[\Phi(\mathbf{x})\,;\,\mathbf{A}(\mathbf{x}')] = -(4\pi i\hbar c/\kappa_0^2)\nabla\delta(\mathbf{x}-\mathbf{x}'), \quad (23)$$

$$[\Phi(\mathbf{x})\,;\,\psi(\mathbf{x}')] = (4\pi e/\kappa_0^2)\delta(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}). \quad (24)$$

From (16) the Hamiltonian is formed in the usual way by $\mathcal{H} = \int d\mathbf{x}\{-\mathbf{E}\cdot\dot{\mathbf{A}}/4\pi c + i\hbar\psi^\dagger\dot{\psi} - L\}$; it is expressed in terms of the canonical variables by means of the identities (20)–(21). This gives

$$\mathcal{H} = \mathcal{H}_f + \mathcal{H}_m + \mathcal{W}, \quad (25)$$

with

$$8\pi\mathcal{H}_f = \int d\mathbf{x}\{\mathbf{E}^2 + \kappa_0^{-2}(\mathrm{div}\mathbf{E})^2 + \kappa_0^2\mathbf{A}^2 + (\mathrm{curl}\mathbf{A})^2\}, \quad (25a)$$

$$\mathcal{H}_m = \hbar c\int d\mathbf{x}\psi^\dagger(\beta\kappa - i\boldsymbol{\alpha}\cdot\nabla)\psi, \quad (25b)$$

$$\mathcal{W} = \int W d\mathbf{x}, \text{ with}$$

$$W = \rho\Phi - \mathbf{j}\cdot\mathbf{A} - 2\pi\kappa_0^{-2}\rho^2. \quad (25c)$$

Finally, the total linear momentum of the field is given in the usual way by

$$\mathcal{P}_l = \int d\mathbf{x}\{\mathbf{E}\cdot\nabla_l\mathbf{A}/4\pi c - i\hbar\psi^\dagger\nabla_l\psi\}. \quad (26)$$

2. THE GENERALIZED SCHROEDINGER EQUATION IN INTERACTION REPRESENTATION, AND ITS INTEGRABILITY

Following the scheme used in quantum electrodynamics[1,2] we shall now introduce a new representation by a canonical transformation $U[\sigma]$ or U_σ depending on a space-like three-dimensional surface σ in four-dimensional space-time. If $Q(x)$ is in Heisenberg representation some q-number (function of field variables) at the point x in space-time, and σ is some space-like surface *through* x, then the transformed of $Q(x)$ in interaction representation shall be given by

$${}^\sigma Q(x) = U_\sigma Q(x) U_\sigma^{-1}. \quad (27)$$

Similarly, the constant state vector χ of Heisenberg's representation shall be transformed into a σ-dependent Schroedinger functional $\Psi[\sigma]$ by means of

$$\Psi[\sigma] = U_\sigma\chi, \quad (28)$$

so that

$$\langle Q\rangle = (\chi^*, Q\chi) = (\Psi[\sigma]^*, {}^\sigma Q\Psi[\sigma]). \quad (29)$$

The transformation U_σ shall satisfy the variational equation

$$i\hbar c\{\delta U_\sigma/\delta\sigma(x)\} = {}^\sigma W(x)U_\sigma = U_\sigma W(x), \quad (30)$$

where $\delta U_\sigma/\delta\sigma(x) = \lim_{\sigma'\to\sigma}\{(U_{\sigma'} - U_\sigma)/\omega\}$, if ω is the volume between the two space-like surfaces σ and σ' and if the limit $\sigma'\to\sigma$ is taken in such a way that the region where σ' does not coincide with σ shrinks together to the infinitesimal neighborhood of the point x on σ. As we assume σ' as well as σ always to be space-like, the volume ω will always have a flat shape and the dimensions of the volume ω in spatial directions cannot be shrunk faster than the time-like distance between σ and σ'. From (30) follows the generalized Schroedinger equation

$$i\hbar c\{\delta\Psi[\sigma]/\delta\sigma(x)\} = {}^\sigma W(x)\,\Psi[\sigma]. \quad (31)$$

While here the transformed interaction operator ${}^\sigma W$ is used, we shall first use (30) in its second form with $U_\sigma W$ rather than with ${}^\sigma W U_\sigma$, as we do know the properties of W but did not yet discuss the properties of ${}^\sigma W$.

We should postulate that U_σ, once given on some arbitrarily given space-like initial surface, should follow for any other surface uniquely from the "equation of motion" (30). A look at (25c), however, shows at once that (30) taken without comment would not even determine uniquely the effect on U_σ by an infinitesimal change of σ. For W as given by (25c) is no scalar, so that the infinitesimal increment of U_σ from σ to σ', as given by (30), will depend on the coordinate system (Lorentz frame), in which $W(x)$ in (30) is taken.

In order to give (30) a definite and unique meaning, therefore, we have to specify explicitly in which frame or reference $W(x)$ in (30) is to be taken. Obviously there is for that only one Lorentz-invariant choice, and that is *that in Eqs. (30)–(31) $W(x)$ should be taken in that Lorentz frame, for which the xyz-space (the surface t = constant) is tangential to the surface σ at the point x.* This choice of the Lorentz frame indeed depends on σ at x alone and is independent of the particular coordinate system, in which surfaces like σ may be described as $t' = f(x', y', z')$. It is also seen that W is invariant under spatial rotations of the Lorentz frame, that keep its time-axis four-dimensionally perpendicular to σ at x. Thus, Eq. (30) has obtained a unique meaning, which could also be given in covariant notation (without particular choice of a Lorentz frame) as $i\hbar c\{\delta U_\sigma/\delta\sigma(x)\} = U_\sigma W(x, \sigma)$, with

$$W(x, \sigma) = -j^\nu A_\nu - 2\pi\kappa_0^{-2}j_\mu N^\mu j_\nu N^\nu, \quad (32)$$

where N^μ is a time-like unit four-vector at x four-dimensionally perpendicular to σ and normalized by $N_\mu N^\mu = -1$.

One may perhaps think that this method is cumbersome. Why not replace W say by $\rho\Phi - \mathbf{j}\cdot\mathbf{A}$? Let us for a moment leave open this question and take W in (30) equal to

$$W = \rho\Phi - \mathbf{j}\cdot\mathbf{A} - \lambda\cdot 2\pi\kappa_0^{-2}\rho^2. \tag{33}$$

For $\lambda \neq 0$, we take (33) to be a correct expression for W in (30)–(31) only if (33) is calculated in the special Lorentz frame described above.[17] We shall now determine the value of the not yet specified constant λ in (33) from the *postulate that Eq. (30) be integrable.* We shall see that this leads to $\lambda = 1$ rather than to $\lambda = 0$.

In order to verify the integrability of our variational equation (30), it is sufficient to consider a variation of σ in two infinitesimal steps, and to show that the change of U_σ resulting from this by (30) is independent of the order of sequence of these steps, as long as the final surface is the same. To facilitate our work, we shall assume that the surfaces are given in some fixed coordinate system, say $\{\xi\eta\zeta\tau\}$, where τ may take the place of x^0 or ct. We denote the initial surface $\tau = f(\xi, \eta, \zeta)$ by 0, the final surface by $\bar{1}$. The shift of σ from 0 to $\bar{1}$ takes place in two steps, first from 0 to 1, later from 1 to $\bar{1}$. The surface 1 be given by $\tau = f(\xi, \eta, \zeta) + \epsilon(\xi, \eta, \zeta)$, where the infinitesimal function ϵ is the variation from 0 to 1 in the τ-direction. Similarly, the τ-shift from 1 to $\bar{1}$ may be given by the infinitesimal function $\varphi(\xi, \eta, \zeta)$. One may also perform first the variation of σ over a distance φ in the τ-direction, which may give an intermediate surface $\bar{0}$; then take the variation ϵ as the second step from $\bar{0}$ to $\bar{1}$. It is now sufficient to check that, for infinitesimal values of both ϵ and φ, taking into account only terms linear in ϵ and linear in φ (including the bilinear term in both), one obtains the same change of U_σ from 0 to $\bar{1}$ independent of the choice between 1 and $\bar{0}$ for the intermediate step.[18]

By $W(1)$ we shall understand the value of $W(x, \sigma)$ in a point on 1; that is, a value of W as found from Eq. (33) in a "local" Lorentz frame "tangent to" 1 (=with its time-axis perpendicular to 1). For $W(0)$ we shall often write simply W. By $\partial_\tau W$, we shall understand the derivative with respect to τ of W as given by (33), calculated as if the Lorentz frame is not changed with τ. This is the derivative of $W(x, \sigma)$ of Eq. (32), with respect to τ, if the N^μ-vector is thought to be constant. As the local Lorentz frames on 0 and on 1 are not identical, however, we must then write

$$W(1) = W(0) + \epsilon\cdot\partial_\tau W + (\mathbf{b}\cdot\mathbf{T}). \tag{34}$$

Here, the last term represents the change of W from 0 to 1 as far as due to the variation of N^μ, so that it may be regarded as the effect, on W as given by Eq. (33), of the infinitesimal Lorentz transformation from the local Lorentz system tangent to 0, to the local Lorentz frame on 1 in the corresponding point (with same ξ, η, ζ). Let $\mathbf{b}$ be defined as the infinitesimal "velocity" (in unit c) of the local frame on 1 with respect to the local frame on 0. Then, $\mathbf{T}$ in (34) is given, *in the local frame on* 0, by[19]

$$\mathbf{T} = 4\pi\lambda\kappa_0^{-2}\rho\mathbf{j}. \tag{35}$$

Let $\mathbf{v}$ be the (finite) velocity (with respect to the fixed $\{\xi\eta\zeta\tau\}$-frame) of the local 0-frame, tangent to the surface 0 that was given as $\tau = f(\xi\eta\zeta)$. Then it is easily seen that $\mathbf{v}/c = \boldsymbol{\partial} f$, if $\boldsymbol{\partial}$ denotes the gradient in $\xi\eta\zeta$-space. Thence, the difference in velocity (in units c) of the 1-frame and the 0-frame, with respect to this $\xi\eta\zeta$-space, is given by $\boldsymbol{\partial}\epsilon$.

