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Cathode Field in Diodes under Partial Space-Charge Conditions

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A method for calculating the electric field at the cathode of a diode for any condition of current flow in the diode is presented. A "universal" curve gives the cathode field as a function of diode current. The equation for this curve is

$$\frac{I}{I_s}=\frac{1}{2}\left[1\pm\left\{1-\frac{27}{4}\left(\frac{E}{E_0}\right)^2\left(1-\frac{E}{E_0}\right)\right\}^{\frac{1}{2}}\right],$$

where I is the diode current, I_s is the saturation current corresponding to the applied anode voltage as calculated from the Child-Langmuir equation, E is the cathode field, and E_0 is the value of E in the absence of current flow.

The only restrictions on the derivation are those of an equipotential, smooth cathode, and negligible initial velocities of emission. Only plane and cylindrical diodes have been considered, but the results are believed to be applicable to any geometry. Application of the results to the study of thermionic cathodes in the Schottky emission region is very straightforward.

The variation of the cathode field as a function of anode voltage is also discussed. A method for calculating the potential distribution in a plane diode for any value of diode current is given in the Appendix.

I. INTRODUCTION

THE characteristics of diodes under conditions of complete space charge, i.e., space-charge density so great that the electric field at the cathode is zero, were calculated long ago. The results are expressed by the familiar Child-Langmuir equations for plane[1,2] and cylindrical[2,3] geometries,

$$J_s=(1/9\pi)(2e/m)^{\frac{1}{2}}V^{\frac{3}{2}}/d^2, \tag{1}$$

and

$$I_s'=(2/9)(2e/m)^{\frac{1}{2}}V^{\frac{3}{2}}/r_a\beta^2, \tag{2}$$

respectively. β^2 is the usual Langmuir-Blodgett function[3] which has been tabulated as a function of the ratio of electrode radii, r_a/r_c, V is the anode voltage, J indicates current density, and I' indicates current per unit axial length.

The problem of the diode under partial space-charge conditions has, however, received little attention. Most textbooks, for example, contain only a simple statement that as the current, and therefore the space-charge density, in a diode with constant anode voltage is increased from zero to the saturation value, the field at the cathode decreases continuously from that in the absence of space charge to zero. In many cases, and in particular the study of electron emission from cathodes in the Schottky region,* a knowledge of the field actually existing at the cathode of the experimental diode under all conditions is necessary. This paper presents a solution of this problem.

The discussion is restricted to cases where initial velocities of emission can be neglected. This condition is not, however, a real limitation, since the anode voltages employed in practical investigations of the Schottky effect are so high that the initial energies of a few tenths of an electron volt are indeed negligible. The cathode is also assumed to be equipotential and ideally smooth (i.e., the surface roughness is neglected).

II. THE PLANE DIODE

For simplicity, consider first a diode consisting of infinite parallel plane electrodes separated by a distance d. Let the space potential U and the distance x be measured from the cathode so the $U=0$ at $x=0$. The distribution of the potential in the space between the electrodes is determined by Poisson's equation, which in this case involves only the coordinate x,

$$\nabla^2U=d^2U/dx^2=4\pi\rho, \tag{3}$$

where ρ is the electron space-charge volume density (considered a positive quantity). As auxiliary relations one has also the current flow equation,

$$J=\rho v, \tag{4}$$

where J is considered positive for electron flow, and the conservation of energy,

$$mv^2/2=eU, \tag{5}$$

where v is the electronic velocity at the point x and the initial velocities of emission have been assumed negligible.

With the aid of these equations ρ and v can be eliminated and a new equation obtained giving the relation between U and J,

$$d^2U/dx^2=aU^{-\frac{1}{2}}, \tag{6}$$

where

$$a=4\pi(m/2e)^{\frac{1}{2}}J. \tag{7}$$

The variable "a" can be further simplified by substituting from Eq. (1),

$$a=\frac{4J}{9J_s}\cdot\frac{V^{\frac{3}{2}}}{d^2}=\frac{4I}{9I_s}\cdot\frac{V^{\frac{3}{2}}}{d^2}. \tag{8}$$

If Eq. (6) is multiplied by the factor $2(dU/dx)$ it may be integrated directly to give

$$(dU/dx)^2=4aU^{\frac{1}{2}}+C_1. \tag{9}$$

[1] C. D. Child, Phys. Rev. **32**, 492 (1911).
[2] I. Langmuir, Phys. Rev. **2**, 450 (1913).
[3] I. Langmuir and K. B. Blodgett, Phys. Rev. **22**, 347 (1923).
* By Schottky emission is meant thermionic emission under conditions such that the potential barrier at the emitter surface is lowered significantly by the electric field at the cathode.

