Phase Shifts in Proton-Alpha-Scattering

C. L. CRITCHFIELD AND D. C. DODDER
Department of Physics, University of Minnesota, Minneapolis, Minnesota
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Phase-shifts in S and P waves are found that fit the observed scattering of protons by helium (4) within the experimental accuracy of the recent Minnesota results. The S wave phase is negative and rather small. The $P_{\frac{1}{2}}$ and $P_{3/2}$ waves have different phase-shifts, both positive, but there are two sets of possible values, one corresponding to a normal doublet in Li^5, the other to an inverted doublet. In both cases there is resonant scattering and strong polarization of the proton beam. Measurement of the polarization would decide between the doublets. The values of the phase-shifts presented are those that minimize the sum of squares of percent differences between observed and calculated differential cross sections.

THE scattering of protons by helium (4) has been measured recently at Minnesota by Freier, Lampi, Sleator, and Williams.[1] Their results for the differential cross sections are more extensive and more accurate than those obtained in earlier investigations.[2] Within the estimated accuracy, however, there is no essential discrepancy between the earlier and the later results. The Minnesota group bombarded helium gas with protons from the electrostatic generator at eight different energies ranging from 0.95 Mev to 3.58 Mev. The scattered protons were counted at a number of angles relative to the incident beam. The angular range extended from 12.5° to 168° (in the center of mass system of coordinates) for most of the energies used.

The accuracy and completeness of the recent results make it possible to determine the phase shifts involved as functions of the energies and to an estimated accuracy of a degree or two. It is the purpose of this paper to present the phase-shift analysis for the $p-He^4$ collisions. No evidence for anomalous phase-shifts in collisions with angular momentum greater than that for P waves was found. The S wave phase-shifts are small and negative; the P wave phase shifts are double valued, but the $P_{3/2}$ and the $P_{\frac{1}{2}}$ waves are refracted very differently above 1.5 Mev. It is impossible to conclude from the differential cross sections alone whether the $P_{\frac{1}{2}}$ shift is larger than the $P_{3/2}$ (i.e., a normal doublet in Li^5) or whether the doublet is inverted.

Values of the P wave phase-shifts are of particular interest. First, it may be hoped to interpret them in terms of spin-orbit forces between nucleons. Since the nuclear forces between He^4 and a proton in a P state are attractive (the phase-shifts are positive), a normal doublet would fit, qualitatively, with Dancoff's[3] calculations based on tensor forces between nucleons, whereas, an inverted doublet would be evidence for the spin-orbit splitting from the Thomas precession as suggested by Inglis.[4] Secondly, the fact that there is definite spin-orbit splitting means that the scattered protons are partially polarized[5] and this effect is of interest not only as a possible source of fast, polarized protons but also as a means of resolving the ambiguity in the P wave phase-shifts. The polarization to be expected from the normal doublet turns out to be quite different from that in the inverted doublet.

The method of analysis of the scattering data follows the customary lines of considering the incident beam as an infinite plane wave in relative coordinates, expanding in eigenfunctions of orbital angular momentum and introducing the effects of nuclear refraction as phase shifts in the asymptotic form at large separation. In the absence of spin-orbit effects the scattered amplitude takes the familiar form

$$A(\theta,\eta,\cdots\delta_l\cdots)=k^{-1}\{-\tfrac{1}{2}\eta\csc^2\tfrac{1}{2}\theta\exp(i\eta\ln\csc^2\theta/2)+\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\exp(i\delta_l+i\phi_l)\sin\delta_l\},\quad(1)$$

where θ is the angle of scattering (center-of-mass system), the δ_l are the phase-shifts induced by non-Coulomb forces in the partial waves of orbital angular momentum $l\hbar$, the angular dependence of the latter being given by the Legendre polynomials $P_l(\cos\theta)$. The quantities, k, η, and ϕ_l are related to the reduced mass M and to the velocity at infinite separation, v, by

$$\begin{aligned} &k=Mv/\hbar\quad \eta=2e^2/\hbar v=0.31612E_{\mathrm{Mev}}^{-\frac{1}{2}}\\ &k\eta=5.5450\times10^{11}\ \mathrm{cm}^{-1},\quad \phi_0=0,\\ &e^{i\phi_l}=\frac{1+i\eta}{1-i\eta}\cdots\frac{l+i\eta}{l-i\eta}\quad l>0.\end{aligned}\quad(2)$$

In Eq. (2), E_{Mev} is the energy of the proton beam in millions of electron volts and $\hbar$ is Planck's constant divided by 2π.

