therefore be calculated by means of Eq. (31). The results of such calculations are shown in Fig. 4. The curves of this figure permit any one of the three variables (x/d), (U/V), and (I/I_s) to be determined if the other two are known. The intersection of the contours of constant (U/V) with the horizontal line corresponding to a given value of (I/I_s) gives the potential distribution in the diode for that value of current.

Note Added in Proof: Since this paper was submitted for publication it has been found that as early as 1920 G. Jaffe (Ann. d. Phys. **63**, 145) considered the plane diode under partial space-charge conditions and obtained an equation identical, except for notation, to Eq. (17) above. However, Jaffe did not express his results in the convenient manner of Fig. 1 nor did he consider the cylindrical diode.

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Non-Equilibrium Phenomena in a Bose-Einstein Gas. I. Transmission of Second Sound

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A model Bose-Einstein gas is considered in which numerical perturbations in the population of the lowest state have a relaxation time that is long compared with the relaxation time for perturbations in the symmetry of the velocity distribution in the excited states. Oscillations of the population of the lowest state about its equilibrium value are transmitted as second sound waves. The velocity of transmission is found as a function of temperature below the lambda-point and compared with that of second sound in liquid helium. In the gas there is a temperature dependent coupling between pressure waves and thermal waves; the normal modes of propagation are mixed. The high speed mode is pure pressure wave near T_λ but goes over gradually to pure thermal wave as T goes down towards 0°K; the low speed mode is pure thermal wave near T_λ but goes over gradually to pure pressure wave as T goes down towards 0°K. For $T \ll T_\lambda$ the thermal wave has a higher speed of propagation than the pressure (ordinary sound) wave.

INTRODUCTION

In a recent paper[1] the formal first order perturbation theory of transport phenomena in a Bose-Einstein gas was considered. In that theory it was assumed that at every point in the gas the numerical populations of the various energy states accessible to the gas atoms remain equal to their equilibrium populations. In other words the regression of numerical fluctuations was assumed to be rapid compared with that of asymmetry fluctuations in the velocity distribution. This assumption is implicit in most applications of first-order perturbation theory in statistical mechanics and is probably valid in general. However, in the degenerate Bose-Einstein gas below a certain transition temperature the number of atoms in the lowest state becomes comparable with the total number of atoms in the gas. The general theory of fluctuations[2] then leads one to expect that the numerical fluctuations in the population of the lowest state become of major importance below the transition temperature. This may be seen in the following way.

The standard deviation Δn of the population of any one state from the mean population $\bar{n}$ is given by

$$\text{Fermi-Dirac gas} \quad \Delta n/\bar{n} = (1/\bar{n} - 1)^{\frac{1}{2}},$$
$$\text{Bose-Einstein gas} \quad \Delta n/\bar{n} = (1/\bar{n} + 1)^{\frac{1}{2}}.$$

At very low temperatures in the Fermi-Dirac gas $1/\bar{n} \rightarrow 1$ and the fluctuations tend to vanish as T approaches absolute zero. In the Bose-Einstein gas, on the other hand, the fluctuations remain of the order unity; in particular the lowest state has a population $n_0 \rightarrow N$ as $T \rightarrow 0$ where N is the total number of atoms in the gas, and $\bar{n}_0$ is comparable with N immediately below the transition temperature T_λ. Thus

$$\Delta n_0/\bar{n}_0 = 1 \text{ for all } T < T_\lambda.$$

These finite fluctuations at low T would be of no importance if all the states had similar populations and were closely spaced in the energy spectrum; it could then be supposed that there would be no correlation between the fluctuations of neighboring states so that in any appreciable energy range there would be no appreciable fluctuation in the total population in that range. But if the system contains one state or one degenerate set of states, in the present case the single lowest state, with a population comparable with that of the whole gas and, therefore, enormous compared with the population of any other state, the finite value of $\Delta n_0/\bar{n}_0$ becomes extremely serious. For example, if a Gaussian distribution is assumed for the actual deviations of the population from its mean value, the fact that $\Delta n_0/\bar{n}_0 = 1$ means that the number of atoms in the lowest state may be 20 percent above or below the mean value for roughly 23 percent of the time. At tem-

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[1] W. Band, Phys. Rev. **76**, 1937 (1949).

[2] R. H. Fowler, *Statistical Mechanics* (Cambridge University Press, London, 1936), Chapter 20.

peratures not more than 40 percent below T_λ there are already over one-half the atoms of the gas involved in this uncertainty.

The foregoing equations are exact only so long as $\bar{n}_0 \ll N$ because the fundamental theorems of statistical mechanics are exact only under this restriction. As soon as $\bar{n}_0$ becomes comparable with N it would be more appropriate to treat the lowest state as a separate phase and the above conclusions can hardly be accurate. The approximation is still good, however, if we are not too far below T_λ and comparatively large fluctuations are to be expected even well below the transition. Because the condensation into the lowest state occurs only in momentum space, these very large fluctuations have only secondary effects on the density of the same order of magnitude as the classical fluctuations near the critical point of a vapor.[3] Perhaps for this reason these large fluctuations in the Bose-Einstein gas have hitherto been regarded more as a mathematical curiosity than a physical reality. But if, for some reason, the relaxation time for these fluctuations happens to be long compared with the relaxation time for asymmetrical perturbations of the velocity distribution, then it is clear that such a situation would place the problem completely outside the scope of ordinary first-order perturbation theory and alter the whole character of transport phenomena to be expected in the assembly.