From this, the relative velocity $c\mathbf{b}$ of 1 with respect to 0 is found according to the rules of addition of velocities in special relativity theory. Thus, if $\mathbf{v}_1$ is the velocity of the 1-frame with respect to the fixed $\xi\eta\zeta$-system, then the relative velocity of 1 with respect to 0 is given by

$$c\mathbf{b} = \frac{\mathbf{v}\{v^{-2}(\mathbf{v}\cdot\mathbf{v}_1) - 1\} + \{\mathbf{v}_1 - \mathbf{v}v^{-2}(\mathbf{v}\cdot\mathbf{v}_1)\}(1 - v^2/c^2)^{\frac{1}{2}}}{1 - \mathbf{v}\cdot\mathbf{v}_1/c^2}.$$

For $\mathbf{v}_1 = \mathbf{v} + c\boldsymbol{\partial}\epsilon$, with infinitesimal $\boldsymbol{\partial}\epsilon$, this amounts to

$$\mathbf{b} = \frac{(\boldsymbol{\partial}\epsilon)_v}{1 - v^2/c^2} + \frac{\boldsymbol{\partial}\epsilon - (\boldsymbol{\partial}\epsilon)_v}{(1 - v^2/c^2)^{\frac{1}{2}}}.$$

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[17] In covariant notation this might be written as

$$W(x, \sigma) = -j^\nu A_\nu - \lambda\cdot 2\pi\kappa_0^{-2} j_\mu N^\mu j_\nu N^\nu.$$

[18] One may, of course, also postulate that this be true for *any* intermediate surface $\bar{0}$, where the τ-shifts between 0 and $\bar{0}$ and between $\bar{0}$ and $\bar{1}$ be given by $\bar{\varphi}$ and $\bar{\epsilon}$, provided that $\bar{\varphi} + \bar{\epsilon} = \epsilon + \varphi$. In that case, however, it is necessary to take into account not only bilinear terms, but all terms quadratic in one of these infinitesimal τ-shifts as well, which complicates the proof considerably. In this case, Eq. (37) should be replaced by the more accurate

$$U[1] = U[0]\cdot\exp\int_0^1 d\omega\cdot Wa$$
$$= U[0]\cdot\left\{1 + \int \epsilon Wa + \tfrac{1}{2}\int \epsilon^2\cdot\partial_\tau W\cdot a + \tfrac{1}{2}\int \epsilon(\mathbf{b}\cdot\mathbf{T})a + \tfrac{1}{2}\int\int' \epsilon\epsilon' W W' a^2\right\}.$$

The further calculations run in principle like those in the text, only more terms have to be taken into account. This is left to the reader. The complications introduced in this way are, of course, completely needless, as one could always introduce another intermediate surface (1′) at a distance $(\epsilon + \bar{\varphi})$ from 0, and then consider the change from 0 to $\bar{1}$ as occurring in three steps, once $\underline{0 \to 1 \to 1'} \to \bar{1}$ and once $\underline{0 \to \bar{0} \to 1'}$ $\to \bar{1}$. Here only the underlined parts of the shifts occur in a different order of sequence and they definitely satisfy now the simplification made in the text, that $1 \to 1'$ and $0 \to \bar{0}$ are equal τ-shifts.

[19] This follows from the fact that (up to terms linear in $\mathbf{b}$) the charge density ρ in Eq. (33) transforms, from the local Lorentz frame $\{x\,y\,z\,ct\}$ on 0 to the local Lorentz frame $\{x'y'z'ct'\}$ on 1, by $\rho' = \rho - \mathbf{b}\cdot\mathbf{j}$ (just as $ct' = ct - \mathbf{b}\cdot\mathbf{x}$).

Here, $(\partial\epsilon)_v \equiv \mathbf{v}v^{-2}(\mathbf{v}\cdot\partial\epsilon)$ is the component of $\partial\epsilon$ in the direction of $\mathbf{v}$, and $\partial\epsilon-(\partial\epsilon)_v$ the component of $\partial\epsilon$ perpendicular to $\mathbf{v}$. Introducing the symbolic notation w for a multiplication by $(1-v^2/c^2)^{-1}$ of the components parallel to $\mathbf{v}$, and by $(1-\mathbf{v}^2/c^2)^{-\frac{1}{2}}$ of the components perpendicular to $\mathbf{v}$, we may then write for the above result

$$\mathbf{b}=w\partial\epsilon. \tag{36}$$

Similarly, the velocity (in units c) of $\bar{0}$ with respect to 0 will be given by $\bar{\mathbf{b}}=w\partial\varphi$.

Now, integration of (30) from 0 to 1 gives, up to linear terms in ϵ:

$$U[1]=U[0]\left\{1+\int \epsilon Wa\right\}. \tag{37}$$

Here $a=(i\hbar c)^{-1}$, while $\int$ stands for an integration over ξ, η and ζ. Similarly,

$$U[\bar{1}]=U[1]\left\{1+\int \varphi W(1)a\right\}. \tag{38}$$

Into (38) we substitute both (37) and (34). Up to the bilinear term this gives

$$U[\bar{1}]=U[0]\left\{1+\int \epsilon Wa+\int \varphi Wa +\int \varphi\epsilon\cdot\partial_\tau W\cdot a+\int \varphi(\mathbf{b}\cdot\mathbf{T})a +\int\int' \epsilon\varphi' WW'a^2\right\}. \tag{39}$$

Here, W' stands for $W(\xi',\eta',\zeta')$. Similarly, if $\bar{0}$ instead of 1 is taken as intermediate surface:

$$U[\bar{1}]=U[0]\left\{1+\int \varphi Wa+\int \epsilon Wa +\int \epsilon\varphi\cdot\partial_\tau W\cdot a+\int \epsilon(\bar{\mathbf{b}}\cdot\mathbf{T})a +\int'\int \varphi'\epsilon W'Wa^2\right\}. \tag{40}$$

The integrability of (30) then requires the vanishing of the difference between (39) and (40), that is,

$$a\int\int' \epsilon\varphi'[W;\,W']=\int\{\epsilon\bar{\mathbf{b}}-\varphi\mathbf{b}\}\cdot\mathbf{T}. \tag{41}$$

The commutator of $W(\xi\eta\zeta)$ with $W(\xi'\eta'\zeta')$ on the surface 0 can be calculated easily with the help of (22)–(24) *in terms of the coordinates on this surface itself*, for which W is simply given by (33). Thus, the commutators of ρ and $\mathbf{j}$ with Φ yield delta-functions (symmetric in $\mathbf{x}-\mathbf{x}'$) multiplied by q-number functions of $\mathbf{x}$ and $\mathbf{x}'$ that must then obviously be antisymmetric in $\mathbf{x}$ and $\mathbf{x}'$, so that they vanish after integration in the left-hand member of (41) for $\mathbf{x}=\mathbf{x}'$ due to the delta-functions. Therefore, these terms are of no interest to us. The only interesting terms are products of q-number functions of $\mathbf{x}$ and $\mathbf{x}'$ with *gradients* of delta-functions, which arise from the commutators of Φ with $\mathbf{A}'$ and of $\mathbf{A}$ with Φ' by (23). This gives

$$[W;\,W']=4\pi i\hbar c\kappa_0^{-2}\{\rho\mathbf{j}'\cdot\nabla-\rho'\mathbf{j}\cdot\nabla'\}\delta(\mathbf{x}-\mathbf{x}'). \tag{42}$$

We remark that $\delta(\mathbf{x}-\mathbf{x}')=(1-v^2/c^2)^{-\frac{1}{2}}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}')$, and that the gradient (∇) with respect to $\mathbf{x}$ on the surface 0 is the gradient (∂) with respect to $\boldsymbol{\xi}$ multiplied by another factor $(1-v^2/c^2)^{-\frac{1}{2}}$ as far as this gradient is calculated in the direction of $\mathbf{v}$. Combining these results, we get

$$\nabla\delta(\mathbf{x}-\mathbf{x}')=w\partial\delta(\boldsymbol{\xi}-\boldsymbol{\xi}'). \tag{43}$$

We substitute (42) with (43) into the left-hand member of (41) and integrate each term once by parts. Into the right-hand member of (41) we substitute Eqs. (35) and (36). Thus we obtain, after division by $4\pi\kappa_0^{-2}=4\pi a i\hbar c\kappa_0^{-2}$:

$$\int\int'\{\epsilon\mathbf{j}\cdot\partial'(\varphi'\rho'w')-\varphi'\mathbf{j}'\cdot\partial(\epsilon\rho w)\}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}') =\lambda\int\rho\mathbf{j}\{\epsilon w\partial\varphi-\varphi w\partial\epsilon\}. \tag{44}$$

Performing the integral over $\boldsymbol{\xi}'$ and remarking that the terms with $\partial(\rho w)$ drop out, we find that (44) *is an identity, if* $\lambda=1$.

This means that *the postulate of integrability of the Schroedinger equation for U_σ excludes a scalar interaction operator $W(x)$ in this meson theory* and that *the variational Eq. (30) for the transformation U_σ is integrable just with the choice* (25c) *or (32) for the interaction operator.*[20]

3. THE RELATIVISTIC TRANSFORMATION PROPERTIES OF THE q-NUMBERS IN INTERACTION REPRESENTATION

In the preceding chapter we have defined by Eq. (27) the transformed q-numbers in interaction representation as certain functions of x as well as of a surface of transformation σ through x. We

[20] We found the operator $W(x,\sigma)$ which makes (30) integrable simply by postulating that the interaction integrand $W(x)$ from (25c) be taken in a Lorentz frame "tangent" to σ in x. There was no need of finding an auxiliary term to be added to $W(x)$. This seems a simplification as compared to the methods suggested by the Japanese school (see footnote 6). Remark that we do *not* yet confine ourselves here to flat surfaces: $\mathbf{v}$ and then w above may be functions of ξ, η, ζ.

shall first investigate in how far these ${}^{\sigma}Q(x)$ depend on σ.

For this purpose, let us consider a variation $\epsilon(\xi, \eta, \zeta)$ of σ in the τ-direction, like in the preceding chapter. Then, U again is varied according to (37), or

$$\delta U = U \int' \epsilon' W' a, \qquad (45)$$

where W' is to be taken at the point of the surface σ with coordinates $\boldsymbol{\xi}'(\xi', \eta', \zeta')$ in the fixed coordinate system, and to be calculated there either in a tangent Lorentz frame, or by Eq. (32). From (45), by $U^{-1}\cdot\delta U + \delta U^{-1}\cdot U = 0$, we find

$$\delta U^{-1} = -U^{-1}\cdot\delta U\cdot U^{-1} = -\int' \epsilon' W' a \cdot U^{-1}. \quad (46)$$

Assuming that the point x on σ is not varied, we find by (45) and (46) that an infinitesimal variation of the surface σ without change of the fixed point P with coordinates x (or $\boldsymbol{\xi}$) on it, changes ${}^{\sigma}Q(x)$ by an amount

$$\delta\, {}^{\sigma}Q(x) = \delta U\cdot QU^{-1} + UQ\cdot\delta U^{-1}$$
$$= U\cdot \int' a\epsilon' [W'Q - QW']\cdot U^{-1}, \quad (47)$$

where the integral is to be taken over all points (P') of the surface σ and where the coordinates $\xi'\eta'\zeta'$ are taken as the integration variables.

