Penetration and Diffusion of Hard X-Rays through Thick Barriers.
I. The Approach to Spectral Equilibrium*

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When some of the secondary components of a complex radiation are barely softer than the primary, the process of formation and decay of these components alone governs the approach to equilibrium. It is then possible to determine the asymptotic trend of the total radiation intensity at great depths of penetration. Under broad assumptions, this trend is $x^{K_0} \exp(-\mu_0 x)$.

T HE penetration of hard x- or γ-rays through thick barriers involves progressive simultaneous processes of energy degradation, multiple scattering and outright absorption. An adequate theoretical study of this phenomenon appears to require the application of a series of different approximation procedures, each of them appropriate to the treatment of one among the various effects which determine the course of the phenomenon. This note deals specifically with the effect of secondary components slightly less penetrating than the primary x-rays.

The generation of a softer secondary radiation component from a harder primary one usually leads, as is well known, to the establishment of "equilibrium" between the secondary and the primary. Equilibrium involves, in general, two different features, namely:

(a) The ratio of the *intensity* of the secondary to the intensity of the primary radiation, as a function of the depth of penetration, approaches asymptotically a maximum value.

(b) The *quality* of the secondary radiation also becomes independent of the depth of penetration.

These circumstances simplify the study of further penetration. The depth of penetration in a homogenous material required to approach equilibrium to a desired extent—e.g., such that the secondary-to-primary ratio attains 90 percent of maximum—is inversely proportional to the difference of the absorption coefficients of the primary and secondary radiation.

The penetration of hard x-rays presents a special problem with regard to the trend toward equilibrium. Some of the secondary x-rays arising from Compton scattering are only *infinitesimally softer* than the primaries, and they are deflected by an infinitesimal angle only. This circumstance does not merely slow down the approach to equilibrium; it actually suppresses the feature (a) in that the intensity ratio of the secondary to a monochromatic primary may grow beyond any limit.

Still, some sort of equilibrium should be attained among those secondary components which are substantially softer than the primary radiation. In fact, the spectral distribution of these components does become independent of the depth of penetration, as required by (b) above. The problem is therefore reduced to the treatment of the secondary x-rays whose frequency is not much lower than that of the primary. Once this problem is solved the rest should no longer offer the same difficulty.

This limitation of the frequency range affords an opportunity to treat the problem approximately by *expansion into powers* of the frequency change. Furthermore, the *deflections* of the secondaries may also be regarded as very *small*.

Note added in proof: The effect of angular deflections is not as small as one might surmise, even when the energy of the primaries is large. Further investigation, to be reported, indicates that the structure of the theory presented here is correct but that the effective value of the constant K_0 should be reduced by a substantial amount, owing to deflection effects.

CALCULATION

Within the limits of this approximation, that is disregarding deflection effects, the simultaneous penetration and energy degradation of a beam of x-rays in a material reduces to a one-dimensional problem. The number $N(\nu, x)d\nu$ of photons per unit volume traveling in the x direction, with a frequency between ν and $\nu+d\nu$ is governed by the equation

$$\partial N/\partial x = -\mu N + \int_{\nu}^{\nu_0} k(\nu', \nu)N(\nu')d\nu'. \tag{1}$$

Here $\mu(\nu)$ is the total ("narrow beam") absorption coefficient for x-rays of frequency ν in the material under consideration, ν_0 is the highest x-ray frequency in the beam and $k(\nu', \nu)d\nu$ is the probability for a photon of frequency ν' to be Compton scattered per unit length in such a way that its frequency after scattering falls

between ν and $\nu+d\nu$. The absorption coefficient μ of each photon may serve conveniently to characterize its frequency provided it is a monotonic function of the frequency (which is true only if one excludes the high frequencies where pair production is predominant). The frequency distribution in the beam may be described by the "integral spectrum" $I(\nu, x) = \int_\nu^{\nu_0} N(\nu', x)d\nu'$ rather than by N itself. Following these changes of variable, Eq. (1) takes the form:

$$\partial^2 I/\partial x\partial\mu = -\mu\partial I/\partial\mu + \int_{\mu_0}^\mu K(\mu', \mu)(\partial I/\partial\mu')d\mu', \quad (2)$$

where

$$K(\mu', \mu) = -k(\nu',\nu)d\nu/d\mu. \quad (2')$$

At this point we rely on the circumstance that our interest centers on a narrow spectral range corresponding to values of μ not much larger than μ_0. In this paper we limit ourselves to a first approximation in which the variations of the kernel $K(\mu', \mu'')$ in the range $\mu_0 \leq (\mu', \mu'') \leq \mu$ are disregarded altogether. The assumed constant value of the kernel will be indicated as K_0 and may be taken as equal to $K(\mu_0, \mu_0)$. Under this assumption Eq. (2) reduces to

$$\partial^2 I/\partial x\partial\mu = -\mu\partial I/\partial\mu + K_0 I. \quad (3)$$