To make the mathematical problem as simple as possible, consider a quasi-ideal Bose-Einstein gas in which the lowest state is separated by a finite energy gap, say e_z from the next state, and let e_z be large (by a factor of the order 10) compared with the average spacing between adjacent levels.

With such a model it may reasonably be assumed that the relaxation time for re-adjustments of equilibrium following an imposed change in the populations of the lowest state is comparatively long.* We shall call this the population relaxation time. It will further be assumed that the model has the following convenient properties: let there be three relaxation times, τ_0 the population relaxation time for the lowest state, τ_a the asymmetry relaxation time for the velocity distribution in the whole gas, and τ_1 the mean population relaxation time for the individual excited states; it is assumed only that

$$\tau_0 \gg \tau_a \gg \tau_1. \tag{A}$$

Under (A) there may conceivably occur physical processes for which significant times t lie well within the range $\tau_0 > t > \tau_a$. For such processes, which will be called "t-processes," the atoms in the lowest state will behave as an almost independent assembly having no appreciable interaction with the remainder of the gas, while the latter, which will be called the "high energy component," remains separately in equilibrium with respect to all external perturbations because of the relation $t > \tau_a \gg \tau_1$.

For t-processes this Bose-Einstein gas model behaves like a mixture of two fluids similar to the two fluid model proposed by Tisza[4] for his phenomenological theory of helium II.

The conditions under which this two-fluid model is applicable to the Bose-Einstein gas are, therefore, the following: first, there must exist three relaxation times of the type described under assumption (A), and secondly, (B) the mean population of the single low energy state or degenerate group of states must be large compared with that of any other single state.

Any physical system that conforms to conditions (A) and (B) will be superfluid for t-processes if the states of lowest energy can include particle transport.[5] Such a system may conform to (A) and (B) because it is a Bose-Einstein gas of the model here described. But there may be other physical systems that conform to (A) and (B) for entirely different reasons. In particular, electrons are known not to obey Einstein-Bose statistics, and yet assemblies of electrons can, under appropriate conditions, become superfluid. The statistics are evidently not in themselves basically decisive, but rather conditions (A) and (B) enunciated above, and it has to be determined in each case of superfluidity why the particular assembly happens to satisfy these conditions.

The calculations to be reported in the present paper were undertaken with the familiar idea that helium II may possibly happen to satisfy conditions (A) and (B) because the statistics applicable to the gas phase influence the behavior of the liquid phase in some way not yet fully understood. The results, namely the existence of second sound propagation of thermal waves in the Bose-Einstein gas, would appear to add to the gradually accumulating evidence that tends to support this point of view.

The proof that this model also yields supercon-

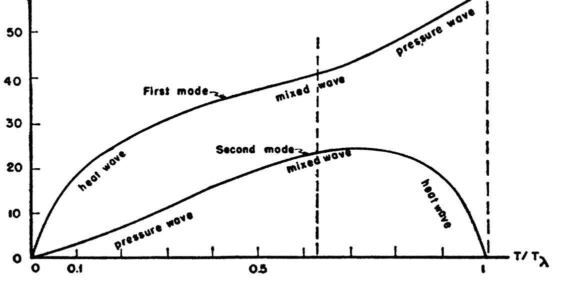

FIG. 1. Sound modes in B.-E. gas.

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[3] L. I. Schiff, Phys. Rev. **57**, 844 (1940).

* In other words, it is assumed that the large energy gap reduces the transition probability between the lowest state and the other states, while no such anomaly on average exists between other pairs of adjacent states.

[4] L. Tisza, Phys. Rev. **72**, 838 (1947); Phys. Rev. **75**, 885 (1949).

[5] Compare K. Mendelssohn, Proc. Phys. Soc. London **LVII**, 371 (1945), and N. F. Mott, Phil. Mag. **40** 61 (1949).

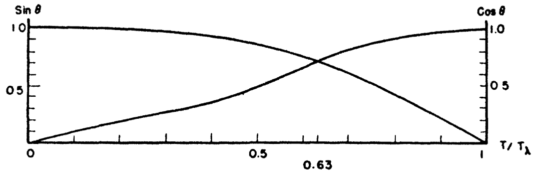

FIG. 2. Phase relation between superfluid and high energy component displacements in the normal modes of sound in B.-.E. gas.

ductivity for heat of the same general character as that observed in helium II is reserved for a later paper.