In the presence of spin-orbit coupling the waves will be refracted differently, depending upon the relative orientation of spin and orbital momentum. Let the normalized spin wave function be designated by $\chi_{\frac{1}{2}}$ for

[1] Freier, Lampi, Sleator, and Williams, Phys. Rev. **75**, 1345 (1949).
[2] N. Heydenburg and N. Ramsey, Phys. Rev. **60**, 42 (1941); C. B. O. Mohr and G. E. Pringle, Proc. Roy. Soc. **160**, 190 (1937).
[3] S. M. Dancoff, Phys. Rev. **58**, 326 (1940).
[4] D. Inglis, Phys. Rev. **50**, 783 (1936).
[5] L. Wolfenstein, Phys. Rev. **75**, 1664 (1949); J. Schwinger, Phys. Rev. **69**, 681 (1946); **73**, 407 (1948). The authors are indebted to Dr. Wolfenstein for the opportunity of reading his manuscript before publication and for several helpful discussions concerning the polarization.

spin pointing in the direction of the incident proton beam (z axis) and $\chi_{-\frac{1}{2}}$ for spin pointing in the opposite direction. The component of the incident beam that is proportional to $(2l+1)\chi_{\frac{1}{2}}P_l(\cos\theta)$ must be considered as a superposition of eigenstates of total angular momentum, ψ_j, with $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$:

$$\begin{aligned}(2l+1)\chi_{\frac{1}{2}}P_l(\cos\theta)&=\psi_{l+\frac{1}{2}}+\psi_{l-\frac{1}{2}},\\ \psi_{l+\frac{1}{2}}&=(l+1)\chi_{\frac{1}{2}}P_l(\cos\theta)-\chi_{-\frac{1}{2}}\sin\theta e^{i\Phi}P_l'(\cos\theta),\\ \psi_{l-\frac{1}{2}}&=l\chi_{\frac{1}{2}}P_l(\cos\theta)+\chi_{-\frac{1}{2}}\sin\theta e^{i\Phi}P_l'(\cos\theta),\end{aligned}\tag{3}$$

where Φ is the azimuthal angle about the direction of bombardment and

$$P_l'(\cos\theta)=dP_l(\cos\theta)/d\cos\theta.$$

If we denote the anomalous phase-shift in $\psi_{l+\frac{1}{2}}$ by δ_l^+ and in $\psi_{l-\frac{1}{2}}$ by δ_l^-, the scattered amplitude arising from incident particles of spin $\chi_{\frac{1}{2}}$ is proportional to

$$\begin{aligned}A_{\frac{1}{2}}(\theta,\eta,\cdots)=k^{-1}\chi_{\frac{1}{2}}\{&-\tfrac{1}{2}\eta\csc^2\tfrac{1}{2}\theta\exp[i\eta\ln\csc^2\tfrac{1}{2}\theta]\\ &+\sum_{l=0}^{\infty}[(l+1)\exp(i\delta_l^+)\sin\delta_l^++l\exp(i\delta_l^-)\sin\delta_l^-]\\ &\times\exp(i\phi_l)P_l(\cos\theta)\}+k^{-1}\chi_{-\frac{1}{2}}\sum_{l=0}^{\infty}[\exp(i\delta_l^-)\sin\delta_l^-\\ &-\exp(i\delta_l^+)\sin\delta_l^+]\exp(i\phi_l+i\Phi)\sin\theta P_l'(\cos\theta).\end{aligned}\tag{4}$$

