Radiation from a Uniformly Accelerated Charge*

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It has been stated that there is no radiation from a charge moving in the relativistic equivalent of uniform acceleration. This proves to be not the case when means of measuring the radiation are used which are suitable to the infinite extent of the path.

The fields for a uniformly accelerated charge as deduced by conformal transformation from rest are found to correspond to one-half the sum of advanced and retarded potentials for two charges.

THE classical radiation from a charge in so simple a motion as uniform acceleration would hardly seem to merit attention at this time, but the fact remains that the published work on the relativistic equivalent of uniform acceleration contains a direct contradiction. The same theory, which states that a charge at rest, but accelerated, radiates at a rate $2e^2\ddot{x}^2/3c^3$, states that when this "rest system acceleration," a, is constant, the rate of radiation is not $2e^2a^2/3c^3$, henceforth called R_0, but zero.[1–3] It is the purpose of this note to show that the radiation rate R_0 is indeed correct, and that the arguments for the zero rate, though attractive, are not correct.

As stated above, the relativistic equivalent of uniform acceleration is a motion such that the acceleration, when viewed from a system in which the particle is instantaneously at rest, has some value, a, which is a constant of the motion. It must be noted that no single frame of reference exists in which the acceleration, a, is always found, but rather there is a different rest system for every instant: no single unaccelerated system can keep pace with the accelerated particle.

The equations of motion have been determined,[3,4] and the fields for a uniformly accelerated point charge have been calculated.[2,3] The mathematical expression for the fields constitutes the basis for one of the arguments for the zero radiation rate. In the special case that the acceleration is parallel to the velocity, the particle decelerates from the speed of light to rest, then accelerates back to the speed of light retracing its path. At the instant that the particle is at rest, the turning point, the magnetic field, vanishes everywhere. As a consequence, the energy flux, $\mathbf{E}\times\mathbf{H}/4\pi$, vanishes everywhere at this instant. At any instant, however, a suitable Lorentz transformation will bring the charge to rest so that there is never an instant when the energy flux is other than zero. It is thus concluded that there is never any radiation by an argument almost exactly parallel to that of Zeno, who once argued that an arrow in flight is at every instant at only one point, so that there is never an instant in which it is in motion. The specific fallacy in the argument here is that there is no single unaccelerated system in which the energy flux *remains* zero.

The same formulas for the fields used to construct the argument recounted above do, in fact, reveal that the rate of radiation is R_0 and not zero. The fields originating from the particle at some given time in the course of its motion will be found at some later time on the surface of a sphere centered on the point of origin and of a radius corresponding to the propagation of the signal with the speed of light. The fields from a slightly later point of the motion will be found at the same field time on the surface of a second sphere correspondingly smaller and centered on the second source point. All contributions to the radiation arising from the section of path between these two source points must then be contained in the eccentric spherical shell between the two spheres, Fig. 1. The energy density, $(E^2+H^2)/8\pi$, may be integrated over this shell for which the volume element proves to be:

$$dV=r^2(1-\beta\cos\theta)drd\phi d\cos\theta$$

with the polar axes about the velocity and β the ratio of the particles' velocity to that of light. Using the known fields to compute the energy density (reference 2, Eq. (244)) and doing the integration, it is found that there is a contribution, constant in that time, from that

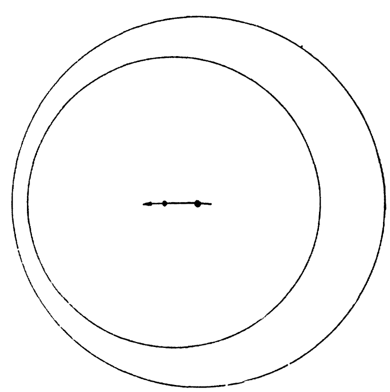

FIG. 1.

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[1] H. A. Lorentz, Enc. d. Math. Wiss. sect. 5, Vol. **12**, p. 168.

[2] W. Pauli, Enc. d. Math. Wiss. sect. 5, Vol. **19**, pp. 648, 653.

[3] M. Born, Ann. d. Physik **30**, 39 (1909).

[4] E. L. Hill, Phys. Rev. **72**, 143 (1947).

section of the path. This contribution thus flows outward undiminished with the speed of light and is radiation. As might be expected, this contribution agrees with that calculated from the rate of radiation, R_0.

This same result is obtained if the Poynting vector, $\mathbf{E}\times\mathbf{H}/4\pi$ is computed and its normal component summed over the surface of a "light sphere" about some given source point correcting for the convection of energy due to the motion of the particle. The correction has the form, $1-\beta\cos\theta$.[5]

Finally, one may consider a finite motion composed of three parts: a gradual start from rest, as long a period of uniform acceleration as desired, and a gradual return to rest. It is then found that the beginning and ending portions can be made to contribute as little as desired to the radiation, and that the amount contributed by the uniformly accelerated portion is in accordance with the rate of radiation R_0. Since the motion in question is bounded, a sphere of sufficiently large radius can contain all of the currents of the problem and can be as far from them as desired, so that the usual method of evaluating the Poynting vector is justified. If the actual uniformly accelerated motion were to have a zero rate of radiation, it would thus be necessary to imagine complete destructive interference of the contributions from the finite and infinite portions of the path.

It may thus be shown by several devices specifically adapted to the nature of the problem that the radiation from a uniformly accelerated charge is not anomalous. The special devices have proven necessary because of the infinite extent of the path. They yield the same answers as more common methods when they are applied to ordinary problems. The prediction of zero radiation does not seem justified on the basis of classical field theory.

A second argument is cited[2] to demonstrate the nonexistence of radiation from the motion. It is stated that there is no radiation reaction force on a uniformly accelerated particle. Since the particle would do no work against the reaction, it could not emit energy by radiation.

This is not the case: Consider the expression derived by Lorentz for rate of radiation in a rest system.[1] The energy radiated during a given interval is:

$$\Delta E=\int_{t_1}^{t_2}\frac{2e^2\ddot{x}^2}{3c^3}dt.$$

This can be integrated by parts to give:

$$\Delta E=\left\{\frac{2e^2\dot{x}\ddot{x}}{3c^3}\right\}_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{2e^2\dot{x}\dddot{x}}{3c^3}dt.$$

The second term is the work done against the radiation reaction and vanishes for uniform acceleration, but the first term, the change in a quantity characteristic of the instantaneous state of the motion and called by Schott the acceleration energy, just accounts for the radiation previously predicted. This term, usually neglected because attention is generally confined to periodic motions or to those bounded in time, accounts for the entire energy in this problem.

Recently it has been suggested in another connection that uniform acceleration does not lead to radiation. In the course of his investigations of the application of the general conformal transformation to electrodynamics, Hill[6] has obtained the fields by conformal transformation from rest to uniform acceleration. The explanation as to the error in radiation rate leads to nothing new, but it is of interest to investigate the fields obtained by this method. One of the transformations takes a point at rest into one in uniform acceleration. Since the Maxwell equations are invariant under the transformation, the idea that the fields of a resting point charge might be transformed into those of a charge in uniform acceleration naturally presents itself for consideration, although its validity is by no means assured. In fact, the fields obtained prove to correspond to one-half of the sum of advanced and retarded potentials for two charges. The second charge is the reflection of the first, both in sign of charge and in position. The reflection is in a plane normal to the path and a distance c^2/a beyond the turning point where a is the rest acceleration and c the velocity of light.

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[5] Cf. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1949), p. 20.

[6] E. L. Hill, Phys. Rev. **72**, 148 (1947).