In the first section of this paper some of the thermodynamic functions of the high energy component of the Bose-Einstein gas are calculated on the assumption that there does exist an equilibrium parameter λ_n and a temperature T_n defining the distribution in energy among the excited states, but that there does not exist any necessary relation between these parameters and the ratio ρ_s/ρ. In other words, three independent variables exist as required for t-processes in describing the assembly as a mixture of high and low energy components.

In the second section the Lagrangian for oscillatory t-processes is expressed in terms of the above thermodynamic functions with special reference to oscillations that carry the state of the assembly back and forth through an equilibrium center that would, if the ordered kinetic energy could be removed, correspond to an internal thermodynamic equilibrium. The Euler equations for this Lagrangian are the wave equations for first and second sound.

In the third section the velocities of propagation of the two modes of propagation are evaluated as functions of mean temperature, and in the last section the velocity of propagation of ordinary sound above the lambda-point is shown to be continuous with the curve for first sound at the lambda-point. Figure 1 shows the two curves calculated for the gas whose atoms are equal in mass to the helium atom. Both the shape of the curve and the magnitude of the velocity for second sound are remarkably like those observed in liquid helium.

It turns out that when the Lagrangian is expressed in terms of the most obvious variables—velocity of the mass center of the mixture and relative velocity between the two components, respectively—there is a strong coupling between the two. In other words, the normal modes of wave propagation are not in general pure pressure waves (mass center oscillations) and pure concentration waves (relative velocity oscillations). The coupling is removable formally by transforming to other variables, and mathematically this is equivalent to treating the two modes as plane polarized components that can each be resolved into two "directions," one a pure pressure and the other a pure concentration wave. Figure 2 shows the "polarizing angle" as a function of temperature. The first sound mode at T_λ is pure pressure wave and the second mode has zero phase velocity. As T drops below T_λ the "plane of polarization" of the first mode rotates gradually until at $T_1=0.63T_\lambda$ it is 45° which means that here the first mode involves both pressure and concentration changes in equal proportions. Below T_1 the first mode becomes predominantly concentration wave and as $T\rightarrow 0$ the proportion of pressure oscillation approaches zero. The second mode on the other hand starts at just below T_λ as almost pure concentration wave, passes through the neutral mixture at T_1 and becomes predominately pressure wave as T approaches zero.

This would mean, for example, that if a source of either pure pressure or pure heat oscillations were present in the gas, the latter would act like a doubly-refracting medium; in particular, if the temperature were near T_1, the two modes would carry roughly equal shares of the transmitted energy, and a "beat" phenomenon between the two might be possible.

It is particularly interesting to note that below T_1 the heat wave would propagate most of its energy in the high speed first mode, while the pressure wave would be found predominately in the lower speed second mode of propagation, thus reversing the relationship existing near the lambda-point.

If one tries to fit the Bose-Einstein gas model more closely to helium II by arbitrarily assigning to (ρ_n/ρ) a $(T/T_\lambda)^6$ law in place of the ideal $(T/T_\lambda)^{\frac{3}{2}}$ law, the present calculations yield the somewhat surprising result that second mode of propagation then has a pure imaginary phase velocity. In other words, such a model might simulate liquid helium in its C_v-curve, but it would fail to give second sound propagation.

NON-EQUILIBRIUM THERMODYNAMIC FUNCTIONS

Under conditions (A) of the last section, the high energy component of the Bose-Einstein gas below T_λ, considered separately, forms an assembly in internal equilibrium and the number of atoms in volume V is

$$N_n = V(2\pi mkT_n)^{\frac{3}{2}}h^{-3}\Lambda_n(1, \tfrac{3}{2}). \tag{2.1}$$

In this equation the expression $\Lambda_n(1, \frac{3}{2})$ is the same as $\Lambda(1, \frac{3}{2})$ of the previous paper,[1] written in terms of T_n and λ_n instead of T and λ; T_n is the temperature appropriate to the normal component separately, and λ_n is its logarithmic free energy. The expression $\Lambda_n(1, \frac{3}{2})$ is a series in powers of λ_n whose coefficients depend only on the quantity e_z/kT_n.

If the whole gas were in equilibrium, the low energy component would have the same λ and T. Actually the low energy component must always have $\lambda=1$. The radius of convergence of the λ_n-series in the expression $\Lambda_n(1, \frac{3}{2})$, however is at

$$\lambda_n = 1/(1-e_z/kT_n) > 1 \tag{2.2}$$

so that it is possible for λ_n to oscillate both above and below its equilibrium value of unity. It is interesting that the existence of the finite energy gap e_z is formally necessary to permit oscillations about equilibrium, for if e_z were zero, λ_n would only vary *below* unity. The temperature and λ-values for the low energy component do not have to appear formally: given N/V, T_n and λ_n, the population N_s of the low energy state can always be calculated from the identity

$$N=N_n+N_s. \tag{2.3}$$

The energy per unit volume in the normal component is

$$E_n/V=(\tfrac{3}{2})kT_n(2\pi mkT_n)^{\frac{3}{2}}h^{-3}\Lambda_n(2,\,5/2). \tag{2.4}$$

where $\Lambda_n(2, 5/2)$ is another λ-series defined in the first paper[1] and here expressed in terms of λ_n and T_n. The energy per excited particle is obtained from (2.4) and (2.1):

$$E_n/N_n=(3/2)kT_n\Lambda_n(2,\,5/2)/\Lambda_n(1,\,3/2). \tag{2.5}$$

It is clear that if both T_n and λ_n vary in such a way as to keep N_n constant, the energy E_n will vary, alternatively E_n could be kept fixed but then N_n will vary. In equilibrium λ_n would be determined at unity. Here fluctuations about equilibrium are being considered and evidently λ_n-1 may be regarded as a measure of departure from equilibrium.