The analogous calculation for incident particles of spin $\chi_{-\frac{1}{2}}$ leads to the scattered amplitude

$$\begin{aligned}A_{-\frac{1}{2}}(\theta,\eta,\cdots)=k^{-1}\chi_{-\frac{1}{2}}\{&-\tfrac{1}{2}\eta\csc^2\tfrac{1}{2}\theta\exp[i\eta\ln\csc^2\tfrac{1}{2}\theta]\\ &+\sum_{l=0}^{\infty}[(l+1)\exp(i\delta_l^+)\sin\delta_l^++l\exp(i\delta_l^-)\sin\delta_l^-]\\ &\times\exp(i\phi_l)P_l(\cos\theta)\}-k^{-1}\chi_{\frac{1}{2}}\sum_{l=1}^{\infty}[\exp(i\delta_l^-)\sin\delta_l^-\\ &-\exp(i\delta_l^+)\sin\delta_l^+]\exp(i\phi_l-i\Phi)\sin\theta P_l'(\cos\theta).\end{aligned}\tag{5}$$

The differential scattering cross section for an unpolarized beam is then the average of the absolute squares of the amplitudes given by Eqs. (4) and (5), summed over spin components. In the following, we shall use only the terms involving $l=0$ and $l=1$; the formula for the cross section $\sigma(\theta,\eta)$, multiplied by k^2 then becomes:

$$\begin{aligned}k^2\sigma(\theta,\eta)=|&-\tfrac{1}{2}\eta\csc^2\tfrac{1}{2}\theta\exp[i\eta\ln\csc^2\tfrac{1}{2}\theta]+e^{i\delta_0}\sin\delta_0\\ &+\cos\theta[2\exp(i\delta_1^+)\sin\delta_1^++\exp(i\delta_1^-)\sin\delta_1^-]\\ &\times\exp(i\phi_1)|^2+\sin^2\theta\sin^2(\delta_1^--\delta_1^+).\end{aligned}\tag{6}$$

For convenience of calculation we define the real quantities ρ and β such that

$$\begin{aligned}\rho e^{i\beta}&=2i[2\exp(i\delta_1^+)\sin\delta_1^++\exp(i\delta_1^-)\sin\delta_1^-]\\ &=2\exp(2i\delta_1^+)+\exp(2i\delta_1^-)-3.\end{aligned}\tag{7}$$

We also define three new quantities, α, ζ, and R, such that

$$\begin{aligned}\alpha&=2\delta_0-\phi_1\\ R\sin(\zeta+\phi_1)&=\eta\csc^2\tfrac{1}{2}\theta\cos(\eta\ln\csc^2\tfrac{1}{2}\theta)\\ R\cos(\zeta+\phi_1)&=1-\eta\csc^2\tfrac{1}{2}\theta\sin(\eta\ln\csc^2\tfrac{1}{2}\theta).\end{aligned}\tag{8}$$

In the variables defined in Eqs. (7) and (8), and multiplying by four, we obtain the equation that is used in the following analysis:

$$\begin{aligned}4k^2\sigma(\theta)=R^2-2R\{&\cos(\alpha-\zeta)+\rho\cos\theta\cos(\beta-\zeta)\}\\ &+1+2\rho\cos\theta\cos(\alpha-\beta)\\ &+\tfrac{1}{2}\rho^2(3\cos^2\theta-1)-3\rho\cos\beta\sin^2\theta.\end{aligned}\tag{9}$$

The unknown quantities at each energy are α, β, and ρ which are related to the desired phase-shifts by

$$\begin{aligned}\delta_0&=\tfrac{1}{2}(\alpha+\phi_1),\\ 2\sin2\delta_1^++\sin2\delta_1^-&=\rho\sin\beta,\\ 2\cos2\delta_1^++\cos2\delta_1^-&=\rho\cos+3.\end{aligned}\tag{10}$$