The magnitude of the electric field, dU/dx, at the cathode ($x=0$, $U=0$) is designated by E. Using this as a boundary condition, $C_1=E^2$, and one can rewrite Eq. (9) as

$$dx=(E^2+4aU^{\frac{1}{2}})^{-\frac{1}{2}}dU. \qquad (10)$$

This can be integrated to give[4]

$$x=-(1/6a^2)(E^2-2aU^{\frac{1}{2}})(E^2+4aU^{\frac{1}{2}})^{\frac{1}{2}}+C_2. \qquad (11)$$

The constant of integration can be determined from the requirement that $U=0$ at $x=0$, or $C_2=E^3/6a^2$. Substituting this value and rearranging terms,

$$x=\frac{2V^{\frac{3}{4}}}{3a^{\frac{1}{2}}}\left(\frac{U^{\frac{1}{2}}}{V^{\frac{1}{2}}}+\frac{E^2}{4aV^{\frac{1}{2}}}\right)^{\frac{1}{2}}\left(\frac{U^{\frac{1}{2}}}{V^{\frac{1}{2}}}-\frac{E^2}{2aV^{\frac{1}{2}}}\right)+\frac{E^3}{6a^2}. \qquad (12)$$

This equation expresses the potential distribution in the diode. It can be put into more useful form by introducing the new dimensionless variable,

$$u=\frac{E^2}{4aV^{\frac{1}{2}}}=\left(\frac{I_s}{I}\right)\left(\frac{3E}{4E_0}\right)^2, \qquad (13)$$

where E_0, the field in the absence of space charge ($I=0$), is given by

$$E_0=V/d, \qquad (14)$$

and rearranging to obtain[5]

$$\left(\frac{x}{d}\right)\left(\frac{I}{I_s}\right)^{\frac{1}{2}}=\left[\left(\frac{U}{V}\right)^{\frac{1}{2}}+u\right]^{\frac{1}{2}}\left[\left(\frac{U}{V}\right)^{\frac{1}{2}}-2u\right]+2u^{\frac{3}{2}}. \qquad (15)$$

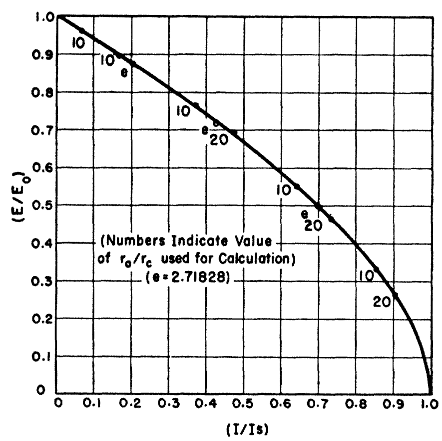

FIG. 1. Cathode field in a diode as a function of diode current.

[4] Equation (11) has also been integrated to essentially the same form by Stern, Gossling, and Fowler, Proc. Roy. Soc. **A124**, 699 (1929) in a study of field emission.

[5] Equation (15) and data for the curve of Fig. 1 were also obtained by R. Cockburn, Proc. Phys. Soc. London **47**, 810 (1935), who also considered the effect of initial velocities, but the data was not plotted in the manner of Fig. 1. Cockburn's work was not known to the author at the time this paper was first written (12/23/48). In the discussion resulting when this paper was presented at the M.I.T. Electronics Conference (April 7, 1949), it developed that the case of the plane diode has also been treated by Mr. Esterson of the English Electric Valve Company, by Professor Dow of the University of Michigan, and by Mr. W. M. Brubaker of the Westinghouse Research Laboratories. None of these workers investigated the cylindrical diode, however, and none has published his work.