The entropy of the high energy component may be defined as

$$S_n=k\textstyle\sum_r \ln(1-\lambda_n e^{-\epsilon_r/kT_n})^{-1}+E_n/T_n-N_nk\ln\lambda_n \tag{2.6}$$

where the sum over ϵ_r extends from the lowest normal state upward. Since the state $\epsilon_0=0$ is excluded, the logarithm may be expanded

$$\ln(1-\lambda_n e^{-\epsilon_r/kT_n})^{-1}=\textstyle\sum_{j=1}(1/j)\lambda_n{}^j e^{-j\epsilon_r/kT}$$

and it is easy to show that in terms of the notation used previously

$$S_n=kV(2\pi mkT_n)^{\frac{3}{2}}h^{-3}\Lambda_n(1,\,5/2) \\ +E_n/T_n-N_nk\ln\lambda_n. \tag{2.7}$$

The extra "j" factor in the denominator yields $\Lambda_n(1, 5/2)$ in place of $\Lambda_n(1, 3/2)$. Neglecting $N_nk\ln\lambda_n$ because λ_n is almost unity for all $T<T_\lambda$, (2.7) can be rewritten in the form

$$S_n/V=k(2\pi mkT_n)^{\frac{3}{2}}h^{-3}\{\Lambda_n(1,\,5/2) \\ +(3/2)\Lambda_n(2,\,5/2)\}. \tag{2.8}$$

The entropy per excited particle, later needed, is obtained from (2.8) and (2.1), namely:

$$\sigma=S_n/N_n=k\{\Lambda_n(1,\,5/2) \\ +(3/2)\Lambda_n(2,\,5/2)\}/\Lambda_n(1,\,3/2). \tag{2.9}$$

This is a function both of T_n and λ_n; and again these two quantities may vary in such a way as to keep N_n constant, when of course σ will vary; or vice-versa T_n and λ_n may vary in such a way as to keep σ constant, when N_n will vary. Incidentally the value of σ under equilibrium for the whole gas, $T_n=T$ and $\lambda_n=1$ is

$$\sigma=\sigma_e=k\{\Lambda_c(1,\,5/2) \\ +(3/2)\Lambda_c(2,\,5/2)\}/\Lambda_c(1,\,3/2). \tag{2.10}$$

Here the Λ_c expressions are evaluated at $\lambda=1$, but they are still functions of T. Thus under equilibrium σ becomes a function of T only[***] while in the absence of equilibrium both T_n and λ_n must be known to find σ.

The most obvious variables to choose in specifying a non-equilibrium state of the whole system are N/V, T_n, and λ_n. However, by using (2.9) and (2.1) all functions can be expressed at least formally in terms say of the three variables ρ, $x=\rho_n/\rho$, and σ, where $\rho_n=mN_n/V$ and $\rho=mN/V$.

THE LAGRANGIAN EQUATIONS FOR OSCILLATORY *t*-PROCESSES

In the foregoing argument an additional independent variable has been introduced which becomes dependent on the true thermodynamic state variables under the proper conditions. For convenience the ratio $x=\rho_n/\rho$ may be chosen as this new variable. Let S and V be the thermodynamic state variables and consider a three-dimensional space (S, V, x). The equilibrium dependency of x on S and V:

$$x=x(S,\,V). \tag{3.1}$$

Let the surroundings be such that the gas can be supplied heat or work without altering the temperature T or the pressure P of the surroundings, no matter how rapidly the exchanges may be effected. For slow changes the gas will also be kept at T and P, and the energy exchange is

$$\Delta_e E=T\Delta S-P\Delta V, \tag{3.2}$$

where ΔS and ΔV refer to the gas. During this displacement the representative point of the gas in (S, V, x)-space remains on the surface (3.1). Now consider an oscillatory exchange in which the surroundings lose entropy to the gas by an amount that fluctuates $\pm\Delta S_0$ while its volume varies $\pm\Delta V_0$. If this oscillation is at sufficiently low frequency the representative point of the gas traces out a curve in (3.1); but if the frequency is rapid enough it will trace out a curve that lies outside of this surface, intersecting it only at the "equilibrium center." The energy of the gas at one extreme of this oscillation can be expressed in terms of the three displacements ΔS, ΔV, Δx to second-order terms

$$\begin{aligned}\Delta E_1={}&(\partial E/\partial S)_{xv}\Delta S+(\partial E/\partial V)_{xs}\Delta V+(\partial E/\partial x)_{sv}\Delta x \\ &+\tfrac{1}{2}(\partial^2E/\partial S^2)_{xv}\Delta S^2+\tfrac{1}{2}(\partial^2E/\partial V^2)_{xs}\Delta V^2 \\ &+\tfrac{1}{2}(\partial^2E/\partial x^2)_{sv}\Delta x^2+(\partial^2E/\partial x\partial V)_s\Delta x\Delta V \\ &+(\partial^2E/\partial x\partial S)_v\Delta x\Delta S+(\partial^2E/\partial S\partial V)_x\Delta S\Delta V.\end{aligned} \tag{3.3}$$

