

Theory of the Effective Range in Nuclear Scattering

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The scattering of neutrons up to about 10 or 20 Mev by protons can be described by two parameters, the scattering length at zero energy, a , and the effective range, r_0 . A formula (16), expressing the phase shift in terms of a and r_0 is derived; it is identical with one previously derived by Schwinger but the derivation is very much simpler. Reasons are given why the deviations from the simple formula are very small, as shown by the explicit calculations by Blatt and Jackson.

The theory is then applied to proton-proton scattering, with a similarly simple result. Moreover, a method is developed to compare proton-proton and proton-neutron scattering without explicit calculation of a nuclear potential.

The most recent experimental results are evaluated on the basis of the theory, and accurate values for the effective ranges are obtained for the triplet scattering of neutrons, and for proton-proton scattering. The nuclear force between two protons is found to differ by a slight amount, but beyond doubt, from that between neutron and proton in the singlet state. All actual results agree with those obtained by Breit and collaborators, and by Blatt and Jackson.

I. INTRODUCTION

SCHWINGER¹ has shown that the phase shift δ in the triplet neutron-proton scattering is related to the wave number k by the relation

$$k \cot \delta = -\gamma + \frac{1}{2}(\gamma^2 + k^2)r_0 + O(k^4 r_0^3), \quad (1)$$

where γ is related to the deuteron binding energy, ϵ , by

$$\epsilon = (\hbar^2/M)\gamma^2, \quad (2)$$

and r_0 is a constant of the dimension of a length, which is called the "effective range" of the nuclear forces. The term of order $k^4 r_0^3$ is negligible at "classical"² neutron energies. Therefore, if δ is obtained from experiment and $k \cot \delta$ plotted against the neutron energy (i.e., k^2), a straight line will result. The slope of this line determines the effective range r_0 ; the intercept at $k=0$ gives the scattering length³ at zero energy, a , by means of the relation

$$1/a \equiv \alpha = \gamma(1 - \frac{1}{2}\gamma r_0). \quad (3)$$

The singlet scattering can be treated in an analogous way.

It has long been suspected that "classical" nuclear scattering experiments can only determine two parameters in the nuclear potential, let us say an effective range and an effective depth. This was demonstrated with the greatest thoroughness by Breit and his collaborators⁴ for proton-proton

scattering. For any reasonable shape of the nuclear potential, they could choose two parameters (depth and range) in such a way as to fit the observed scattering essentially within experimental error. This showed that practically no information could be obtained, from classical scattering experiments, on the shape of the potential. Subsequently, Landau and Smorodinsky⁵ surmised the relations (12) and (51) which are similar to (1). However, Schwinger was the first to give a general proof of these relations.

Schwinger's proof is based on a variational principle and is quite complicated. It is the main purpose of this paper to give a greatly simplified derivation. Because of the simplicity, it is then easy to calculate higher terms in Eq. (1), and in particular to show generally that the neglected term is not only of order k^4 but in addition has a very small coefficient. For the proton-proton scattering, a similar equation (Eq. (51)) can be derived with very little trouble; moreover, the relation between proton-proton and proton-neutron force can be established without explicit calculation of the nuclear potential. In a subsequent paper, it will be shown that the photoelectric and the photomagnetic effects of the deuteron can be calculated in terms of the effective ranges, and that the theory can be generalized to the case of tensor forces.

Blatt¹ has applied Schwinger's theory to an evaluation of the experiments on neutron-proton scattering. Blatt and Jackson⁶ have analyzed all the implications of the theory and have, in addition, given very useful curves and formulas showing the relation between the parameters of potentials of

* This work was done while the author was a Visiting Professor at Columbia University, New York, New York.

¹ J. Schwinger, Harvard lecture notes (holographed only). Quoted by J. Blatt, Phys. Rev. **74**, 92 (1948). See reference 6, Sections 2 and 3.