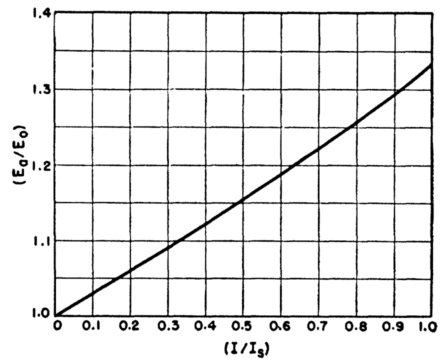

FIG. 2. Variation of anode field with current in a plane diode.

In the Appendix it is shown how this equation can be utilized to determine the potential distribution in the diode.

By definition, $U=V$ when $x=d$. When this substitution is made in the expression above one obtains the relation

$$(I/I_s)^{\frac{1}{2}}=(1+u)^{\frac{1}{2}}(1-2u)+2u^{\frac{3}{2}}. \qquad (16)$$

This expression and Eq. (13) may be considered as a parametric representation of the relation between (I/I_s) and (E/E_0), where u is the parameter. However, these equations can be combined to give the explicit relation[6]

$$\frac{I}{I_s}=\frac{1}{2}\left[1\pm\left\{1-\frac{27}{4}\left(\frac{E}{E_0}\right)^2\left(1-\frac{E}{E_0}\right)\right\}^{\frac{1}{2}}\right]. \qquad (17)$$

Here the negative sign is to be used for values of (E/E_0) greater than $\frac{2}{3}$, and the positive sign for values less than $\frac{2}{3}$.

Values of cathode field as a function of diode current calculated by this method are plotted in Fig. 1 as the solid curve. It is seen, as might be expected, that the major portion of the drop in cathode field occurs in the region close to complete space-charge saturation. Thus, for a current density which is one-half of the saturation value, the cathode field has fallen only to two-thirds of its zero-current value; for a current density 90 percent of the saturation value, the cathode field is still 27 percent of the zero-current value.

It is of interest also to investigate the field at the anode of a plane diode under conditions of partial space charge. From Eq. (10) one may write

$$E_a=(dU/dx)_{x=d}=(E^2+4aV^{\frac{1}{2}})^{\frac{1}{2}}. \qquad (18)$$

Substituting the value of "a" from Eq. (8) and introducing E_0 from Eq. (14) gives the resulting expression

$$E_a/E_0=[(E/E_0)^2+(16/9)(I/I_s)]^{\frac{1}{2}}. \qquad (19)$$

Use of the data obtained above for (E/E_0) as a function

[6] First pointed out to the author by Mr. Esterson of the English Electric Valve Company. Also obtained by Mr. Brubaker of the Westinghouse Research Laboratories.

of (I/I_s) yields the results shown in Fig. 2. Inspection of the figure shows that the variation of (E_a/E_0) with (I/I_s) is given approximately by the linear relation

$$E_a/E_0 = 1 + \tfrac{1}{3}(I/I_s). \tag{20}$$

III. THE CYLINDRICAL DIODE

Most experimental investigations are made in cylindrical diodes rather than plane diodes, so that this is a very practical case. In the case of a cylindrical diode under partial space-charge conditions, Poisson's equation cannot be integrated directly. Rose[7] has used series expansion methods to investigate the field at the cathode of such a diode but the results are not easy to apply. Crank, Hartree, Ingham, and Sloane,[8] however, have made a numerical investigation of the potential distribution in such diodes with the aid of the differential analyzer. The data of their Table I may be utilized to find the electric field at the cathode.[9] In their notation,

$$U = 1667(rI')^{\frac{2}{3}}\Phi, \tag{21}$$

and

$$\frac{dU}{dr} = 1667\left(\frac{I'^2}{r}\right)^{\frac{1}{3}}\left(\frac{d\Phi}{d\xi} + \frac{2}{3}\Phi\right), \tag{22}$$

where

$$\xi = \log_e(r/r_c). \tag{23}$$

In Table I they give values of Φ (which may be considered as a reduced voltage variable) as a function of ξ for the boundary conditions $\Phi = 0$ at $r = r_c$ and for various values of $(d\Phi/d\xi)_{r=r_c}$, that is, the potential distribution in the diode for various values of diode current.