On the other hand the energy received from the surroundings is exactly

$$\Delta E_2=T\Delta S_0-P\Delta V_0, \tag{3.4}$$

[***] The expressions Λ_c and therefore σ_e depend on T because e_z/kT differs from zero. If we neglect the energy gap, the entropy per particle σ becomes independent of T below the transition temperature, and for a first approximation in numerical calculation it is sufficiently accurate to replace (2.10) by

$$\sigma=(5/2)k\sum_{j=1}^{\infty}1/j^{5/2}\div\sum_{j=1}^{\infty}1/j^{3/2}=1.285k.$$

where $\Delta V_0 = -\Delta V$ exactly. The second-order terms in ΔE_2 are absent because under the "bath" conditions assumed in the surroundings T and P are independent of the displacements.

The additional work needed to drive the gas through the displacement $(\Delta S, \Delta V, \Delta x)$ is

$$\Delta W = \Delta E_1 - \Delta E_2. \tag{3.5}$$

This, in practice, must be obtained from the ordered kinetic energy present in the gas as it passes through the equilibrium center in the course of the oscillations. It is clear that the equilibrium center must, by definition, be such that the first order terms in ΔW vanish identically when ΔS, ΔV, ΔX are measured from that center. Thus,

$$\begin{aligned}-T\Delta S_0 + P\Delta V_0 = (\partial E/\partial S)_{xv}\Delta S + (\partial E/\partial V)_{xs}\Delta V \\ + (\partial E/\partial x)_{sv}\Delta x.\end{aligned} \tag{3.6}$$

This is essentially a projective relation between two vectors in (S, V, x)-space, one in the surface (3.1) and one lying in a direction intersecting that surface. The relation tells us the point on the surface towards which the gas in the non-equilibrium state $(\Delta S, \Delta V, \Delta x)$ tends to relax. In the present paper this relaxation question will not be discussed.

Using (3.6) in (3.5) remembering (3.3) we have

$$\begin{aligned}\Delta W = {}& \tfrac{1}{2}(\partial^2 E/\partial S^2)_{xv}\Delta S^2 + \tfrac{1}{2}(\partial^2 E/\partial V^2)_{xs}\Delta V^2 \\ & + \tfrac{1}{2}(\partial^2 E/\partial x^2)_{sv}\Delta x^2 + (\partial^2 E/\partial x\partial V)_s \Delta x\Delta V \\ & + (\partial^2 E/\partial x\partial S)_v \Delta x\Delta S + (\partial^2 E/\partial S\partial V)_x \Delta S\Delta V.\end{aligned} \tag{3.7}$$

To make use of this it is first necessary to transform the variables to σ, ρ and x in place of S, V, and x, $\sigma = S/x$ and $\rho = 1/V$. The calculation is elementary and it is found that

$$\begin{aligned}\Delta W = {}& \tfrac{1}{2}(\partial^2 E/\partial \sigma^2)_{x\rho}\Delta\sigma^2 + \tfrac{1}{2}\{(\partial^2 E/\partial\rho^2)_{x\sigma} \\ & + (2/\rho)(\partial E/\partial\rho)_{x\sigma}\}\Delta\rho^2 + \tfrac{1}{2}(\partial^2 E/\partial x^2)_{\sigma\rho}\Delta x^2 \\ & + \{(\partial^2 E/\partial x\partial\sigma)_\rho - (1/x)(\partial E/\partial\sigma)_{\rho x}\}\Delta x\Delta\sigma \\ & + (\partial^2 E/\partial x\partial\rho)_\sigma \Delta x\Delta\rho + (\partial^2 E/\partial\sigma\partial\rho)_x \Delta\sigma\Delta\rho.\end{aligned} \tag{3.8}$$

The present paper considers only the high frequency limit when $S \propto N_n$ so that $\Delta\sigma = 0$, implying absence of transitions between the two components. Thus

$$\begin{aligned}\Delta W = {}& \tfrac{1}{2}\{(\partial^2 E/\partial\rho^2)_{x\sigma} + (2/\rho)(\partial E/\partial\rho)_{x\sigma}\}\Delta\rho^2 \\ & + \tfrac{1}{2}(\partial^2 E/\partial x^2)_{\sigma\rho}\Delta x^2 + (\partial^2 E/\partial x\partial\rho)_\sigma \Delta x\Delta\rho.\end{aligned} \tag{3.9}$$

This has been written out on the assumption of complete absence of transitions between the two components, but in calculating the coefficients appearing in (3.9) full equilibrium will be assumed. This is the same approximation as used by Tisza[2] in his first derivation of second sound in helium II; in that work an error of some 15 percent resulted.[6] In the present problem the error will be similar with the difference between σ and σ_e; it vanishes with e_z/kT. On this understanding first approximations will be obtained for the various terms appearing in (3.9), writing E_n for E.