² The expression "classical nuclear physics" was introduced by S. Allison to describe the energy region up to, say, 10 or 20 Mev.

³ E. Fermi and L. Marshall, Phys. Rev. **71**, 666 (1947).

⁴ Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939); Hoisington, Share, and Breit, Phys. Rev. **56**, 884 (1939).

⁵ Landau and Smorodinsky, J. Phys. Acad. Sci. U.S.S.R. **8**, 154 (1944); Smorodinsky, J. Phys. Acad. Sci. U.S.S.R. **8**, 219 (1944); **11**, 195 (1947).

⁶ J. M. Blatt and J. D. Jackson, Phys. Rev. this issue. Referred to as BJ. As far as possible, the same notation is used here as in BJ.

various shapes, and the effective range and scattering length. They have also evaluated the coefficient of $k^4 r_0^3$ in (1) for various potential shapes. We shall therefore refer to their paper with regard to quantitative numerical results obtainable with the present theory.

II. NEUTRON-PROTON SCATTERING, EFFECTIVE RANGE

Consider a neutron of energy E_1 and wave number k_1 . If E_1 is in the laboratory system,

$$E_1 = 2\hbar^2 k_1^2 / M. \quad (4)$$

Let u_1 be the radial wave function multiplied by r , for an S state; then u_1 satisfies the Schrödinger equation

$$d^2 u_1 / dr^2 + k_1^2 u_1 - V(r) u_1 = 0, \quad (5)$$

where V is the potential energy, multiplied by $M/\hbar^2$. For another energy, we have

$$d^2 u_2 / dr^2 + k_2^2 u_2 - V(r) u_2 = 0. \quad (5a)$$

Multiply (5) by u_2 and (5a) by u_1 , subtract and integrate; then we get

$$u_2 u_1' - u_1 u_2' \Big|_0^R = (k_2^2 - k_1^2) \int_0^R u_1 u_2 dr, \quad (6)$$

where the upper limit R is arbitrary.

If R is infinity, the orthogonality relation results. If R is chosen equal to the range of the nuclear forces, one obtains the relation of Bethe and Peierls⁷ between scattering phase shift and k . We shall not use (6) directly, but first introduce a comparison function ψ which represents the asymptotic behavior of u for large distances, *viz.*

$$\psi_1 = A_1 \sin(k_1 r + \delta_1), \quad (7a)$$

where δ_1 is the phase shift⁸ for energy E_1 . It is most convenient to choose the normalizing factor A_1 so as to make $\psi = 1$ at the origin, thus:

$$\psi_1 = \sin(k_1 r + \delta_1) / \sin \delta_1. \quad (7)$$

This will at the same time determine the normalization of u since u is supposed to approach ψ asymptotically for large r including normalization.

For the ψ 's, a relation analogous to (6) will hold, *viz.*:

$$\psi_2 \psi_1' - \psi_1 \psi_2' \Big|_0^R = (k_2^2 - k_1^2) \int_0^R \psi_1 \psi_2 dr. \quad (8)$$

Now subtract (6) from (8). Then, if the upper limit R is chosen large compared with the range of the nuclear forces, each function u_i will be equal to

⁷ H. A. Bethe and R. E. Peierls, Proc. Roy. Soc. **A148**, 146 (1935).

⁸ All considerations refer to S waves; therefore we do not need to put the orbital momentum in evidence by writing δ_l . The subscript in this paper will generally refer to the energy, not to l .

its asymptotic form ψ_i and there will therefore be no contribution to the integrated term (left-hand side) from the upper limit R . For the same reason, the integral on the right-hand side can now be extended to infinity. At the lower limit, $u_1 = u_2 = 0$ so that this term does not contribute. This leaves

$$(\psi_1 \psi_2' - \psi_2 \psi_1')_{r=0} = (k_2^2 - k_1^2) \int_0^\infty (\psi_1 \psi_2 - u_1 u_2) dr. \quad (9)$$

Now we have normalized ψ to unity at $r=0$ (Eq. (7)), and the derivative of ψ can easily be obtained from (7) so that we find

$$k_2 \cot \delta_2 - k_1 \cot \delta_1 = (k_2^2 - k_1^2) \int_0^\infty (\psi_1 \psi_2 - u_1 u_2) dr. \quad (10)$$

This equation is exact and is the fundamental equation of our theory.