As a preliminary step in the reduction of the data, the cross sections at only three angles of scattering were used, *viz.*: $\theta=90°$, 54° 44′, and 125° 16′. The latter two angles are those at which the D wave component vanishes and, hence, also the term in Eq. (9) that is quadratic in ρ. By reading values for $\sigma(\theta)$ at these angles from graphs of the experimental results and by assuming a value for α, say α', we obtain a pair of equations in $\rho\sin\beta$ and $\rho\cos\beta$. Solving these and substituting in Eq. (9) as applied at $\theta=90°$, we then solve for α. By plotting the assumed values of α' against the calculated values, the roots were located. At $E=1.49$ Mev, for example, there were three sets of solutions which may be labeled by the corresponding values of α, *viz.*, $\alpha=-67.3°$, $-14.0°$, and 36.5°. Using the values of α, β, and ρ for each set, the cross section was computed for all the angles at which observations were made.[1] In this way it was found that only the set belonging to $\alpha=-67.3°$ predicted cross sections that were close enough to the observed values to be acceptable. For example, the percent difference between calculation and observation at $\theta=25°$ was found to be -1.5, 39.6, and -39.5, in the sets with $\alpha=-67.3°$, $-14°$, and 36.5°, respectively.

Consideration of the continuity of the phase-shifts leads one to conclude that negative α characterizes the correct set of roots at all energies. At $E=3.58$ Mev, there are four sets of roots two of which have negative α, *viz.*, $\alpha=-91.0°$ and $-132.9°$. Again comparing results of computation with the observations at various angles we found that only $\alpha=-91.0°$ gave an acceptable fit. The percent difference at $\theta=25°$ was 3.4 with $\alpha=-91.0°$

TABLE I. Results of least square reduction.

Energy (Mev)	α	$\rho\cos\beta$	$\rho\sin\beta$
0.95	-59.9 ± 5.6	-0.010 ± 0.060	0.344 ± 0.042
1.49	-65.2 ± 2.0	-0.493 ± 0.034	1.445 ± 0.020
1.70	-62.3 ± 3.1	-1.077 ± 0.053	1.914 ± 0.016
2.02	-74.3 ± 6.6	-2.238 ± 0.103	2.274 ± 0.073
2.22	-77.4 ± 2.6	-3.089 ± 0.037	2.033 ± 0.059
2.53	-78.9 ± 2.3	-3.954 ± 0.028	1.202 ± 0.072
3.04	-84.4 ± 2.2	-4.097 ± 0.018	0.068 ± 0.063
3.58	-89.4 ± 1.5	-3.958 ± 0.024	-0.374 ± 0.059

Note.—± quantities are estimated standard deviations.

TABLE II. Percent deviations in final fit.

θ	$E=0.95$	1.49	1.70	2.02	2.22	2.53	3.04	3.58
12° 38′	4.1	4.3	2.5	2.4	2.6	−1.5	−1.4	1.0
18 52	2.4	4.7	−0.3	3.2	−0.2	0.2	−0.8	1.1
25 3	3.3	0.8	0.4	2.3	0.6	0.6	1.2	1.3
31 13	3.1			−3.8	−1.8	−3.4	1.6	
37 20	0.6	−4.0	−1.8	3.1	1.7			0.3
43 23						−1.8	−2.2	0.1
49 23	−1.5	2.3	−0.6		2.4	4.5		
55 21							1.4	−1.0
61 10	−2.7	1.1	−0.5	−0.4		−0.4		
66 54					2.1			
72 37	−0.7	0.7		1.7		2.0	−2.3	−1.6
90 7	2.3	−1.4	2.1	0.1	0.4			2.5
94 22						0.5	1.6	
99 28	0.1							
104 36				−4.7				
110 14								3.4
114 22		−2.4	3.1	−1.2	−2.1	2.7	0.5	2.0
120 14								−2.8
132 37		1.5	3.4	0.8	−1.3	1.9	−0.5	−1.5
140 14							−2.2	2.1
149 22		0.5	−1.0	0.5	−1.7	0.6	−0.4	2.5
150 14						−3.1	−4.4	−1.3
160 14						1.1	2.2	2.1
164 59							0.5	
168 2			−2.4	0.1	−2.2	1.0	−1.2	−3.4
s	2.9	3.1	2.3	2.7	2.0	2.4	1.9	2.2

and 34.8 with $\alpha=-132.9°$. In contrast to the result at $E=1.49$, it is the next lowest value of α that gives the best fit at $E=3.58$, instead of the lowest value. Hence, it appears that the two series of negative roots cross over, there being a double root at some intermediate energy. This interpretation is corroborated by the fact that at $E=2.53$ Mev there are no real, negative roots for α, a possible result of being near the double root and making a small error in estimating the cross sections at the three angles used.