To the approximation here accepted the Λ-sums in (2.1), (2.4) and (2.9) can also be regarded as constants. Then from (2.1)

$$x = \rho_n/\rho = (T_n/T_\lambda)^{\frac{3}{2}} \doteq (T/T_\lambda)^{\frac{3}{2}}, \tag{3.10}$$

while from (2.4) $E_n\rho \propto T_n^{5/2}$. Therefore,

$$E_n \propto x^{5/3}/\rho, \tag{3.11}$$

where the constant of proportionality depends only on the mass of the atom, etc. From this relation

$$(\partial E_n/\partial\rho)_x = -E_n/\rho \quad \text{and} \quad (\partial^2 E_n/\partial\rho^2)_x = 2E_n/\rho^2,$$

so that

$$\tfrac{1}{2}(\partial^2 E_n/\partial\rho^2)_x + (1/\rho)(\partial E_n/\partial\rho) = 0. \tag{3.12}$$

Also, $(\partial E_n/\partial x)_\rho = (5/3)E_n/x$ so that

$$(\partial^2 E_n/\partial x^2)_\rho = (10/9)E_n/x^2, \tag{3.13}$$

and

$$(\partial^2 E_n/\partial x\partial\rho) = -(5/3)E_n/x\rho. \tag{3.14}$$

Let the displacement of the mass center of the mixture of low energy and high energy components be $\mathbf{Y}$ and that of the high energy component relative to the mass center be $\mathbf{X}$. Then make the transformation

$$\mathbf{Z} = (\rho_n/\rho_s)^{\frac{1}{2}}\mathbf{X}, \tag{3.15}$$

where ρ_s means the partial density, or mass per cm^3 due to the low energy component. Then the complete Lagrangian for unit mass of the mixture, namely, $L =$ K.E. per gm—ΔW becomes:[6]

$$L = \tfrac{1}{2}\dot{\mathbf{Y}}^2 + \tfrac{1}{2}\dot{\mathbf{Z}}^2 + f(\nabla\cdot\mathbf{Y})^2 + g(\nabla\cdot\mathbf{Z})^2 + h(\nabla\cdot\mathbf{Y})(\nabla\cdot\mathbf{Z}), \tag{3.16}$$

where the coefficients f, g, h are constants expressed in terms of the partial derivatives (3.12)–(3.14) as follows:

$$\left.\begin{aligned} f &= -\tfrac{1}{2}\rho^2(\partial^2 E_n/\partial\rho^2)_x - \rho(\partial E_n/\partial\rho)_x \\ &\qquad - \tfrac{1}{2}x^2(\partial^2 E_n/\partial x^2)_\rho, \\ g &= -\tfrac{1}{2}x(1-x)(\partial^2 E_n/\partial x^2)_\rho, \\ h &= -[x(1-x)]^{\frac{1}{2}}\rho(\partial^2 E_n/\partial x\partial\rho) \\ &\qquad - 2x[x(1-x)]^{\frac{1}{2}}(\partial^2 E_n/\partial x^2)_\rho. \end{aligned}\right\} \text{(3.17)**}$$

Using (3.12)–(3.14) in these relations

$$\left.\begin{aligned} f &= -5E_n/9 = -(5/6)P/\rho, \\ g &= -(5/6)[(1-x]P/\rho, \\ h &= -(5/6)[(1-x)/x]^{\frac{1}{2}}P/\rho. \end{aligned}\right\} \tag{3.18}$$

The h-term represents coupling between density and thermal fluctuations and it is evident that it is not generally small. It is, therefore, necessary to transform (3.16) so that the second quadratic form becomes diagonal while the kinetic energy form remains invariant. This can be done by means of a simple rotation in $\mathbf{Y}-\mathbf{Z}$ space:

$$\mathbf{Y} = \mathbf{Y}^* \cos\theta - \mathbf{Z}^* \sin\theta, \quad \mathbf{Z} = \mathbf{Z}^* \cos\theta + \mathbf{Y}^* \sin\theta, \tag{3.19}$$

where θ is a constant for a given equilibrium tem-

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[6] W. Band and L. Meyer, Phys. Rev. 74, 386 (1948).