In general, the commutator $[W(P'); Q(P)]$ appearing in (47) will be zero for *finite* space-like distances PP', so that we conclude that the expression (47) is independent of variations $\epsilon(\boldsymbol{\xi}')$ of the surface at points P' at a finite distance from the fixed point P. However, (47) may easily depend on the value of the spatial derivatives of $\epsilon(\boldsymbol{\xi}')$ at the point P (at $\boldsymbol{\xi}' = \boldsymbol{\xi}$), and surely will do so as soon as $[W'; Q]$ contains gradients of delta-functions.

This means that the transformed q-numbers in interaction representation in general will depend on the local slope of the surface of transformation σ, or even on the gradients or higher spatial derivatives of this slope, if $[W'; Q]$ involves higher derivatives of delta-functions. (The value of ϵ itself at P must of course be zero, as we kept the varied surface through P.)

We have to see the question of the Lorentz transformation of these transformed q-numbers ${}^{\sigma}Q(x)$ now in the light of this σ-dependency of ${}^{\sigma}Q(x)$. We may, in particular, distinguish the following two questions:

(A) Let $Q(P)$ be given in a Lorentz frame K and ${}^{\sigma}Q(P)$ be obtained from it by a transformation U_σ belonging to a certain "transformation surface" σ (not related to K in any particular way). Let ${}^{\sigma}Q'(P)$ be the Lorentz-transformed of ${}^{\sigma}Q(P)$, obtained by the same transformation surface σ from $Q'(P)$ in a new Lorentz frame K'. What is the relation between ${}^{\sigma}Q'(P)$ and ${}^{\sigma}Q(P)$?

The answer is obviously, by Eq. (27), that under these circumstances ${}^{\sigma}Q(P)$ will transform in exactly the same way as the q-number $Q(P)$ in Heisenberg representation, for U_σ in (27) is determined here independent of the Lorentz frame and depends only on the surface σ, which was thought to be constant.

(B) Let ${}^{\circ}Q(P)$ be the value which ${}^{\sigma}Q(P)$ takes for σ in P sufficiently flat and "tangent" to the Lorentz frame K, in which the components of Q are taken. Let ${}^{\circ}Q'(P)$ be the value which ${}^{\sigma'}Q'(P)$ takes for a new σ' in P sufficiently flat and tangent to the new Lorentz frame K', in which the components of Q' are measured. What is the transformation in this case from ${}^{\circ}Q(P)$ to ${}^{\circ}Q'(P)$?

This is the question we are going to discuss now. Its importance lies in the fact that in most practical examples, one will turn the σ-dependent quantity ${}^{\sigma}Q(x)$ into a simple point function ${}^{\circ}Q(x)$ by taking for σ the surface t = constant, which indeed is completely flat and is tangent to the Lorentz frame, in which t is measured. Then, a Lorentz transformation is automatically a transformation of the type (B) rather than one of the type (A).

In the following, we shall often call ${}^{\circ}Q$ **the** value of Q in interaction representation, and the transformations (B) **the** Lorentz transformations of the q-numbers in interaction representation.

Again we consider infinitesimal transformations only. (The transformation formulas for finite transformations would then follow from them by integration.) We follow the notations of the preceding chapter. For the initial surface we may take now a completely flat surface and take this flat surface at once as our $\xi\eta\zeta$-space without any loss of generality, so that the complications with powers of $(1-v^2/c^2)$ of the preceding chapter this time are avoided, while for W we can now take simply (25c) without making a distinction between a $\xi\eta\zeta$- and an xyz-space.[21]

The transformed surface $\bar{\sigma}$ in a frame of reference $\bar{K}$ moving with a velocity $c\mathbf{b}$ with respect to K is then given near P by $\tau = \mathbf{b}\cdot\mathbf{r}'$ (with $\mathbf{r}' = \mathbf{x}' - \mathbf{x}$), so that the "$\tau$-shift" of $\bar{\sigma}$ with respect to σ is given by

$$\epsilon' = \mathbf{b}\cdot\mathbf{r}' = \mathbf{b}\cdot(\mathbf{x}' - \mathbf{x}) \text{ for } \mathbf{r}' \text{ not too large.} \quad (48)$$

At a larger distance, the shape of $\bar{\sigma}$ does not matter and need not be flat, so that we may consider ϵ' as approaching in some way to zero for $\mathbf{x}'$ moving to infinity.

[21] We are allowed to flatten out the initial σ, because this process does not change ${}^{\sigma}Q(P)$ as long as we do not change the *first few* spatial derivatives of the function $\tau(\xi, \eta, \zeta)$ first determining σ. The exact number of these *first few* derivatives depends on the orders of derivatives of delta-functions that might appear in $[W(P'); Q(P)]$, as explained above. If the surface σ originally was "sufficiently flat" (first few partial derivatives of $\boldsymbol{\partial}\tau$ equal to zero), a change of these derivatives of $\boldsymbol{\partial}\tau$ at P itself need not be made indeed while flattening the surface. As in ${}^{\circ}Q(P)$ the transformation surface should at P be tangent to the Lorentz frame used, we conclude that the surface σ used in K, when flattened out, simply becomes the surface t = constant, and ordinary x, y, z-coordinates take the place of ξ, η, ζ.

Let us now think $[W(P'); Q(P)]$ to be expanded in derivatives of delta-functions of $(\mathbf{x}'-\mathbf{x})$, by

$$[W(P'); Q(P)] = \mathbf{f}_0(P', P)\delta(\mathbf{r}') + \mathbf{f}_1(P', P)\cdot\nabla'\delta(\mathbf{r}') + \sum_k\sum_l \mathbf{f}_{kl}(P', P)\nabla_k'\nabla_l'\delta(\mathbf{r}') + \cdots, \quad (49)$$

and substitute this expansion into (47). Adding to this the variation of ${}^\sigma Q(x)$ due to the Lorentz transformation of Q itself in Heisenberg representation (which we shall denote by δQ), we find for the total change of ${}^\circ Q(x)$ under such an infinitesimal transformation, after integrations by parts and with use of (48) for ϵ' near x,

$$\delta\,{}^\circ Q(x) = U\{\delta Q(x) - a\mathbf{b}\cdot\mathbf{f}_1(P, P) + a\sum_k\sum_l[(b_k\nabla_l' + b_l\nabla_k')\mathbf{f}_{kl}(P', P)]_{(P'=P)} - \cdots\}U^{-1}. \quad (50)$$

We shall apply this formula now for the calculation of the transformation properties of ${}^\circ\mathbf{A}$, ${}^\circ\Phi$, ${}^\circ\mathbf{E}$, ${}^\circ\mathbf{H}$, ${}^\circ\psi$, ${}^\circ\psi^\dagger$, ${}^\circ\rho$ and ${}^\circ\mathbf{j}$ for our neutral Proca meson field interacting with Dirac electrons. In Heisenberg representation, the infinitesimal Lorentz transformation ($c\bar{t} = ct - \mathbf{b}\cdot\mathbf{r}'$, $\bar{\mathbf{r}}' = \mathbf{r}' - \mathbf{b}ct$) from K to $\bar{K}$ gives

$$\begin{aligned} &\delta\mathbf{A} = -\mathbf{b}\Phi, \quad \delta\Phi = -\mathbf{b}\cdot\mathbf{A}, \quad \delta\mathbf{E} = [\mathbf{b}\times\mathbf{H}], \\ &\delta\mathbf{H} = -[\mathbf{b}\times\mathbf{E}], \quad \delta\psi = -\tfrac{1}{2}\mathbf{b}\cdot\boldsymbol{\alpha}\psi, \\ &\delta\psi^\dagger = -\tfrac{1}{2}\psi^\dagger\boldsymbol{\alpha}\cdot\mathbf{b}, \quad \delta\mathbf{j} = -\mathbf{b}\rho, \quad \delta\rho = -\mathbf{b}\cdot\mathbf{j}. \end{aligned} \quad (51)$$

Of these quantities, $\mathbf{A}$ and Φ by (23) yield first derivatives of delta-functions, when commuted with W:

$$[W(\mathbf{x}'); \mathbf{A}(\mathbf{x})] = -4\pi i\hbar c\kappa_0^{-2}\rho(\mathbf{x}')\nabla'\delta(\mathbf{r}'), \quad (52)$$

$$[W(\mathbf{x}'); \Phi(\mathbf{x})] = +4\pi i\hbar c\kappa_0^{-2}\mathbf{j}(\mathbf{x}')\nabla'\delta(\mathbf{r}'). \quad (53)$$

Taking the values of $\mathbf{f}_1$ for $\mathbf{A}$ and for Φ from (52)–(53) by comparison with (49), and substituting this with (51) into (50), we find (with $ai\hbar c = 1$):

$$\delta\,{}^\circ\mathbf{A} = U\{-\mathbf{b}\Phi + 4\pi\kappa_0^{-2}\mathbf{b}\rho\}U^{-1} = -\mathbf{b}\{{}^\circ\Phi - 4\pi\kappa_0^{-2}\,{}^\circ\rho\}, \quad (54)$$

$$\delta\,{}^\circ\Phi = U\{-\mathbf{b}\cdot\mathbf{A} - 4\pi\kappa_0^{-2}\mathbf{b}\cdot\mathbf{j}\}U^{-1} = -\mathbf{b}\cdot\{{}^\circ\mathbf{A} + 4\pi\kappa_0^{-2}\,{}^\circ\mathbf{j}\}. \quad (55)$$