Although the method described above serves to locate the correct branch of the multiple solutions, the existence of a double root makes it useless as a way of obtaining preliminary values of α, β, and ρ at intermediate energies. With the values of α at $E=1.49$ and 3.58, however, those for the other energies can be estimated. The method used is that presented by Breit and his collaborators[6] wherein we rely on the fact that the function

$$f(\eta,\delta_0)=(e^{2\pi\eta}-1)^{-1}\pi\cot\delta_0-\ln\eta+1.202\eta^2$$

should be very closely a linear function of the energy. By drawing a line through the points obtained at $E=1.49$ and 3.58, the other δ_0's, and hence α's, were found. Then, from two observed points, preliminary values of β and ρ were obtained.

With the preliminary values of α, β, and ρ as obtained by the procedure described above, values of $4k^2\sigma(\theta)$ were computed from Eq. (9) for every angle and for every energy for which observations were made. The percent differences, ϵ_θ, between calculated and experimental values of $4k^2\sigma(\theta)$ were then taken. First-order corrections to α, $\rho\cos\beta$, and $\rho\sin\beta$ were found at each energy so as to minimize the sum of squares taken over all angles, i.e.,

$$\mathrm{Min}\sum_\theta\left\{\frac{\partial\ln\sigma(\theta)}{\partial\alpha}\Delta\alpha+\frac{\partial\ln\sigma(\theta)}{\partial\rho\cos\beta}\Delta(\rho\cos\beta)+\frac{\partial\ln\sigma(\theta)}{\partial\rho\sin\beta}\Delta(\rho\sin\beta)-\frac{\epsilon_\theta}{100}\right\}^2.$$

No allowance is made for possible uncertainty in the value of the energy, or of angle, and all results are considered to be of the same weight. With the first-order corrections made, new values of ϵ_θ were computed and the procedure repeated. For the most part, the corrections indicated by the second calculation were of the order of one percent.

The semistatistical procedure in minimizing the sum of squares of the deviations permits rough estimates of the standard deviations of the individual observations and of the values obtained for α, $\rho\cos\beta$, and $\rho\sin\beta$. The results of the calculation are presented in Table I and the resulting percent deviations, observed less calculated, are given in Table II for each angle and energy presented in reference 1. The quantities labeled s in Table II are the estimated standard, percent deviations of the single observations. It is apparent that the magnitudes of s are in accord with the estimated probable error of 1.5 percent claimed by the experimentalists of reference 1. The fact that the deviations at the higher energies are no worse, and in fact somewhat better, than those at lower energy indicates that higher partial waves are not important to the scattering by nuclear forces.

Using the values in Table I and solving Eq. (10), we obtain the values of the phase-shifts. As remarked above, there are two sets of solutions for the P waves, one a normal doublet, the other an inverted doublet. The results are shown in Table III which includes the estimated standard deviations. In the event that the doublet is inverted, only the phase-shift of the $P_{3/2}$ wave passes through 90°, and that occurs at about 2.8 Mev. For the normal doublet, however, both P waves pass through resonance, the $P_{\frac{1}{2}}$ at a proton energy of 2.3 Mev and the $P_{3/2}$ at 3.4 Mev. The first resonance level is presumably the lowest quantum state of the Li^5 nucleus which, therefore, lacks being bound by about 1.8 Mev. The splitting caused by spin-orbit coupling would be 0.9 Mev. The width of the $P_{\frac{1}{2}}$ level at half-maximum is also about 0.9 Mev and the width of the $P_{3/2}$ levels are even larger, perhaps by 50 percent.

TABLE III. Phase-shifts in S and P waves.