** See Eq. (C9) of reference 6 where the term in $(\partial E/\partial\rho)_x$ was inadvertently omitted.

perature and pressure, namely,

$$\tan 2\theta = h/(f-g). \tag{3.20}$$

The Lagrangian then becomes

$$L=\tfrac{1}{2}\dot{\mathbf{Y}}^{*2}+\tfrac{1}{2}\dot{\mathbf{Z}}^{*2}+F(\nabla\cdot\mathbf{Y}^*)^2+G(\nabla\cdot\mathbf{Z}^*)^2, \tag{3.21}$$

where

$$\begin{aligned} F&=f\cos^2\theta+g\sin^2\theta+h\sin\theta\cos\theta,\\ G&=g\cos^2\theta+f\sin^2\theta-h\sin\theta\cos\theta. \end{aligned} \tag{3.22}$$

THE TWO MODES OF SOUND PROPAGATION

The Lagrangian (3.21) gives Euler equations in the form of two-wave equations, one in terms of $\mathbf{Y}^*$ and the other in terms of $\mathbf{Z}^*$. The former will be called first sound, and roughly corresponds to normal sound; it has a velocity of propagation V_n given by

$$V_n^2=-2F, \tag{4.1}$$

and this happens to coincide with the velocity of normal sound at the lambda-transition, as will be proved later. The other form of wave propagation will be called second sound and has a velocity of propagation V_s given by

$$V_s^2=-2G. \tag{4.2}$$

The expressions (3.22) and (3.20) have to be examined to find the temperature dependence of the two velocities. After some manipulation it is found that

$$\left.\begin{aligned} F&=\tfrac{1}{2}(f+g)+\tfrac{1}{2}((f-g)^2+h^2)^{\frac{1}{2}},\\ G&=\tfrac{1}{2}(f+g)-\tfrac{1}{2}((f-g)^2+h^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{4.3}$$

Using (3.18) these can be expressed as follows:

$$\left.\begin{aligned} F&=-(5/12)(P/x\rho)\{1+(1-3x(1-x))^{\frac{1}{2}}\},\\ G&=-(5/12)(P/x\rho)\{1-(1-3x(1-x))^{\frac{1}{2}}\}. \end{aligned}\right\} \tag{4.4}$$

The following asymptotical expressions are of interest:

$$T\rightarrow T_\lambda \quad \left.\begin{aligned} &F\rightarrow-(5/6)P/\rho,\\ &G\rightarrow-(5/8)(1-x)P/\rho\rightarrow 0. \end{aligned}\right\} \tag{4.5}$$

$$T\rightarrow 0 \quad \left.\begin{aligned} &F\rightarrow-(5/6)P/x\rho\rightarrow 0,\\ &G\rightarrow-(5/8)P/\rho\rightarrow 0. \end{aligned}\right\} \tag{4.6}$$

Figure 1 shows the two curves.

Just below T_λ first sound is almost pure pressure wave, but (3.19) and (3.20) show that as T falls, the proportion of thermal wave present in first sound increases until the temperature at which $f=g$, or $2\theta=\frac{1}{2}\pi$, $Y^*=(Y+Z)/\sqrt{2}$. This temperature is given by writing $f=g$ in (3.18): thus say $T=T_1$ when $(1-x)/x=1$, or $x=\frac{1}{2}$: $T_1=0.63T_\lambda$. Below this temperature first sound becomes predominately heat wave, while second sound becomes predominately pressure wave. The values of $\cos\theta$ and $\sin\theta$ as functions of temperature are plotted in Fig. 2.

The relation (3.19) is identical with that between two polarized components of light waves in an optically active medium. The difference of velocity between the two polarized components in the present case is, however, comparable with the velocity of both, instead of being small. If pure pressure fluctuations were produced by a source in the gas at some temperature in the neighborhood of T_1 it would split up into two "polarized" components on transmission in the gas, which would recombine at a receiver either as heat waves or as pressure waves, or as some mixture of the two, depending on the phase relation of the two components, i.e., on the length of the path at the receiver. A receiver sensitive only to thermal waves that could be moved along the direction of propagation, could be used to measure the coupling by observation of the "beats" between the two waves. The heat wave in this case would be of the type that carries only the kinetic energy of relative motion between the two fluid components alternately hotter and colder than the equilibrium temperature.[7]

It has occasionally been suggested that by suitable modification of the density of the lower energy states the Bose-Einstein gas could be made to behave more like He II in respect to the temperature dependence of the normal fluid concentration, namely to make

$$x=(T/T_\lambda)^k, \tag{4.7}$$

where k has a value somewhere in the neighborhood of 5 or 6. It is of interest to see what such a relation as this would do to the transmission of second sound in such a modified gas.

In order to simulate the equilibrium thermal properties of liquid helium it would also have to be assumed that the energy spectrum did not modify the relation

$$E_n\rho \propto xT$$

so that, using (4.7)

$$E_n \propto x^{(1+1/k)}/\rho \tag{4.8}$$

in place of (3.11). By the same type of argument as used before, it is now quite easily shown that

$$\begin{gathered} f=-\tfrac{1}{2}(1+k)E_n/k^2, \quad g=-\tfrac{1}{2}(1+k)(1-x)E_n/xk^2,\\ h=-(1+k)(2-k)[(1-x/x)]^{\frac{1}{2}}E_n/k^2. \end{gathered} \tag{4.9}$$

The discriminant of the quadratic form is $-\frac{1}{4}h^2+fg$ and if this is negative, the form reduces to two straight lines and is no longer positive definite; the velocity of transmission of the second sound would become purely imaginary. This restricts k to the range

$$3\geqslant k\geqslant 1.$$

The value 5.5 for k required to simulate the equilibrium properties of liquid helium would eliminate the possibility of second-sound transmission in the model.

