Two-Component Wave Equations

C. W. KILMISTER
Queen Mary College, London, England
July 6, 1949

IN a recent letter Jehle[1] has considered a two-component wave equation; it is of interest to know the relationship of such equations to Dirac's equations. The object of this letter is to show that by modifying Eddington's[2] method of deriving Dirac's equations we can obtain an equation additional to Dirac's and simpler in form, which may, under certain circumstances, be put in two-component form. This differs somewhat from Jehle's equation, but its derivation suggests that it is the two-component equation most likely to have physical interest.

It is convenient to use quaternion notation; f_i, $(i=1, 2, 3)$, are the usual quaternion units, and $f_0=i$. For a quaternion q, where

$$q=q^k f_k,$$

we write

$$Sq=q^0 f_0, \quad Vq=q-Sq.$$

When the position quaternion,

$$q=x^k f_k \tag{1}$$

undergoes the orthogonal transformation

$$q'=aqb, \quad (|a|=|b|=1), \tag{2}$$

a "wave quaternion," ψ, is defined as one which undergoes the transformation,

$$\psi'=a\psi.$$

If φ is any other quaternion and

$$p=V\cdot\psi\varphi, \quad q=V\cdot\varphi\psi, \tag{3}$$

the obvious identities

$$p\psi=\psi q, \quad \varphi p=q\varphi, \tag{4}$$

are equivalent to Eddington's[3] simple wave equation; and their covariance under (2) is equivalent to

$$P'=apa^{-1}, \quad q'=q, \quad \bar{\varphi}'=a\bar{\varphi}, \tag{5}$$

where $\bar{\varphi}$ is the conjugate quaternion to φ.

The divergence of a mixed (see Silberstein[4]) quaternion, j, is the invariant

$$\mathrm{div} j=S(\partial j/\partial x^0-\nabla j); \tag{6}$$

since the four quaternions,

$$j(\alpha)=S\cdot(\psi f_\alpha \varphi)+V\cdot\psi\varphi, \quad (\alpha=0\cdots 3), \tag{7}$$

are mixed, the identification of one of them with the four-dimensional probability vector is suggested. Using (6) leads to the equations

$$\nabla\psi-(\partial\psi/\partial x^0)f_\alpha=M\psi, \quad \varphi\nabla'-f_\alpha(\partial\varphi/\partial x^0)=-\bar{M}\varphi, \tag{8}$$

where M is a linear quaternion function of a quaternion, $\bar{M}$ its conjugate (Hamilton[5]) and under field-free conditions M, $\bar{M}$ have special forms so that iteration produces the second-order wave equation. *If* $\alpha\neq 0$, *these are identical with Dirac's equations* as written by Conway,[6] and are well known to admit a 4×4 matrix representation. *If* $\alpha=0$, *however*, the ψ-equation, for instance, becomes

$$\nabla\psi-i(\partial\psi/\partial x^0)=M\psi, \tag{9}$$

or, it may be written

$$f^k(\partial\psi/\partial x^k)=-M\psi, \tag{10}$$

where

$$f^k=g^{k\lambda}f_\lambda=\bar{f}_k, \quad \tfrac{1}{2}(f^k\bar{f}^\lambda+f^\lambda\bar{f}^k)=S(f^k\bar{f}^\lambda)=-g^{k\lambda}. \tag{11}$$

The special form of the left-hand side of (10) allows of a 2×2 matrix representation of the f^k, and ψ is then represented as a two-component column matrix. Such a representation of M is not, in general, possible, but when it is, a two-component equation is true. One such possibility, suggested by considering the case when ψ has real quaternion components (i.e., $S\cdot\psi f_i$), $(i=1\cdots 3)$, $S\cdot\psi$, real), is the equation, where now f^k and λ, μ, ψ represent matrices,

$$f^k(\partial\psi/\partial x^k)=\lambda\psi+\mu\psi^*. \tag{12}$$

Under field-free conditions iteration must give the usual wave equation; this requires $\lambda=0$, and if we choose the representation

$$f_1=\begin{pmatrix} i & \cdot \\ \cdot & -i \end{pmatrix}, \quad f_2=\begin{pmatrix} \cdot & -1 \\ 1 & \cdot \end{pmatrix}, \quad f_3=\begin{pmatrix} \cdot & -i \\ -i & \cdot \end{pmatrix}, \tag{13}$$

then

$$\mu=\nu f_2,$$

where ν is a scalar. Writing now

$$f^2 f^k=\gamma^k, \quad \lambda=\lambda_k\gamma^k, \tag{14}$$

we find that

$$\bar{\gamma}^k=-\gamma^k \text{ for } k\neq 2, \quad \bar{\gamma}^2=\gamma^2,$$

and the γ^k obey the same rules as the f^k. The equation becomes

$$\gamma^k[(\partial/\partial x^k)-\gamma_k]\psi=\nu\psi^*,$$

where the λ_k describe the applied field. Jehle's choice of a special form for s^k, involving complex conjugates, restricts the λ_k to the form, $i\varphi_k$, which he considers; however, Eddington[7] has remarked that, in Dirac's theory, this is an unnecessary survival of Hermitic conditions from the Schrödinger theory, and it seems as if the same should apply here.

[1] H. Jehle, Phys. Rev. **75**, 1609 (1949).
[2] Sir A. S. Eddington, *Relativity Theory of Protons and Electrons* (Cambridge University Press, London, 1936), §8.2.
[3] Reference 2, §5.4.
[4] L. Silberstein, Phil. Mag. (6) **23**, 798 (1912), Eq. (13).
[5] Sir W. R. Hamilton, *Elements of Quaternions* (Longmans Green and Company, New York, 1866).
[6] A. W. Conway, Proc. Roy. Soc. (**A**)**162**, 147) 1937).
[7] Reference 2, p. 131.

On the Temperature Dependence of Counter Characteristics in Self-Quenching G-M Counters*

OM PARKASH
E.P.U. Physics Department, Government College, Hoshiarpur (E.P.), India
July 5, 1949

IT has been reported[1] that the self-quenching counters, usually employing argon or some other gas mixed with some organic vapor, show a temperature dependence of their counting rate which can be sometimes troublesome. Korff, Spatz, and Hilberry[2] during their experiments on argon-alcohol counters found a marked dependence of the counting rate on temperature. They also found that the plateau disappeared at lower temperatures. Putman,[3] in trying to eliminate variations of counting rate with temperature, found that there is a bodily shifting of the plateau towards high operating potentials as the temperature was increased. In his case, however, when the plateau was flattened with an external quenching circuit, the counting rate became largely independent of temperature.

The temperature dependence of the counting rate has been explained in terms of the action of the quenching vapors in a counter. Most of the organic compounds are vapors at room temperature. At lower temperature, however, some of these will condense out with the following result: (1) There may be insufficient quenching vapor left. Under such conditions, the number of vapor molecules and consequently the effect of quenching will depend on temperature. (2) The liquid might condense near the electrodes in such a way as to cause semi-conducting paths across the insulating material between the wire and the cylinder. Leakage across such paths can manifest itself as spurious counts. It is therefore very important to know the range of temperatures within which we can safely use the G-M counters, and the present investigation was undertaken to this effect.

Two counters were chosen, one with internal and the other with external cathode. The counter with internal cathode, prepared in this laboratory, is a usual type of glass enveloped counter with a thin oxidized copper cylinder cathode, filled with commercial