For $\mathbf{H}$ we find, by (20) and (52),

$$[W(\mathbf{x}'); \mathbf{H}(\mathbf{x})] = \mathrm{curl}[W(\mathbf{x}'); \mathbf{A}(\mathbf{x})] = 0, \quad (56)$$

by $\mathrm{curl}\nabla = 0$. The other quantities listed give merely ordinary delta-functions (if anything). As $\mathbf{f}_0(P', P)$ does not appear in (50), these delta-functions do not contribute to $\delta\,{}^\circ Q$ either. Therefore, these quantities transform in interaction representation exactly in the same way as in Heisenberg representation. For instance, ${}^\circ\mathbf{E}$ forms a tensor with ${}^\circ\mathbf{H}$, and ${}^\circ\psi$ and ${}^\circ\psi^\dagger$ are undors. Further, ${}^\circ\rho$ forms a four-vector with ${}^\circ\mathbf{j}$. In particular,

$$\delta\,{}^\circ\rho = -\mathbf{b}\cdot{}^\circ\mathbf{j}. \quad (57)$$

Subtracting $4\pi\kappa_0^{-2}$ times (57) from Eq. (55), we obtain

$$\delta\{{}^\circ\Phi - 4\pi\kappa_0^{-2}\,{}^\circ\rho\} = -\mathbf{b}\cdot\mathbf{A}. \quad (58)$$

Comparing this with (54), we see that

$${}^\circ V = {}^\circ\Phi - 4\pi\kappa_0^{-2}\,{}^\circ\rho \textit{ with } {}^\circ\mathbf{V} = {}^\circ\mathbf{A} \quad (59)$$

forms a four-vector, which we shall denote by ${}^\circ V^\mu$. It is the transformed, by (27), of

$$V^0 = V = \Phi - 4\pi\kappa_0^{-2}\rho \text{ with } V^k = A^k, \quad (k = 1, 2, 3), \quad (60)$$

but, while in interaction representation *${}^\circ V^\mu$ is a four-vector in sense* (B), *${}^\circ A^\mu$ is NO four-vector in this sense*, and while *A^μ is a four-vector in Heisenberg representation, V^μ is NO four-vector*. This means that *the transformation properties of these quantities in Heisenberg representation and in interaction representation are different.*

Remark that, by (21) and (60), V satisfies the "identity" $\kappa_0^2 V = -\mathrm{div}\,\mathbf{E}$, thence,

$$\kappa_0^2\,{}^\circ V = -\mathrm{div}\,{}^\circ\mathbf{E}. \quad (61)$$

From this follows (for $t = t'$)

$$[V(\mathbf{x}); \psi(\mathbf{x}')] = 0, \text{ thence, } [{}^\circ V(\mathbf{x}); {}^\circ\psi(\mathbf{x}')] = 0. \quad (62)$$

This difference in behavior between V and Φ (which by (24) did *not* commute with ψ) is important for the establishing of the free commutativity of meson field and matter field in interaction representation, as we shall see in Chapter 5.

4. THE FIELD EQUATIONS IN INTERACTION REPRESENTATION

We shall now calculate the derivatives of quantities ${}^\circ Q(x)$ in interaction representation. They are in principle given by

$$\partial\,{}^\circ Q(x) = \lim_{\bar{x}\to x}\left\{\frac{{}^{\bar\sigma}Q(\bar{x}) - {}^\sigma Q(x)}{\bar{x} - x}\right\}, \quad (63)$$

where $\bar\sigma$ and σ are both taken "sufficiently flat" and parallel at $\bar{x}$ or x to the $t = 0$ surface of the Lorentz frame used. We shall here first treat separately the spatial gradient and the derivative with respect to time of ${}^\circ Q$ in this frame of reference.

For the spatial derivative, σ can be taken as passing through $\bar{x}$ as well as through x, so that there is no need here to take $\bar\sigma$ different from σ at all. Thence, by (63) and (27),

$$\nabla\,{}^\circ Q(x) = U_\sigma\nabla Q(x)U_\sigma^{-1}, \quad (64)$$

that is, the spatial derivative of the transformed is the transformed of the spatial derivative. (We did already use this in the second part of Eq. (61).)

For the derivative of ${}^\circ Q$ with respect to $x^0 = ct$, it is necessary to take $\bar\sigma$ through the point $\bar{x}$, which may be shifted from x over a distance τ in the

x^0-direction. Thence, the shift from σ to $\bar{\sigma}$ can be given by a function $\epsilon(x', y', z')$ like in the preceding chapters, with $\epsilon = \tau$ at the point $\mathbf{x}' = \mathbf{x}$. The change of $°Q$ as far as due to this shift of σ is given again by (47); in addition to that, there is a change $\tau\partial_0 Q(x)$ already in Heisenberg representation, so that the total increment of $°Q(x)$ from x^0 to $x^0+\tau$ is given by

$$\delta\, °Q = U\left\{\tau\partial_0 Q(x) + a\int \epsilon'[W'; Q]\right\}U^{-1}. \quad (65)$$

Here, $[W'; Q]$ may contain delta-functions *and derivatives of delta-functions*, as shown in Eq. (49). The derivatives of delta-functions we integrate by parts. A difference with the derivation of Eq. (50) is that this time ϵ' near $\mathbf{x}$ can be considered as a constant $(=\tau)$ instead of the linear function of $\mathbf{r}'$ given by (48). This is due to the fact that the same Lorentz frame is used for $°Q(x^0+\tau)$ as for $°Q(x^0)$, *so that the two surfaces $\bar{\sigma}$ and σ must be parallel*[22] (*and sufficiently flat*) *near* $\mathbf{x}$. Thus we find, by $\partial_0\, °Q = \lim(\delta\, °Q/\tau)$ and by $\epsilon' = \tau$ for $\mathbf{x}'$ near $\mathbf{x}$:

$$\partial_0\, °Q(x) = U\left\{\partial_0 Q(x) + (i\hbar c)^{-1}\int' [W(x'); Q(x)]\right\}U^{-1}$$
$$= U_\sigma \partial_0 Q(x)\, U_\sigma^{-1} + (i\hbar c)^{-1}\int_\sigma [°W(x'); °Q(x)]d\sigma_0', \quad (66)$$

where σ runs through x and is the surface $t=$ constant near $\mathbf{x}$, and where $d\sigma_0' = dx'dy'dz'$. The Eqs. (66) and (64) together can be written in a form that *seems* to show their covariance (though it *does not show* that a change of Lorentz frame for $°Q$ necessitates a change of the surface σ) as[22, 23]

$$\partial_\mu\, °Q(x) = U_\sigma \partial_\mu Q(x)\, U_\sigma^{-1} + (i\hbar c)^{-1}\int_\sigma [°W(x'); °Q(x)]d\sigma_\mu', \quad (67)$$

where the spatial components of the surface element $d\sigma_\mu'$ vanish at least near $\mathbf{x}$, where σ is the surface $x^0 =$ constant.

Substituting (49) into (66), we find

$$\partial_0\, °Q = U\{\partial_0 Q + a\mathrm{f}_0(P, P) - a[\nabla'\cdot \mathbf{f}_1(P', P)]_{(P'=P)} + a\sum_k\sum_l [\nabla_k'\nabla_l' \mathrm{f}_{kl}(P', P)]_{(P'=P)} - \cdots\}U^{-1}. \quad (68)$$

We apply this formula to the various field components. From (52) and (53) we take $\mathbf{f}_1(P', P)$ for $\mathbf{A}$ and for Φ, (for which $\mathrm{f}_0(P, P) = 0$), and thus find by (68):

$$\partial_0\, °\mathbf{A} = U\{\partial_0 \mathbf{A} + 4\pi\kappa_0^{-2}\nabla\rho\}U^{-1},$$

or, by (17), (60), and (27),

$$\partial_0\, °\mathbf{A} = -°\mathbf{E} - \nabla\, °V; \quad (69)$$

similarly,

$$\partial_0\, °\Phi = U\{\partial_0\Phi - 4\pi\kappa_0^{-2}\,\mathrm{div}\mathbf{j}\}U^{-1} = U\{-\mathrm{div}\mathbf{A} + 4\pi\kappa_0^{-2}\partial_0\rho\}U^{-1}. \quad (70)$$

As $\rho(\mathbf{x})$ commutes with $W(\mathbf{x}')$, we find

$$\partial_0\, °\rho = U\{\partial_0\rho\}U^{-1}. \quad (71)$$

From (59), (70), (71), and (64) then follows

$$\partial_0\, °V = \partial_0\, °\Phi - 4\pi\kappa_0^{-2}\partial_0\, °\rho = -\mathrm{div}\, °\mathbf{A}. \quad (72)$$

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[22] Indeed: if $^\sigma Q$ depends on the slope of σ as shown in the preceding chapter, the value of $^{\bar{\sigma}}Q(\bar{x}) - {^\sigma Q}(x)$ would be different, if $\bar{\sigma}$ would be taken with a different slope. Therefore, the equations of motion derived in the following *are only then correct, if one takes all surfaces parallel*, thus giving $^{\bar{\sigma}}Q(\bar{x}) - {^\sigma Q}(x)$ an unambiguous meaning for given x, $\bar{x}$, and σ through x. In the text we use (25c) for W, which is correct, if we take the t-axis perpendicular to these parallel surfaces σ. Therefore, we write $°Q$ rather than $^\sigma Q$ throughout the following discussions in the text.—If one drops this particular choice of the t-axis and calculates $\partial_\mu\, {^\sigma Q}$ under the mere restriction that all surfaces σ be at least parallel to each other (even if not to the x, y, z-plane), one would have to replace $W(x')$ in the text by $W(x', \sigma)$ of Eq. (32), with N^μ perpendicular to σ. Thus, Eq. (67) would become covariant as far as σ is kept invariant, if we write in (67) $^\sigma Q$ instead of $°Q$ and $W(x', \sigma)$ instead of $W(x')$.