Since we require $U = 0$ at $r = r_c$, Eq. (21) shows that $\Phi = 0$ at $r = r_c$ and Eq. (22) may therefore be written

$$E = 1667(I'^2/r_c)^{\frac{1}{3}}(d\Phi/d\xi)_{r=r_c}. \tag{24}$$

E_0, the value of E for zero current (i.e., in the absence of space charge) is given from electrostatics as

$$E_0 = V/r_c \cdot \log_e(r_a/r_c). \tag{25}$$

The anode voltage, V, is given by Eq. (21) as

$$V = 1667(r_aI')^{\frac{2}{3}}\Phi. \tag{26}$$

Combination of these three expressions gives

$$\frac{E}{E_0} = \left(\frac{r_c}{r_a}\right)^{\frac{1}{3}} \log_e\left(\frac{r_a}{r_c}\right) \cdot \frac{1}{\Phi}\left(\frac{d\Phi}{d\xi}\right)_{r=r_c}. \tag{27}$$

Equation (26) shows that for a given diode at a given anode voltage, the quantity $(I'^{\frac{2}{3}}\Phi)$ is a constant, so that

$$I'/I_s' = I/I_s = (\Phi_0/\Phi)^{\frac{3}{2}}, \tag{28}$$

where Φ_0 is the value of Φ corresponding to complete space-charge saturation, i.e., $E = 0$ and $(d\Phi/d\xi) = 0$ from Eq. (24). Φ_0 is given in the second column of Table I in the reference cited.

Equations (27) and (28) permit calculation of corresponding values of (E/E_0) and (I/I_s). Such calculations have been made for cylindrical diodes with various values of r_a/r_c, and a few points are shown plotted as circles in Fig. 1. The values of r_a/r_c used are shown. It is seen that these points fall on the curve obtained for the planar diode, a fact that was not expected beforehand.

IV. UNIVERSAL NATURE OF THE CATHODE FIELD CHARACTERISTIC AND ITS APPLICATION

The fact that the curve of Fig. 1, relating the two dimensionless ratios (E/E_0) and (I/I_s), applies to both plane and cylindrical diodes is very interesting and leads one to believe that this characteristic is of even wider application. Apparently the effect of geometry is entirely taken into account by the two normalizing factors E_0 and I_s, one of which is obtained from the solution of Laplace's equation and the other from the solution of Poisson's equation for complete space charge. Although a rigorous proof cannot be given for the statement, it is believed that the cathode field characteristic of Fig. 1 is "universal" and applies also to other geometries, including those with external cathodes. Indeed, it seems to be a direct consequence of Poisson's equation. Probably the only restrictions on its application are those of negligible initial velocities and an equipotential cathode—the same conditions imposed on the generalized space charge law.[10] The calculated value of E assumes that the cathode is ideally smooth.

The universal cathode field characteristic can be applied very simply to electron emission measurements in the Schottky region. Here the current I, in a diode, is measured as a function of the anode voltage V. For purposes of analysis it is desired to know the current as a function of the electric field at the cathode. The

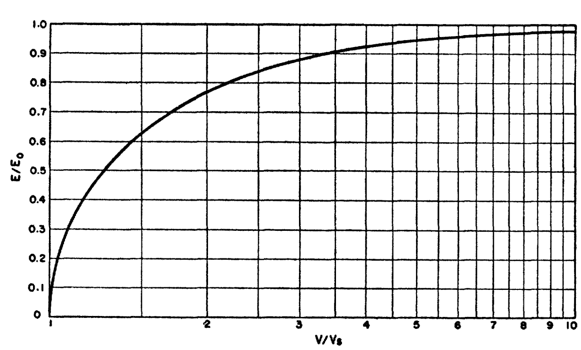

FIG. 3. Maximum cathode field in a diode as a function of anode voltage.

[7] M. E. Rose, Bartol Research Foundation, Contract OEMsr-385, First Progress Report, Supplement I (November 1942).

[8] Crank, Hartree, Ingham, and Sloane, Proc. Phys. Soc. (London) **51**, 954 (1939).

[9] H. M. Schwartz, of the Bartol Foundation (communication from W. E. Danforth), has also suggested use of the data of Crank *et al.*

[10] I. Langmuir and K. T. Compton, Rev. Mod. Phys. **3**, 251 (1931).

saturation current I_s corresponding to the applied voltage V can be calculated from the Child-Langmuir equation or from an extrapolation of an experimental Child-Langmuir plot, and the value of the cathode field in the absence of space charge, E_0 can be determined from the usual formulas of electrostatistics. The existing value of E can then be obtained directly from Fig. 1 and the measured value of the current I. No approximations are involved, and it is not necessary to sum a complicated series expansion.