		Inverted doublet		Normal doublet	
Energy	S	$P_{\frac{1}{2}}$	$P_{3/2}$	$P_{\frac{1}{2}}$	$P_{3/2}$
0.95	−12.0±2.8	3.3± ?	3.3± ?	3.3± ?	3.3± ?
1.49	−18.1±1.0	4.1±1.8	20.4±1.0	25.8±1.7	9.6±0.8
1.70	−17.6±1.6	4.2±2.2	31.1±1.4	40.7±1.4	13.8±0.3
2.02	−24.6±3.3	8.1±1.9	47.8±0.6	63.2±2.1	23.7±0.9
2.22	−26.7±1.3	9.4±1.6	60.6±0.6	83.1±1.6	31.9±0.6
2.53	−28.2±1.2	13.1±2.7	78.8±0.8	115.3±1.4	49.6±1.1
3.04	−32.0±1.1	15.7±2.5	96.6±1.7	160.7±2.0	79.8±1.9
3.58	−35.2±0.8	20.3±4.0	105.4±6.6	181.0±5.6	95.9±2.1

[6] See G. Breit and W. G. Bouricius, Phys. Rev. 75, 1029 (1949).

The S wave phase-shifts are similar to those that would be produced by a weak, repulsive potential of fairly long range. If the S wave potential is assumed to be independent of the energy, the phase-shifts as observed can be obtained by assuming the Coulomb repulsion to apply down to a separation, 5.75×10^{-13} cm, and substituting a constant positive potential of 3.8 Mev at smaller distances. On the other hand, the S wave interaction probably is not energy independent and the square well picture has no immediate significance.

Although the existing results on the complementary system formed by scattering neutrons in helium are not of comparable accuracy, it is of interest to compare them with our results. Staub and Tatel[7] have found that the backward scattering of the neutrons exhibits a double resonance which can be accounted for by either a normal doublet in the P waves and a negative S wave phase-shift or an inverted doublet with a positive S wave phase-shift. They preferred the former possibility which corresponds to our normal doublet if one assumes similarity between neutron-neutron and proton-proton forces. In fact, if we had imposed the condition that both P waves go through resonance in the energy range used, we should have obtained the normal doublet as a unique solution. On the other hand, the proton-alpha-scattering shows only a single maximum in the backward scattering. In any event, it is clear from the analysis above that a description by means of phase shifts will always lead to two solutions for the P waves and it is only by an additional assumption that a unique solution can be derived from the differential cross section alone.

The question as to whether the P wave doublet be normal or inverted can be settled, in principle, by determining the polarization of the scattered protons as a function of the energy of bombardment. Following the line of reasoning presented in reference 5, we compute the expectation value of the x component of the spin of the scattered proton, σ_x,

$$\sigma_x\chi_{\frac{1}{2}}=\chi_{-\frac{1}{2}}\quad \sigma_x\chi_{-\frac{1}{2}}=\chi_{\frac{1}{2}}$$

in each of the waves, Eqs. (4) and (5), average the results and divide by the cross section at the same energy and angle. This gives the fractional polarization which, since it varies as $\sin\theta$ will be near maximum at $\theta=90°$. The fractional polarization in the x direction is also proportional to $\sin\Phi$ and is therefore a maximum along the y axis. Considering only P waves and setting $\theta=\Phi=90°$, we obtain for the percent polarization p:

$$p=200\sin(\delta_1^- - \delta_1^+)\{\eta\sin(\phi_1+\delta_1^-+\delta_1^+-0.69\eta)-\sin\delta_0\sin(\phi_1+\delta_1^-+\delta_1^+-\delta_0)\}/k^2\sigma(90°). \quad (11)$$

TABLE IV. Percent polarization at 90°.

Energy	Inverted doublet	Normal doublet
0.95	0	0
1.49	−70	79
1.70	−90	96
2.02	−98	80
2.22	−92	42
2.53	−68	−32
3.04	−39	−87
3.58	−15	−57

Calculations with the phase-shifts obtained above are presented in Table IV. It is evident from the numerical results that the protons scattered at 90° are almost completely polarized at energies somewhat less (~0.5 Mev) than the resonant values. It will be observed also that the direction of polarization changes sign in the case of the normal doublet. Hence, relatively inexact measurements of the proton polarization should suffice to decide which of the two solutions to the scattering problem is the correct one.

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[7] H. Staub and H. Tatel, Phys. Rev. 58, 820 (1940).