[23] It should be remarked here that our formulas (66)–(67) are essentially different from the (apparently incorrect) formula

$$\partial_\mu {^\sigma Q}(x) = U_\sigma \partial_\mu Q(x)\, U_\sigma^{-1} + (i\hbar c)^{-1}\int_\sigma [{^\sigma W}(x, \sigma); {^\sigma Q}(x')]d\sigma_\mu' \quad (67\mathrm{F})$$

(with the x and x' interchanged in the integrand), at which Schwinger arrives in his paper on the interaction representation (reference 1, Eq. (2.9)). We would have been led to this wrong equation, if in (65) we would have replaced $\delta\, {^\sigma Q}$ by its average over a small surface element of σ near $\mathbf{x}$, and then would have replaced W' by $W(x, \sigma)$ assuming that the region of deviation between $\bar{\sigma}$ and σ could be shrunk spatially to the one point $\mathbf{x}$. Obviously this cannot be done, if yet one wants to calculate $\delta\, {^\sigma Q}$ by its average over a small finite region about $\mathbf{x}$.—In Schwinger's own derivation of Eq. (67F), he forgets that for the calculation of a variation δU by means of (30), one should average (integrate) the interaction operator $W(x')$ over the whole volume ω between the surfaces, (where the spatial dimensions of ω should be larger than the time-like distance τ between the surfaces). Moreover, the term in $^{\bar{\sigma}}Q(\bar{x}) - {^\sigma Q}(x)$, leading to the commutator between W and Q, is the term in which $Q(x)$ is considered constant and the variation of U_σ is taken into account. In such a term the integration of W over ω is more essential than the averaging of Q over any region.—The difference between (67) and (67F) is of the utmost importance, as (67F), even in its simplified form with σ parallel to the x, y, z-plane, by (49) leads wrongly to

$$\partial_0\, °Q = U\{\partial_0 Q + a\mathrm{f}_0(P, P) + a[\nabla'\cdot\mathbf{f}_1(P, P')]_{(P'=P)} + a\,\Sigma_k\Sigma_l[\nabla_k'\nabla_l'\mathrm{f}_{kl}(P, P')]_{(P'=P)} + \cdots\}U^{-1}, \quad (68\mathrm{F})$$

which, in general, is different from (68), as $\mathbf{f}_1$, f_{kl}, etc., are neither antisymmetric nor symmetric in P and P'. Thus, (68F) would in particular not lead to the results (69), (70), (72), (75), and (77).—True enough, there is no difference between (68) and (68F), if its use is confined to cases, where derivatives of delta-functions do not appear, as in Eq. (74), or as in quantum electrodynamics with one of the field variables taken for Q. But even in quantum electrodynamics, as soon as we take for Q, for instance, the spatial derivatives of the electric field strength (compare footnote 5), Schwinger's formula leads to wrong results, giving for instance (wrongly) $\partial_0(\nabla_k\, °\mathbf{E}_l) = \nabla_k(\mathrm{curl}_l\, °\mathbf{H} - 4\pi\, °\mathbf{j}_l)$, instead of (correctly) $\partial_0(\nabla_k\, °\mathbf{E}_l) = \nabla_k\, \mathrm{curl}_l\, °\mathbf{H}$.

From

$$[W(\mathbf{x}'); \mathbf{E}(\mathbf{x})] = 4\pi i\hbar c \mathbf{j}(\mathbf{x}')\delta(\mathbf{r}'), \quad (73)$$

one finds after comparison with (49) that for this case only $f_0(P', P)$ is different from zero. By (68) and by (18) and (64) we thus obtain

$$\partial_0\, {}^\circ\mathbf{E} = U\{\partial_0\mathbf{E} + 4\pi\mathbf{j}\}U^{-1} = \text{curl}\, {}^\circ\mathbf{H} + \kappa_0^2\, {}^\circ\mathbf{A}. \quad (74)$$

Further by (64) "identities" are unchanged in the interaction representation. Thus (20) together with (69) can by (59) be written as

$${}^\circ F_{\mu\nu} = \partial_\mu\, {}^\circ V_\nu - \partial_\nu\, {}^\circ V_\mu, \quad (75)$$

while (61) with (74) gives

$$\kappa_0^2\, {}^\circ V^\nu = \partial_\nu\, {}^\circ F^{\mu\nu}. \quad (76)$$

Further, (72) gives

$$\partial_\nu\, {}^\circ V^\nu = 0. \quad (77)$$

From (75)–(77) then follows the wave equation

$$\{\Box - \kappa_0^2\}\, {}^\circ V^\nu = 0. \quad (78)$$

Finally, by (22b) and (24),

$$[W(\mathbf{x}'); \psi(\mathbf{x})] = e\{\Phi(\mathbf{x}') - \mathbf{A}(\mathbf{x}')\cdot\boldsymbol{\alpha}\}\psi(\mathbf{x}')\delta(\mathbf{r}'). \quad (79)$$

Comparing this with (49) we find again that only $f_0(P', P)$ is different from zero here, and (68) together with (19) and (64) yields

$$\partial_0\, {}^\circ\psi = U\{\partial_0\psi + (e/i\hbar c)[\Phi - \mathbf{A}\cdot\boldsymbol{\alpha}]\psi\}U^{-1} = \{-i\kappa\beta - \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\}\, {}^\circ\psi,$$

or

$$\{\kappa + \gamma^\mu\partial_\mu\}\, {}^\circ\psi = 0. \quad (80)$$

If this is multiplied from the left by $(\gamma^\nu\partial_\nu - \kappa)$, we obtain the Klein-Gordon equation for a field of free electrons:

$$\{\Box - \kappa^2\}\, {}^\circ\psi = 0. \quad (81)$$

From (75)–(78) and (80)–(81) we see that the q-numbers ${}^\circ V^\nu$, ${}^\circ F_{\mu\nu}$, and ${}^\circ\psi$ (similarly ${}^\circ\psi^\dagger$), which, according to the preceding chapter, form a set of tensors and undors, in interaction representation satisfy field equations "as if no interactions were present." This actually is the main advantage of the interaction representation and enables us to solve the field equations (see below) and write down their four-dimensional commutation relations (next chapter). Also it enables us to distinguish the positive and negative frequency part of the field in a relativistically invariant way.[24] An essential difference with quantum electrodynamics, however, lies in the fact that ${}^\circ V^\mu$ is not simply the transformed of the four-vector A^μ used in Heisenberg representation.

[24] See, for instance, J. Schwinger, "Recent developments in quantum electrodynamics," notes published at the occasion of the Summer Symposium at the University of Michigan (Ann Arbor, 1948), or J. Schwinger, Phys. Rev. **75**, 651 (1949).

It has been shown by Schwinger[25] that the Dirac equation (80) for ${}^\circ\psi$ can be solved rigorously, if ${}^\circ\psi$ is given on some space-like surface σ (not necessarily flat), by

$${}^\circ\psi(x) = \int_\sigma S(x-x')\gamma^\mu\, {}^\circ\psi(x')d\sigma_\mu', \quad (82)$$

and similarly,

$${}^\circ\bar{\psi}(x) = \int_\sigma d\sigma_\mu'\, {}^\circ\bar{\psi}(x')\gamma^\mu S(x'-x), \quad (83)$$

where $S(x)$ is a Dirac matrix depending on x and given by

$$S_{\alpha\beta}(x) = \{\gamma^\lambda\partial_\lambda - \kappa\}_{\alpha\beta}\Delta(x). \quad (84)$$

Here, $\Delta(x)$ is a function similar to $\Delta_0(x)$ (see Eq. (11)), but with κ_0 simply replaced by κ. In the integrations (82)–(83), $d\sigma_\mu'$ is a component of a surface element of σ at a point x' on it. On the other hand, x need not lie on σ, nor is σ related in any way to the transformation surfaces used by (27) in the transformation from Heisenberg representation to interaction representation.

Schwinger also solves the wave equation for the potential four-vector in quantum electrodynamics.[26] We shall solve here the wave equations (77)–(78) by a similar method.

We first define a quantity ${}^\circ B_\mu[\sigma, x]$ by

$${}^\circ B_\mu[\sigma, x] = \int [\Delta_0(x-x')\partial_\nu'\, {}^\circ V_\mu(x') - {}^\circ V_\mu(x')\partial_\nu'\Delta_0(x-x')]d\sigma^{\nu\prime}. \quad (85)$$

We remark at once that this quantity actually depends only on x and is independent of the choice of σ. Indeed by Schwinger's[27]

$$\frac{\delta}{\delta\sigma(x'')}\int_\sigma f(x')d\sigma_\nu' = \partial_\nu'' f(x''), \quad (86)$$

and by use of (13a) and (78) for ${}^\circ V_\mu(x')$ in the integrand of (85), it is readily seen that a variation of σ near the point x'' on σ does not change the value of ${}^\circ B_\mu$. Thence, ${}^\circ B_\mu$ may be calculated with σ replaced by a different surface. If one takes simply the surface $t=$constant through the point x itself, it follows then by (14) that

$${}^\circ B_\mu[\sigma, x] = {}^\circ V_\mu(x). \quad (87)$$

Thus ${}^\circ B_\mu[\sigma, x]$ expresses ${}^\circ V_\mu(x)$ in terms of the values ${}^\circ V_\mu(x')$ and $\partial_\nu'\, {}^\circ V_\mu(x')$ of the potential and its derivatives on an arbitrary surface σ (that need not pass through x and that need not be flat).

[25] Reference 1, Eqs. (2.23)–(2.27).
[26] Reference 1, Eq. (2.22).
[27] Reference 1, Eq. (1.46) or (2.8)

Similarly, it is shown that [28]

$$\int_\sigma [\Delta_0(x-x')\cdot\partial_\nu'\partial_\lambda'\,{}^\circ V_\mu(x') - \partial_\lambda'\,{}^\circ V_\mu(x')\cdot\partial_\nu'\Delta_0(x-x')]d\sigma^{\nu\prime} = \partial_\lambda\,{}^\circ V_\mu(x). \quad (88)$$

5. THE COMMUTATION RELATIONS IN INTERACTION REPRESENTATION

In Heisenberg representation, we have given in Chapter 1 the canonical commutation relations between q-numbers in points at equal time. Transforming the q-numbers by means of (27), taking the corresponding surface $t=\text{constant}$ for σ, we find that these canonical commutation relations remain valid in interaction representation. From them follow the commutation relations in interaction representation for unequal times by means of the equations of motion (75)–(81) solved by (82)–(88).

The following commutation relations are easily verified to satisfy the equations of motion (77)–(78) and (80)–(81):

$$[{}^\circ V_\mu(x);\,{}^\circ V_\nu(x')] = 4\pi i\hbar c\{g_{\mu\nu}-\kappa_0^{-2}\partial_\mu\partial_\nu\}\Delta_0(x-x'), \quad (90a)$$

$$\{{}^\circ\psi_\alpha(x);\,{}^\circ\bar{\psi}_\beta(x')\} = -iS_{\alpha\beta}(x-x') = i\{\kappa-\gamma^\lambda\partial_\lambda\}_{\alpha\beta}\Delta(x-x'), \quad (90b)$$

$$\{{}^\circ\psi_\alpha(x);\,{}^\circ\psi_\beta(x')\} = \{{}^\circ\bar{\psi}_\alpha(x);\,{}^\circ\bar{\psi}_\beta(x')\} = 0, \quad (90c)$$

$$[{}^\circ V_\mu(x);\,{}^\circ\psi_\alpha(x')] = [{}^\circ V_\mu(x);\,{}^\circ\bar{\psi}_\alpha(x')] = 0. \quad (90d)$$

Also, for $t=t'$ they are seen to give (22b), (23), (24), and (62) by virtue of (14) and (59). Further, the commutation relations for ${}^\circ F_{\mu\nu}$ calculated from (90a) and (90d) by differentiation check for $t=t'$ with the commutation relations following from (20) and (22a). Thus, the commutation relations (90) must be generally valid. They could, of course, have been derived directly from the canonical commutation relations by means of Eqs. (82)–(83) and (85)–(88) in the way discussed by Schwinger.[29]

Remark that the possibility of commutation relations that make the Proca field completely commutative with the electron field (see (90d)) is based on the fact that we replaced the scalar potential ${}^\circ\Phi$ by the different quantity ${}^\circ V$ given by (59), thus avoiding the complication (24) existing in Heisenberg representation.[30]

6. THE HEISENBERG FORM OF THE FIELD EQUATIONS IN INTERACTION REPRESENTATION

We shall now show that on account of the relativistic commutation relations (90), the field equations in interaction representation can be written in the form

$$i\hbar\partial_\lambda\,{}^\circ q = [{}^\circ\Pi_\lambda;\,{}^\circ q]. \quad (91)$$

For ${}^\circ q \equiv {}^\circ\psi$ this has been proved by Schwinger;[31] for ${}^\circ q\equiv {}^\circ V_\nu$ we shall prove it here by a method analogous to the one used by Schwinger for the potential in quantum electrodynamics.