If the primary radiation source is monochromatic, of frequency ν_0, is located at $x=0$ and emits radiation with a density of one photon per unit volume, the initial condition for I is

$$I(\mu, 0) = 1 \quad \text{for} \quad \mu \geq \mu_0.$$
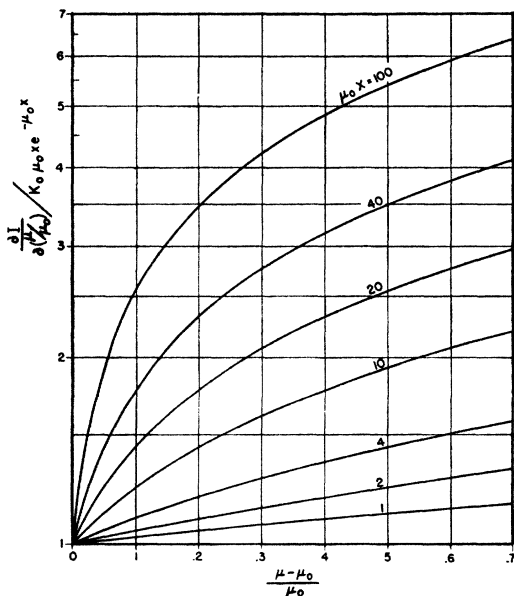
The solution of (3) reduces then to the product of an

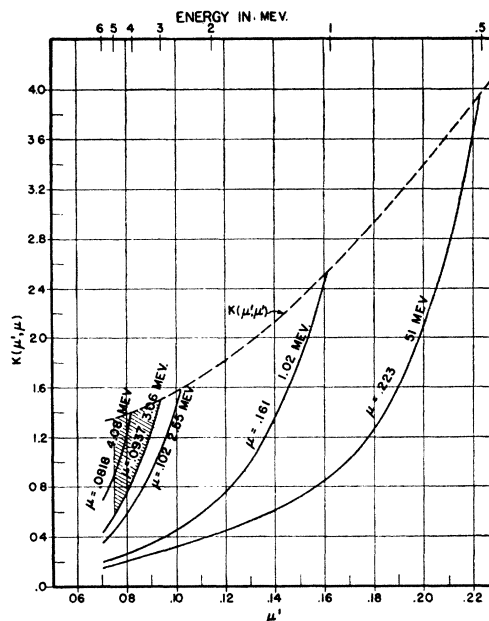FIG. 1. Differential spectra of secondary x-rays, normalized to unity at $\mu=\mu_0$.

FIG. 2. The function $K(\mu', \mu)$ for aluminum.

exponential and of a confluent hypergeometric, or Laguerre, function

$$I(\mu, x) = F(-K_0, 1, -(\mu-\mu_0)x) \exp(-\mu_0 x)$$
$$= L_{K_0}(-(\mu-\mu_0)x) \exp(-\mu_0 x). \quad (4)$$

The differential spectrum $\partial I/\partial\mu = (-d\nu/d\mu)\partial I/\partial\nu$ corresponding to this spectral distribution function is plotted in Fig. 1 for different values of the depth of penetration x, and for the representative value 1.5 of the parameter K_0 (see Fig. 2). The figure shows the trend of the spectral distribution toward equilibrium, as x and μ increase.

According to our introductory remarks equilibrium may be approached at a given depth x and for a spectral component whose absorption coefficient is μ when $(\mu-\mu_0)x$ is a large number. This condition does not require that μ be much larger than μ_0, provided $\mu_0 x$ is sufficiently large. The hypergeometric function in (4) may be expanded into inverse powers of $(\mu-\mu_0)x$.[1] The leading term for large values of $(\mu-\mu_0)x$ yields

$$I(\mu, x) \sim x^{K_0}(\mu-\mu_0)^{K_0} \exp(-\mu_0 x). \quad (5)$$

This expression describes an equilibrium distribution within the meaning of (b) above, since it consists of the product of a function of x and of a function of μ. The fact that the integral spectrum of an equilibrium distribution near its high energy limit μ_0 must behave as $(\mu-\mu_0)^{K_0}$ has been pointed out previously by Feshbach.[2] However, this result no longer holds, at least for a

[1] See Jahnke-Emde, *Tables of Functions* (B. G. Teubner, Leipzig, 1933 and Dover Publications, New York, 1943), p. 275.
[2] OSRD Rept. No. 4488, Vol. 4. Photostats available from U. S. Library of Congress.

monochromatic primary radiation, when μ is so close to μ_0 that $(\mu-\mu_0)x\leq 1$.