As a further extension of the universal cathode field characteristic, consider now a diode with an emitter for which the Schottky effect is negligible so that for values of anode voltage V above the saturation voltage V_s corresponding to the temperature-limited emission I_0, the current is constant and equal to I_0. It is desired to find the variation in cathode field E as a function of V. For V less than V_s, the diode is completely space charge limited and E is zero. For V greater than V_s, the current I_0 is always less than the current I_s which would flow under complete space-charge limitation and is related to I_s by

$$I_s/I_0=(V/V_s)^{\frac{3}{2}}. \tag{29}$$

Here I_0 is analogous to I, the current flowing in the diode, of the previous considerations, so that

$$V/V_s=(I/I_s)^{-\frac{2}{3}}, \tag{30}$$

and (E/E_0) can be calculated as a function of (V/V_s) from the data of Fig. 1 giving (E/E_0) as dependent on (I/I_s).

The results of this calculation are given in Fig. 3. It is seen that under these conditions an anode voltage 5.4 times the saturation voltage is required to bring the cathode field to within 5 percent of its space-charge-free value, while the value to attain 98 percent of the zero-current field is ten times V_s.

In this section it has been assumed thus far that the saturation is ideal and that the Schottky effect (and also field emission) is negligible. If this is not the case, then the diode current will be increased over the value I_0 and the cathode field will always be less than that predicted by Fig. 3. This figure should therefore be interpreted in this case as giving the maximum cathode field which can exist in a diode as a function of the ratio of anode voltage to the voltage required to obtain maximum space-charge-limited emission (the "*MSCLE* point"). The exact value can be obtained, of course, from Fig. 1 as a function of the diode current.

V. SUMMARY

A method has been presented for calculating the electric field at the cathode of a diode under conditions of partial space charge. The results are shown by the "universal" characteristic of Fig. 1, which gives the cathode field as a function of the diode current and is in a particularly convenient form for use. An analytical expression has also been given for this universal characteristic (Eq. 17). The derivation has been applied only to plane and cylindrical diodes, but the results are believed to be applicable to any geometry. The only restrictions are an equipotential, smooth cathode and negligible initial velocities of emission. No approximations are involved in the derivation.

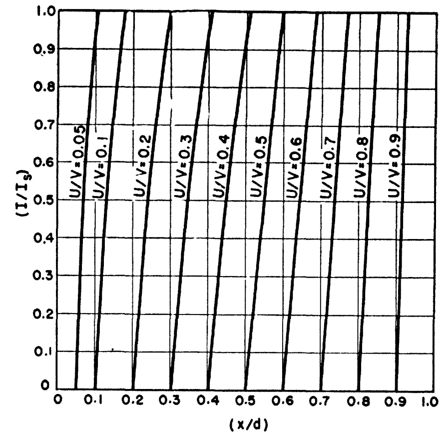

FIG. 4. Potential distribution in a plane diode.

The maximum cathode field is also plotted in Fig. 3 as a function of anode voltage. If the current saturation in the diode is ideal, this curve gives the correct value of the cathode field, but if this is not the case, the real value of field is less than that indicated.

Variation of the electric field at the anode of a plane diode with the diode current is shown in Fig. 2. The potential distribution in a plane diode under partial space-charge conditions can be obtained from Fig. 4 (see Appendix).

ACKNOWLEDGMENTS

This investigation of the cathode field in diodes was suggested by an approximate solution of the problem by Dr. J. G. Buck.[11] The writer is indebted to Dr. J. W. McNall, of this laboratory, for his advice and criticism concerning the presentation of the results.

APPENDIX

Potential Distribution in a Plane Diode

The analysis above also permits determination of the complete potential distribution in a plane diode. The quantity B, defined as

$$B=(x/d)^2(I/I_s), \tag{31}$$

may be calculated from Eq. (15) as a function of u with the quantity (U/V) as a parameter. For $(U/V)=1.0$, by definition $(x/d)=1.0$, so that this case gives values of (I/I_s) corresponding to the various values of u. The position of a point x in the diode, expressed as (x/d), corresponding to given values of (U/V) and (I/I_s) can

[11] Coomes, Buck, and Petrauskas, Third Progress Report, Contract NObsr-30028, Physics Department, University of Notre Dame (March 19, 1948).

therefore be calculated by means of Eq. (31). The results of such calculations are shown in Fig. 4. The curves of this figure permit any one of the three variables (x/d), (U/V), and (I/I_s) to be determined if the other two are known. The intersection of the contours of constant (U/V) with the horizontal line corresponding to a given value of (I/I_s) gives the potential distribution in the diode for that value of current.