We put (compare footnote 15)

$$ {}^\circ\Pi_\lambda = \frac{1}{4\pi c}\int(\partial^\mu\,{}^\circ V^\rho)(\partial_\lambda\,{}^\circ V_\rho)d\sigma_\mu - \frac{1}{8\pi c}\int\{\kappa_0^2\,{}^\circ V_\rho\,{}^\circ V^\rho + (\partial_\mu\,{}^\circ V_\rho)(\partial^\mu\,{}^\circ V^\rho)\}d\sigma_\lambda + \hbar\int {}^\circ\bar{\psi}\gamma^\mu\partial_\lambda\,{}^\circ\psi\cdot d\sigma_\mu, \quad (92)$$

where the last term according to Schwinger[31] ensures (91) for ${}^\circ\psi$.

We remark at once that this expression (92) is independent of σ. Indeed, by (86) with (80) and $\partial_\mu\,{}^\circ\bar{\psi}\cdot\gamma^\mu = \kappa\,{}^\circ\bar{\psi}$ and by (78), we find

$$\frac{\delta\,{}^\circ\Pi_\lambda}{\delta\sigma(x)} = \frac{1}{4\pi c}\{\Box\,{}^\circ V^\rho\cdot\partial_\lambda\,{}^\circ V_\rho - \kappa_0^2\,{}^\circ V^\rho\partial_\lambda\,{}^\circ V_\rho\} + \hbar\,{}^\circ\bar{\psi}\partial_\lambda\gamma^\mu\partial_\mu\,{}^\circ\psi + \hbar(\partial_\mu\,{}^\circ\bar{\psi}\gamma^\mu)\partial_\lambda\,{}^\circ\psi = 0. \quad (93)$$

Next, we calculate $[{}^\circ\Pi_\lambda;\,{}^\circ V_\nu(x)]$ by means of (90a). We shall indicate by a prime the coordinates

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[28] Equation (88) also follows directly from (85) with (87), if we first calculate $\partial_\lambda\,{}^\circ B_\mu$, replacing ∂_λ by $-\partial_\lambda'$ where it acts on Δ_0. The $(-\partial_\lambda')$ are then transferred to the other factor by $-u\partial_\lambda'v = v\partial_\lambda'u - \partial_\lambda'(uv)$. The last term here gives rise to an integral, which by use of Schwinger's general relation (reference 1, Eq. (1.58))

$$\int\partial_\lambda'f(x')d\sigma_\nu' = \int\partial_\nu'f(x')d\sigma_\lambda', \quad (89)$$

and by subsequent use of (13a) and (78) is seen to vanish. Then, for $\partial_\lambda\,{}^\circ B_\mu$ only (88) is left. According to (87) this must be $\partial_\lambda\,{}^\circ V_\mu$. Summing over $\lambda=\mu$, it is then seen from (88) that the validity of (77) at x follows from its validity on and near the surface σ.

[29] Reference 1, Eqs. (2.28)–(2.29).

[30] Actually, we might have made this the starting point of a large part of our theory. Simply *postulating* that equations of motions (75)–(81) and commutation relations (90) be valid after transformation to interaction representation, it would have followed at once from (24) and the commutativity of **A** with ψ, that Φ should be replaced by V according to (60), thus replacing the identity (21) by (61), but that *no* corresponding change in **A** should be made. As we should expect that yet ${}^\circ$**A** should form some kind of a four-vector together with ${}^\circ V$,—while we know that in sense (A) the vector ${}^\sigma$**A** formed a four-vector together with ${}^\sigma\Phi$,—we could have concluded then that ${}^\sigma$**A** and (or) ${}^\sigma\Phi$ necessarily should depend on σ. This would have shown us that from the insensitivity of ${}^\sigma Q(x)$ as to variations of σ at a finite distance from x, one cannot draw Schwinger's conclusion (reference 1, between Eqs. (2.4) and (2.5)) that ${}^\sigma Q(x)$ should be a point function independent of σ throughout. Thus, one is automatically led to the more careful investigation of the transformation properties in interaction representation discussed in Chapter 3.

[31] Reference 1, Eq. (1.65). Here only the $\bar{\psi}$-dependent part of Schwinger's expression (1.64), identical with the ${}^\circ\bar{\psi}$-dependent part of our expression (92) (compare footnote 15). has been used.

in the integrand of (92), and first write down terms arising from the last term in (90a) only. Omitting a factor $(-i\hbar\kappa_0^{-2})$, and denoting $\Delta_0(x'-x)$ simply by Δ_0', we obtain for these terms:

$$\int \{(\partial'^\mu\partial'^\rho\partial_\nu'\Delta_0')\partial_\lambda' \,{}^\circ V_\rho' + (\partial'^\mu \,{}^\circ V'^\rho)\partial_\lambda'\partial_\rho'\partial_\nu'\Delta_0'\}d\sigma_\mu'$$

$$-\int \{\kappa_0^2 \,{}^\circ V'^\rho\partial_\rho'\partial_\nu'\Delta_0' + (\partial_\mu' \,{}^\circ V_\rho')\partial'^\mu\partial'^\rho\partial_\nu'\Delta_0'\}d\sigma_\lambda'.$$

By (77) we can bring forward all "factors" ∂_ρ' (letting them operate on ${}^\circ V'^\rho$ as well as on Δ_0'). Then, by Eq. (89) (see footnote 28), and using $\partial_\mu'\partial_\lambda' = \partial_\lambda'\partial_\mu'$, we can write the result as

$$\int \{(\Box'\partial_\nu'\Delta_0')\partial_\lambda' \,{}^\circ V'^\rho + (\Box' \,{}^\circ V'^\rho)\partial_\lambda'\partial_\nu'\Delta_0'$$

$$-\kappa_0^2(\partial_\lambda' \,{}^\circ V'^\rho)\partial_\nu'\Delta_0' - \kappa_0^2 \,{}^\circ V'^\rho\partial_\lambda'\partial_\nu'\Delta_0'\}d\sigma_\rho',$$

which vanishes by (13a) and (78). Therefore, only terms arising from the term with $g_{\mu\nu}$ in (90a) remain. They give

$$(i\hbar)^{-1}[{}^\circ\Pi_\lambda\,;\,{}^\circ V_\nu]$$

$$=\int \{(\partial'^\mu\Delta_0')\partial_\lambda' \,{}^\circ V_\nu' + (\partial_\lambda'\Delta_0')\partial'^\mu \,{}^\circ V_\nu'\}d\sigma_\mu'$$

$$-\int \{\kappa_0^2\Delta_0' \,{}^\circ V_\nu' + (\partial_\mu'\Delta_0')\partial'^\mu \,{}^\circ V_\nu'\}d\sigma_\lambda'. \quad (94)$$

The second term we integrate by parts with respect to x'^λ. If again we apply Eq. (89) of footnote 28, this term thus yields

$$\int \partial_\mu'(\Delta_0'\partial'^\mu \,{}^\circ V_\nu')d\sigma_\lambda' - \int \Delta_0'\partial_\lambda'\partial'^\mu \,{}^\circ V_\nu' d\sigma_\mu'.$$

The first of these terms by (78) cancels the last two terms of (94), while the second term, combined with the first term of (94), by (13b) and (88) gives

$$(i\hbar)^{-1}[{}^\circ\Pi_\lambda\,;\,{}^\circ V_\nu] = \partial_\lambda \,{}^\circ V_\nu,$$

which proves (91) for ${}^\circ q \equiv {}^\circ V_\nu$.

As all other q-numbers are expressed algebraically in terms of ${}^\circ\psi$, ${}^\circ\bar{\psi}$, ${}^\circ V_\nu$, and their derivatives, the validity of (91) for any other q-number follows then automatically.

If we write out ${}^\circ\Pi_\lambda$ (as given by Eq. (92)) in three-dimensional notation, taking a surface t = constant for σ, and we eliminate derivatives with respect to time from (92) by means of the equations of motion (69), (72), (80), and make use of the identities (20) and (61) after some integrations by parts, we find

$$c\,{}^\circ\Pi^0 = \frac{1}{8\pi}\int d\sigma_0\{{}^\circ\mathbf{E}^2 + {}^\circ\mathbf{H}^2 + \kappa_0^2\,{}^\circ\mathbf{A}^2 + \kappa_0^2\,{}^\circ V^2\}$$

$$+\hbar c\int d\sigma_0\,{}^\circ\psi^\dagger(\kappa\beta - i\boldsymbol{\alpha}\cdot\nabla)\,{}^\circ\psi; \quad (95)$$

$${}^\circ\Pi_k = \int d\sigma_0\left\{\frac{1}{4\pi c}\,{}^\circ\mathbf{E}\cdot\nabla_k\,{}^\circ\mathbf{A} - {}^\circ\psi^\dagger i\hbar\nabla_k\,{}^\circ\psi\right\},$$

$$(k=1,\,2,\,3). \quad (96)$$

Comparing these results with (25a–b) and with (26), we notice that, by (25),

$$c\,{}^\circ\Pi^0 = {}^\circ\mathcal{H}_f + {}^\circ\mathcal{H}_m = {}^\circ\mathcal{H} - {}^\circ\mathcal{W}, \quad (97)$$

$${}^\circ\Pi_k = {}^\circ\mathcal{P}_k, \quad (98)$$

or, in a *pro forma* more "covariant" notation, but still *for σ = flat surface t = constant only*:

$$c\,{}^\circ\Pi^\lambda = c\,{}^\sigma\mathcal{P}^\lambda + \int_\sigma {}^\sigma W(x,\sigma)d\sigma^\lambda, \quad (99)$$

where ${}^\sigma\mathcal{P}^\lambda$ is in interaction representation the q-number four-vector of the total energy ($\mathcal{H} = c\mathcal{P}^0$) and total momentum ($\mathcal{P}_k$ with $k=1, 2, 3$) transformed by (27) at the surface σ.