Since the density of primary photons at the depth x is $\exp(-\mu_0 x)$, the *ratio* of the total photon density to the primary density increases with the increasing depth according to the law x^{K_0}, that is, it increases beyond any limit. This result indicates that equilibrium within the meaning of (a) above is never attained or even approached asymptotically.

DISCUSSION

This work originates from a preliminary report by one of us (HAB), where a formula equivalent to (5) was obtained. In addition to some unnecessary approximations[3] the earlier work contains an arbitrary reduction of K_0 by a factor "1.5"/2 intended to compensate for variations of $K(\mu', \mu)$ in (2), variations that were disregarded when substituting (3) for (2). The actual values of $K(\mu', \mu)$ for aluminum are plotted in Fig. 2. If one intended to solve Eq. (2) in the energy interval from 5 to 3 Mev, the values of K involved in the integration would come from the shaded area. Their effective mean would in fact be lower than the value of 1.34 of $K(\mu_0, \mu_0)$ at 5 Mev by about 25 percent.

There remains to verify how accurate is the approximation introduced in (3)–(5), with $K_0=K(\mu_0, \mu_0)$ or with a lower estimated effective value of K_0. This verification will be provided by further numerical work now in progress relying on better approximations where the variations of K are taken into account. The purpose of this note is primarily to point out the physical factors that control the approach to equilibrium and the analytical methods for treating them. For the time being, Eq. (5) must be understood to represent the integral spectrum of x-rays approximately, with the following restrictions:

(1) The lowest frequency ν considered should be sufficiently close to the primary frequency ν_0, so that $(\mu-\mu_0)/\mu_0\ll 1$;

(2) The depth of penetration x considered should be sufficiently large so that $(\mu-\mu_0)x\gg 1$.

The restriction (1) should be removed by future work (now started) intended to determine the equilibrium spectral distribution in a form $f(\mu)$ that would be valid for all values of $\mu>\mu_0$ and $\nu<\nu_0$ and would reduce approximately to $(\mu-\mu_0)^{K_0}$ for $(\mu-\mu_0)/\mu_0\ll 1$. Such a function will enable one to calculate by suitable integrations the quantities of practical importance such as the energy flux, the dose in roentgens or the total photon density at various depths. This future work should also indicate the choice of an optimum "effective value" for the exponent K_0 of x in (5).

The calculation performed in this note applies primarily to a monochromatic x-ray source. Any source may, of course, be treated as a superposition of monochromatic sources. However, when the primary x-rays arise from a thick target under electron bombardment, the secondary radiation is likely to approach equilibrium much faster than the secondary of a monochromatic spectrum. The reason is that the intensity spectrum of thick target primary x-rays already tends to zero at the peak energy where $\mu=\mu_0$ as the equilibrium spectrum (5) must eventually do, while the primary monochromatic (differential) spectrum consists of a sharp line just at $\mu=\mu_0$. This effect requires more detailed investigation. The progressive filtration of a non-monochromatic beam tends to make its attenuation curve steeper than $\exp(-\mu_0 x)$, while the addition of secondary x-rays scattered by a small angle tends to make this curve less steep. Owing to these opposite effects, an experimental attenuation curve may not be expected to differ very critically from $\exp(-\mu_0 x)$, in agreement with observation.

The entire treatment of this note assumes that the primary radiation is more penetrating than its secondaries. This condition is not fulfilled when the absorption is primarily due to pair production, since the pair production cross section increases as the photon energy increases. This case requires a separate investigation. From the standpoint of the present treatment, as the primary photon energy increases and pair production becomes increasingly important, the derivative $-d\mu/d\nu$ in the denominator of (2') approaches zero. Then the coefficient K_0 in (3), (4) and (5) diverges showing that our approximation becomes unusable.

[3] General Electric Company Report HAB-1, A-4251, June 24, 1947. According to the report, the numerical values of the exponent K_0 (or a', as it was then called) is lower than one finds by taking $K_0=K(\mu_0, \mu_0)$ for various reasons including: (a) Use of the extreme relativistic approximation for the Klein-Nishina cross section. The error due to this approximation amounts to about 10 percent at 5 Mev. (b) Failure to include the effect of pair production in the numerical evaluation of μ. This effect amounts to a factor of about 1.5 in aluminum at 5 Mev. (Notice that this effect depends on the derivative of the pair production cross section rather than on the cross section itself.)