Note Added in Proof: Since this paper was submitted for publication it has been found that as early as 1920 G. Jaffe (Ann. d. Phys. **63**, 145) considered the plane diode under partial space-charge conditions and obtained an equation identical, except for notation, to Eq. (17) above. However, Jaffe did not express his results in the convenient manner of Fig. 1 nor did he consider the cylindrical diode.

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Non-Equilibrium Phenomena in a Bose-Einstein Gas. I. Transmission of Second Sound

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A model Bose-Einstein gas is considered in which numerical perturbations in the population of the lowest state have a relaxation time that is long compared with the relaxation time for perturbations in the symmetry of the velocity distribution in the excited states. Oscillations of the population of the lowest state about its equilibrium value are transmitted as second sound waves. The velocity of transmission is found as a function of temperature below the lambda-point and compared with that of second sound in liquid helium. In the gas there is a temperature dependent coupling between pressure waves and thermal waves; the normal modes of propagation are mixed. The high speed mode is pure pressure wave near T_λ but goes over gradually to pure thermal wave as T goes down towards 0°K; the low speed mode is pure thermal wave near T_λ but goes over gradually to pure pressure wave as T goes down towards 0°K. For $T \ll T_\lambda$ the thermal wave has a higher speed of propagation than the pressure (ordinary sound) wave.

INTRODUCTION

In a recent paper[1] the formal first order perturbation theory of transport phenomena in a Bose-Einstein gas was considered. In that theory it was assumed that at every point in the gas the numerical populations of the various energy states accessible to the gas atoms remain equal to their equilibrium populations. In other words the regression of numerical fluctuations was assumed to be rapid compared with that of asymmetry fluctuations in the velocity distribution. This assumption is implicit in most applications of first-order perturbation theory in statistical mechanics and is probably valid in general. However, in the degenerate Bose-Einstein gas below a certain transition temperature the number of atoms in the lowest state becomes comparable with the total number of atoms in the gas. The general theory of fluctuations[2] then leads one to expect that the numerical fluctuations in the population of the lowest state become of major importance below the transition temperature. This may be seen in the following way.

The standard deviation Δn of the population of any one state from the mean population $\bar{n}$ is given by

$$\text{Fermi-Dirac gas} \quad \Delta n/\bar{n} = (1/\bar{n} - 1)^{\frac{1}{2}},$$
$$\text{Bose-Einstein gas} \quad \Delta n/\bar{n} = (1/\bar{n} + 1)^{\frac{1}{2}}.$$

At very low temperatures in the Fermi-Dirac gas $1/\bar{n} \to 1$ and the fluctuations tend to vanish as T approaches absolute zero. In the Bose-Einstein gas, on the other hand, the fluctuations remain of the order unity; in particular the lowest state has a population $n_0 \to N$ as $T \to 0$ where N is the total number of atoms in the gas, and $\bar{n}_0$ is comparable with N immediately below the transition temperature T_λ. Thus

$$\Delta n_0/\bar{n}_0 = 1 \text{ for all } T < T_\lambda.$$

These finite fluctuations at low T would be of no importance if all the states had similar populations and were closely spaced in the energy spectrum; it could then be supposed that there would be no correlation between the fluctuations of neighboring states so that in any appreciable energy range there would be no appreciable fluctuation in the total population in that range. But if the system contains one state or one degenerate set of states, in the present case the single lowest state, with a population comparable with that of the whole gas and, therefore, enormous compared with the population of any other state, the finite value of $\Delta n_0/\bar{n}_0$ becomes extremely serious. For example, if a Gaussian distribution is assumed for the actual deviations of the population from its mean value, the fact that $\Delta n_0/\bar{n}_0 = 1$ means that the number of atoms in the lowest state may be 20 percent above or below the mean value for roughly 23 percent of the time. At tem-

[1] W. Band, Phys. Rev. **76**, 1937 (1949).
[2] R. H. Fowler, *Statistical Mechanics* (Cambridge University Press, London, 1936), Chapter 20.