7. THE VACUUM

It is seen from (25a) or (95) that the unperturbed meson field energy $\mathcal{H}_f$ is positive definite.[32] Therefore, if the meson field ${}^\circ V_\mu$ is expanded in plane waves and the terms with the wave factors $\exp(-2\pi i\nu t)$ are separated from those with wave factors $\exp(+2\pi i\nu t)$, the former (denoted by ${}^\circ V_\mu^{(+)}$ in Schwinger's notation)[24] will contain only annihilation operators, and the latter (${}^\circ V_\mu^{(-)}$) will contain only creation operators, if the meson field is expressed in terms of Jordan-Klein matrices.[24]

If empty space is considered as a state, in which in zero-order approximation no mesons are present, we may describe such a state by a situation functional $\Psi_{(0)}$ satisfying in zero-order approximation the condition[33]

$${}^\circ V_\mu^{(+)}\Psi_{(0)} = 0. \quad (100)$$

The two fields ${}^\circ V_\mu^{(+)}$ and ${}^\circ V_\mu^{(-)}$ both satisfy the wave equations (77)–(78). By their definitions and by (90a) with (12) they also obviously satisfy the

[32] This in contrast to the unperturbed field energy of an electromagnetic field, which, with $S = \partial_\nu A^\nu$, is given in quantum electrodynamics by the q-number

$$(8\pi)^{-1}\int\{\mathbf{E}^2 + \mathbf{H}^2 - 2\Phi\,\mathrm{div}\mathbf{E} + 2S\,\mathrm{div}\mathbf{A} - S^2\}.$$

[33] In quantum electrodynamics the corresponding condition is impossible. There only for the transverse part of the vector potential one might postulate $\mathbf{A}_\perp^{(+)}\Psi_{(0)} = 0$ for empty space.

commutation relations

$$[{}^\circ V_\mu{}^{(+)}(x)\,;{}^\circ V_\nu{}^{(+)}(x')] = [{}^\circ V_\mu{}^{(-)}(x)\,;{}^\circ V_\nu{}^{(-)}(x')] = 0 \quad (101)$$

and

$$\begin{aligned}[{}^\circ V_\mu{}^{(+)}(x)\,;{}^\circ V_\nu{}^{(-)}(x')] &= 4\pi i\hbar c\{g_{\mu\nu}-\kappa_\circ{}^{-2}\partial_\mu\partial_\nu\}\Delta_\circ{}^{(+)}(x-x'),\\ [{}^\circ V_\mu{}^{(-)}(x)\,;{}^\circ V_\nu{}^{(+)}(x')] &= 4\pi i\hbar c\{g_{\mu\nu}-\kappa_\circ{}^{-2}\partial_\mu\partial_\nu\}\Delta_\circ{}^{(-)}(x-x'),\end{aligned} \quad (102)$$

where $\Delta_\circ{}^{(\pm)}(x)$ is given, with $\epsilon_k=+(\kappa_\circ{}^2+k^2)^{\frac{1}{2}}$, by

$$\Delta_\circ{}^{(\pm)}(x)=\frac{\pm i}{4\pi^3}\int d\mathbf{k}\,\epsilon_k{}^{-1}\exp(i\mathbf{k}\cdot\mathbf{x}\mp i\epsilon_k ct), \quad (103)$$

so that

$$\Delta_\circ{}^{(+)}(x)+\Delta_\circ{}^{(-)}(x)=\Delta_\circ(x). \quad (104)$$

8. THE FIRST APPROXIMATION SCHWINGER TRANSFORMATION

By (25c), (21), and (59)–(61), the interaction operator in (31) for a surface at x tangent to $t=$constant is given by

$$\begin{aligned}{}^\circ W(x) &= -{}^\circ\mathbf{j}\cdot{}^\circ\mathbf{A}-\kappa_\circ{}^{-2}\,{}^\circ\rho\,\operatorname{div}{}^\circ\mathbf{E}+2\pi\kappa_\circ{}^{-2}\,{}^\circ\rho^2\\ &= -{}^\circ\mathbf{j}_\mu\,{}^\circ V^\mu+2\pi\kappa_\circ{}^{-2}\,{}^\circ\mathbf{j}_\mu N^\mu\,{}^\circ\mathbf{j}_\nu N^\nu. \end{aligned} \quad (105)$$

This interaction causes perturbation of any initial state in which in zero-order approximation a number of given free Dirac particles and mesons was present. In particular, the first term $-{}^\circ\mathbf{j}_\mu{}^\circ V^\mu$ may cause virtual emission of mesons, which in second order may cause a direct interaction between the Dirac particles, competing with the direct interaction through $2\pi\kappa_\circ{}^{-2}\,{}^\circ\rho^2$. This direct interaction we shall now determine by Schwinger's method of canonical transformations.[24]

This method applies in general a canonical transformation of the type

$$\Psi'[\sigma] = \exp(-iS[\sigma])\,\Psi[\sigma], \quad (106)$$

with

$$S[\sigma]=-(\hbar c)^{-1}\int^\sigma d\omega'\,{}^\circ w(x'). \quad (107)$$

Here, ${}^\circ w(x)$ is a point function describing the perturbation (see below). In case of a perturbation, which in first-order causes merely virtual processes, the limits of integration in (107) are given by[24]

$$\int^\sigma d\omega'\cdots = \frac{1}{2}\left\{\int_{-\infty}^\sigma - \int_\sigma^{+\infty}\right\}d\omega'\cdots = -\frac{1}{2}\int_{-\infty}^{+\infty} d\omega'\cdot\epsilon(x',\sigma)\cdots, \quad (108)$$

with $\epsilon(x',\sigma)=+1$ (or $=-1$), if x' lies on the future (or past) side of σ.

The transformed situation functional $\Psi'[\sigma]$ of (106) is then found to satisfy a Schroedinger equation[34]

$$i\hbar c(\delta\Psi'[\sigma]/\delta\sigma(x)) = W'(x)\Psi'[\sigma], \quad (109)$$

with

$$W'={}^\circ W-{}^\circ w+i[({}^\circ W-\tfrac{1}{2}\,{}^\circ w)\,;S] -\frac{1}{2!}[[({}^\circ W-\tfrac{1}{3}\,{}^\circ w)\,;S];S]-\cdots. \quad (110)$$

For ${}^\circ w$ now we choose the operator, of which we want to determine the second-order effect; in our case, the terms linear in e in ${}^\circ W$, that is,

$${}^\circ w=-{}^\circ\mathbf{j}_\mu\,{}^\circ V^\mu. \quad (111)$$

One reason why for ${}^\circ w$ we cannot well take the complete interaction operator ${}^\circ W$ (as done in quantum electrodynamics)[24] is the fact that S in (107) should be a function of the one surface σ at the limit of integration only. Therefore ${}^\circ w(x')$ in (107) should be simply a point function and should not, like the last term in (105), depend on some time-like direction N^μ at x'.[35] Therefore, use of the whole interaction operator for ${}^\circ w$ would make it difficult to consider $\Psi'[\sigma]$ as a functional of the one surface σ only.[36]

For the sake of simplicity we shall now, in our following calculations, take for σ in (106)–(109) simply a surface at x tangent to $x^0=$constant $(=\tau)$. Also we perform the calculation of W' here up to terms quadratic in e only. Thus, one finds from (110) with (111), (105), and with ${}^\circ\mathbf{j}_\mu N^\mu={}^\circ\rho$ for the surface chosen:

$$W'=2\pi\kappa_\circ{}^{-2}\,{}^\circ\rho^2-\tfrac{1}{2}i[{}^\circ\mathbf{j}_\mu\,{}^\circ V^\mu;\,S]\cdots. \quad (112)$$

The last term in (112) gives by (107), (108), (111),

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[34] This follows from

$$i\hbar c\delta\Psi'/\delta\sigma=\{i\hbar c(\delta e^{-iS}/\delta\sigma)e^{iS}+e^{-iS}\,{}^\circ We^{iS}\}\Psi'$$

with

$$\frac{\delta e^{-iS}}{\delta\sigma}e^{iS}=\sum_{n=0}^{\infty}\frac{i^n\cdot[\{\delta S/\delta\sigma\}\,;S]^{(n)}}{(n+1)!}$$

and

$$e^{-iS}\,{}^\circ We^{iS}=\sum_{n=0}^{\infty}\frac{i^n}{n!}[\{{}^\circ W\}\,;S]^{(n)},$$

where $[\{A\}\,;B]^{(0)}=A$, and where

$$[\{A\}\,;B]^{(n+1)}=[\{A\}\,;B]^{(n)}\cdot B-B\cdot[\{A\}\,;B]^{(n)}.$$

Further, $i\hbar c\delta S/\delta\sigma=-i\,{}^\circ w$ by Eq. (107).

[35] As long as we confine ourselves completely to flat surfaces σ, this is of course more an argument of plausibility, as we might still define N^μ at x' then as perpendicular to the distant surface σ.