Incidentally by choosing a model in which $k=2$ it is seen from (4.9) that coupling between the two modes would be zero, and first sound would be pure pressure waves and second sound would be pure heat waves at all temperatures. Second sound would have a higher velocity of transmission below T_1 than first sound,

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[7] W. Band and L. Meyer, Non-Equilibrium States in He II, (communicated to Phys. Rev., April 4, 1949).

however, so that it would still be true that heat waves would travel faster than pressure waves at such low temperatures.

VELOCITY OF NORMAL SOUND ABOVE T_λ

To complete the study of this model it is desirable to know the behavior of the velocity of normal sound V just above T_λ. defined by the equation

$$V^2=(\partial P/\partial\rho)_S. \tag{5.1}$$

Neglecting the effect of the energy gap,

$$N/V=(2\pi mkT)^{\frac{3}{2}}h^{-3}\sum_{j=1}^{\infty}\lambda^j/j^{\frac{3}{2}}, \tag{5.2}$$

$$P=kT(2\pi mkT)^{\frac{3}{2}}h^{-3}\sum_{j=1}^{\infty}\lambda^j/j^{5/2}, \tag{5.3}$$

$$S=Nk\left\{(5/2)\sum_{j=1}^{\infty}\lambda^j/j^{5/2}\div\sum_{j=1}^{\infty}\lambda^j/j^{\frac{3}{2}}-\ln\lambda\right\}. \tag{5.4}$$

One needs to compute

$$(\partial P/\partial\rho)_S=(\partial P/\partial\rho)_T+(\partial P/\partial T)_\rho(dT/d\rho)_S. \tag{5.5}$$

Write (5.2) in the form

$$\sum_{j=1}^{\infty}\lambda^j/j^{\frac{3}{2}}=(\rho h^3/m)(2\pi mkT)^{-\frac{3}{2}},$$

then one can easily obtain the partial derivatives

$$(\partial\lambda/\partial\rho)_T=(\lambda/\rho)\sum_{j=1}^{\infty}\lambda^j/j^{\frac{3}{2}}\div\sum_{j=1}^{\infty}\lambda^j/j^{\frac{1}{2}}, \tag{5.6}$$

$$(\partial\lambda/\partial T)_\rho=-(3\lambda/2T)\sum_{j=1}^{\infty}\lambda^j/j^{\frac{3}{2}}\div\sum_{j=1}^{\infty}\lambda^j/j^{\frac{1}{2}}. \tag{5.7}$$

Using (5.6) one then derives from (5.3)

$$(\partial P/\partial\rho)_T=(kT/m)\Lambda(\tfrac{3}{2})/\Lambda(\tfrac{1}{2}), \tag{5.8}$$

and using (5.7) similarly obtains

$$(\partial P/\partial T)_\rho=(P/T)\times\{5/2-(3/2)\Lambda(3/2)^2/\Lambda(5/2)\Lambda(\tfrac{1}{2})\}. \tag{5.9}$$

Because

$$(dT/d\rho)_S=-(\partial S/\partial\rho)_T\div(\partial S/\partial T)_\rho \tag{5.10}$$

it follows from (5.4) that

$$(dT/d\rho)_S=-(\partial\lambda/\partial\rho)_T/(\partial\lambda/\partial T)_\rho=2T/3\rho. \tag{5.11}$$

Finally substituting (5.11), (5.9) and (5.8) into (5.5) after using (5.2) and (5.3) in the following form:

$$(kT/m)=(P/\rho)\Lambda(3/2)/\Lambda(5/2) \tag{5.12}$$

one finds

$$(\partial P/\partial\rho)_S=5P/3\rho,$$

or

$$V^2=(5/3)P/\rho. \tag{5.13}$$

This curve coincides with the velocity of first-sound at the lambda-point and has the same gradient.

It is of interest that isothermal sound, if it could exist, would theoretically have a velocity given by

$$V_{\mathrm{is}}^2=(\partial P/\partial\rho)_T=(kT/m)\Lambda(\tfrac{3}{2})/\Lambda(\tfrac{1}{2})=(P/\rho)\Lambda(\tfrac{3}{2})^2/\Lambda(5.2)\Lambda(\tfrac{1}{2}). \tag{5.14}$$

But because $\Lambda(\frac{1}{2})$ diverges, as T goes down to T_λ the isothermal velocity would vanish at the lambda-point. Physically this is clear because at this temperature the pressure ceases to depend on the density and becomes a function only of the temperature: "*isothermal*" would then mean "*constant pressure*" and so would eliminate pressure waves.

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