[36] For this reason it would be impossible to add the term $2\pi\kappa_\circ{}^{-2}\,{}^\circ\rho^2$ to ${}^\circ w$ even if one would replace (108) by an integration from the infinite past to the surface σ (see reference 24) in an attempt to take into account the non-virtual processes involved in such a perturbation.

and (90a), and with the abbreviation $\epsilon' = \epsilon(x', \sigma)$,

$$(i/4\hbar c)\int [{}^\circ\mathbf{j}_\mu\, {}^\circ V^\mu;\, {}^\circ\mathbf{j}_\nu'\, {}^\circ V'^\nu]\epsilon' d\omega'$$

$$= \frac{i}{8\hbar c}\int (\{{}^\circ\mathbf{j}_\mu;\, {}^\circ\mathbf{j}_\nu'\}\cdot[{}^\circ V^\mu;\, {}^\circ V'^\nu] + [{}^\circ\mathbf{j}_\mu;\, {}^\circ\mathbf{j}_\nu']\cdot\{{}^\circ V^\mu;\, {}^\circ V'^\nu\})\epsilon' d\omega'$$

$$= -\pi\int \underline{{}^\circ\mathbf{j}_\mu\, {}^\circ\mathbf{j}_\nu'}\cdot(g^{\mu\nu} - \kappa_0^{-2}\partial'^\mu\partial'^\nu) \times\Delta_0(x-x')\cdot\epsilon' d\omega' + W_2, \quad (113)$$

where $\underline{AB} = \frac{1}{2}(AB + BA)$, and where

$$W_2 = (i/4\hbar c)\int [{}^\circ\mathbf{j}_\mu;\, {}^\circ\mathbf{j}_\nu']\cdot\underline{{}^\circ V^\mu\, {}^\circ V'^\nu}\cdot\epsilon' d\omega'. \quad (114)$$

The terms with κ_0^{-2} in (113) we integrate by parts with respect to x_μ'. Then $\partial'^\mu\epsilon'$ will be different from zero on the surface σ itself only. There, however, $x-x'$ is space-like and $\partial'^\nu\Delta_0(x-x')$ will vanish except at $x=x'$ itself, where $\partial'^\nu\Delta_0(x-x') = g^{\nu 0}\delta(\mathbf{x}'-\mathbf{x})$ by (13b) and (14b). Near $x=x'$, however, σ is the surface $x'^0=\tau$, so that there we have

$$\partial'^\mu\epsilon' = 2g^{\mu 0}\delta(x'^0-\tau). \quad (115)$$

Combining these results we get

$$(\partial'^\mu\epsilon')\partial'^\nu\Delta_0(x-x') = 2g^{\mu 0}g^{\nu 0}\delta^{(4)}(x'-x). \quad (116)$$

Using this equation we find for the discussed terms with κ_0^{-2} in (113) exactly $-2\pi\kappa_0^{-2}\,{}^\circ\rho^2$, which cancels the first term in (112). Thus one finds, up to e^2:

$$W' = W_0 + W_2, \quad (117)$$

where W_2 is given by (114), while the terms with $g^{\mu\nu}$ in (113) give

$$W_0(x) = -\tfrac{1}{2}\,\underline{{}^\circ\mathbf{j}^\nu(x)\cdot{}^\circ b_\nu(x)} \quad (118)$$

with (by $\epsilon' = -\mathrm{sgn}(t-t')$)

$${}^\circ b_\nu(x) = -2\pi\int {}^\circ\mathbf{j}_\nu(x')\cdot\Delta_0(x-x') \cdot\mathrm{sgn}(t-t')\cdot d\omega'. \quad (119)$$

By comparison with (9) we see at once that ${}^\circ b_\nu(x)$ is a solution of

$$\{\Box - \kappa_0^2\}\,{}^\circ b_\nu(x) = -4\pi\,{}^\circ\mathbf{j}_\nu(x), \quad (120)$$

and takes for $\kappa_0\to 0$ so-to-say the place of the average between the retarded and the advanced potential (compare Eq. (15)). However remark that ${}^\circ b_\nu(x)$ is *not* simply the transformed of $A_\nu^{\mathrm{direct}}(x)$ in interaction representation. This transformed quantity would be obtained by $U_\sigma A_\nu^{\mathrm{direct}} U_\sigma^{-1}$ with σ through x, while in our present ${}^\circ b_\nu$, the $j_\nu(x')$ in the integrand of (119) have been transformed by $U_{\sigma'}$ with σ' through x'.

For $\kappa_0\to 0$, the first term in (117) goes over into the Møller interaction between electrons. The second term describes two-quanta radiation effects. Both are after all simply scalar point functions independent of the slope of the surface σ through x. Indeed, the only dependency on σ could enter these expressions through the factor ϵ' in (114). But in (114) the commutator $[{}^\circ j_\mu;\, {}^\circ j_\nu']$ will make the integrand zero wherever $x-x'$ is space-like. That is, just the region where σ lies and where ϵ' jumps from -1 to $+1$. If, therefore, the slope of σ through x is changed, this has no influence on the value of (114). Thence, (117) does not depend on the slope of σ through x.

9. THE ENERGY-MOMENTUM FOUR-VECTOR AFTER CANONICAL TRANSFORMATION

The transformation (106) changes observables by

$$Q' = e^{-iS}\,{}^\circ Q e^{iS} = \sum_{n=0}^{\infty}\frac{i^n}{n!}[\{{}^\circ Q\};\, S]^{(n)} = {}^\circ Q + i[{}^\circ Q;\, S] - \tfrac{1}{2}[[{}^\circ Q;\, S];\, S]\cdots. \quad (121)$$

(Compare footnote 34 for the notation.) For the unperturbed energy and momentum four-vector ${}^\circ\Pi_\lambda$ of interaction representation given by (92) (compare (99)) this gives

$$\Pi_\lambda' = {}^\circ\Pi_\lambda + i[{}^\circ\Pi_\lambda;\, S] - \frac{1}{2!}[[{}^\circ\Pi_\lambda;\, S];\, S]\cdots. \quad (122)$$

Here, by (107)–(108) and (91),

$$ic[{}^\circ\Pi_\lambda;\, S] = -\frac{1}{2}\int d\omega'\cdot\epsilon'\cdot\partial_\lambda'\,{}^\circ w(x'). \quad (123)$$

We integrate by parts with respect to x_λ' and use (115):

$$ic[{}^\circ\Pi_\lambda;\, S] = \delta_\lambda{}^0\int d\omega'\cdot\delta(x'^0-\tau)\,{}^\circ w(x') = \delta_\lambda{}^0\int_\sigma d\sigma_0'\,{}^\circ w(x') = \int_\sigma d\sigma_\lambda'\,{}^\circ w(x'). \quad (124)$$

Here, we have obviously omitted minus a similar integral over the infinite past,[37] which corresponds to the assumption that at $t=-\infty$ the perturbation

[37] Or the average over infinite past and infinite future, which is the same, as $(\partial_\lambda\,{}^\circ w)$ as a perturbation would cause virtual processes only.

did not yet exist and later was switched on infinitely slowly.[38]

Substitution of (124) into (122) yields

$$c\Pi_\lambda' = c\,{}^\circ\Pi_\lambda + \int_\sigma d\sigma_\lambda \left\{ {}^\circ w + \frac{i}{2!}[{}^\circ w\,;\,S]\cdots \right\}. \quad (125)$$

Thence, by (99) and (121), the total energy-momentum four-vector is now given by

$$c\mathcal{P}_\lambda' = c\,{}^\circ\Pi_\lambda + \int_\sigma d\sigma_\lambda \cdot \sum_{n=0}^{\infty} \left(\frac{i^n}{(n+1)!}[\{{}^\circ w\}\,;\,S]^{(n)} - \frac{i^n}{n!}[\{{}^\circ W\}\,;\,S]^{(n)} \right), \quad (126)$$

or, by (110) or footnote 34,

$$c\mathcal{P}_\lambda' = c\,{}^\circ\Pi_\lambda - \int_\sigma d\sigma_\lambda \cdot W'(x). \quad (127)$$

This shows that $W'(x)$ is not only the new "interaction operator" by (109), but is also the new interaction energy. On the other hand, the unperturbed energy-momentum four-vector is then always given by the same old (un-transformed) ${}^\circ\Pi_\lambda$, which still can be used by (92) for the calculation of derivatives of functions of field variables. It should be borne in mind, though, that *physically* the four-vector ${}^\circ\Pi_\lambda$ in the first approximation Schwinger representation (with Ψ') is *not* identical with the quantity denoted by this same ${}^\circ\Pi_\lambda$ in interaction representation (with Ψ).[39] The latter quantity would of course be given by Π_λ' (125) in first approximation Schwinger representation.

10. CONCLUSIONS

We have shown in the preceding chapters how a meson theory can be described in an interaction representation and how one can proceed with canonical transformations in close analogy to Schwinger's development of quantum electrodynamics. There were, however, a number of complications, already listed in the introduction, which made it necessary to consider several points with more care than in quantum electrodynamics, and which made it necessary to criticize certain formulas and conclusions of Schwinger (see footnotes 23 and 30).

In particular, we found that Schwinger's formulation of the generalized Schroedinger equation in interaction representation, with the non-scalar interaction energies of meson theories, is only correct, if the interaction operator is taken in a Lorentz frame with its time-axis perpendicular to the surface σ. Then, the Schroedinger equation turned out to be integrable automatically (Chapter 2). The field variables in interaction representation ${}^\sigma Q(x)$ were in general no pure point functions, unless the surfaces σ used were confined to surfaces perpendicular to the time-axis used. The field variables ${}^\circ Q$ defined by such surfaces were then shown to transform different from the original field variables Q in the Heisenberg representation; but certain combinations of the ${}^\circ Q$ could then be defined, which formed tensors and undors among each other (Chapter 3).

These combinations were then shown to satisfy field equations as if no interactions would be present (Chapter 4), and these field equations could be solved in a general form. The commutation relations could then be formulated four-dimensionally (Chapter 5) and were automatically such that the new "meson field"-components (combinations) were commutative with the matter field.

In interaction representation, the field equations could also be given in a Heisenberg form (Chapter 6), using the unperturbed energy as the "Hamiltonian." On the other hand, only the perturbation energy was used as "Hamiltonian" in the Schroedinger equation, and the total energy was the sum of these two expressions.

It was shown in Chapter 7 that the absence of mesons could be formulated by the simple condition (100). In quantum electrodynamics a similar assumption would be meaningless, unless it is restricted to the transverse field. In this regard, the meson field has some advantage of simplicity over the regular Maxwell-Lorentz-Fermi field.

It may perhaps be pointed out here that, by a proper treatment, all the usual electromagnetic properties of matter can be explained completely by means of fields of mesons of vanishing mass.[40]

Finally, it was shown in Chapters 8 and 9 how Schwinger's perturbation method of subsequent canonical transformations can be applied also in meson theory, though also here some complications arise from the fact that in (107) we could not take the complete interaction operator for ${}^\circ w$. In Chapter 9 we verified that the interaction operator also after canonical transformation still can be regarded as the interaction energy, while the unperturbed energy, as a q-number unchanged in form by the transformation, is changed on the other hand in its physical meaning.[39]

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[38] This assumption seems just as good or as bad as the basic idea of Schwinger's perturbation method that, at $t=-\infty$, $\Psi[\sigma]$ should be simply the unperturbed state $\Psi'[\sigma]$.

[39] That is, $(\Psi'^*, \Pi_\lambda\Psi')$ is obviously not identical with $(\Psi^*, \Pi_\lambda\Psi)$.

[40] Compare footnote 7, and reference given there.