We can now apply (10) to the special case $k_1 = 0$. Then

$$k_1 \cot \delta_1 = -\alpha \equiv -1/a, \quad (11)$$

where a is the scattering length of Fermi and Marshall, for zero-energy neutrons, which can be determined with great accuracy. For the triplet state, a is positive, for the singlet state, negative. We shall use subscripts zero for the wave functions referring to zero energy, and we may drop the subscripts for state 2. Then (10) becomes

$$k \cot \delta = -\alpha + \frac{1}{2} k^2 \rho(0, E) \quad (12)$$

with

$$\frac{1}{2} \rho(0, E) = \int_0^\infty (\psi_0 \psi - u_0 u) dr. \quad (13)$$

Clearly, ρ has the dimension of a length. It can also be defined for two arbitrary energies,

$$\frac{1}{2} \rho(E_1, E_2) = \int_0^\infty (\psi_1 \psi_2 - u_1 u_2) dr. \quad (13a)$$

The important point is now that ψ and u differ only inside the range of the nuclear forces. Therefore the integrands in (13) and (13a) will be different from zero only inside the force-range. However, in this region the wave functions u and ψ depend only very slightly on energy, because kr is small and the potential energy is much larger than k^2 . Therefore it will be a good approximation (indeed a very good one, as we shall show in the next section) to replace u by u_0 and ψ by ψ_0 and to write

$$\frac{1}{2} \rho(0, E) \approx \frac{1}{2} \rho(0, 0) \equiv \frac{1}{2} r_0 = \int_0^\infty (\psi_0^2 - u_0^2) dr. \quad (14)$$

This quantity is now a constant, independent of energy, and we call it the *effective range*.⁹ It is

⁹ F. C. Barker and R. E. Peierls, Phys. Rev. **75**, 312 (1949), have given a similar derivation for the effective range. Also similar are the calculations of G. F. Chew and M. L. Goldberger, Phys. Rev. **75**, 1637 (1949).

identical with the effective range used and defined by Blatt and Jackson. (For the shape of the functions ψ_0 and u_0 see their Fig. 3.) Schwinger defined r_0 by

$$\frac{1}{2}r_0 = \int_0^\infty (\psi_0^2 - u_0^2) dr, \quad (14a)$$

where u_0 and ψ_0 refer to the ground state of the deuteron, and in particular ψ_0 is the asymptotic solution

$$\psi_0 = e^{-\gamma r} \quad (15)$$

with γ defined in (2). Since $\rho(E_1, E_2)$ is insensitive to the energies E_1 and E_2 , it makes little difference if both of them are replaced by $-\epsilon$ where ϵ is the binding energy of the deuteron; so Schwinger's effective range is very close to ours. Blatt and Jackson have shown that (14) will give a somewhat closer approximation to the scattering than (14a).

Using (14), then, the fundamental relation (12) reduces to

$$k \cot \delta = -\alpha + \frac{1}{2}k^2 r_0. \quad (16)$$

Schwinger's relation (1) is obtained from (10) if the comparison state is taken to be the ground state. As pointed out by Blatt, and mentioned in the Introduction, these relations are both very convenient for the analysis of neutron-proton scattering data, although some complication is caused by the mixture of triplet and singlet scattering. If these two parts can be separated and δ determined for triplet and singlet separately, a straight line is obtained when the empirically determined quantity $k \cot \delta$ is plotted against the "energy" k^2 .

Equation (16) may be inserted in the formula for the cross section,

$$\frac{\sigma}{4\pi} = \frac{\sin^2 \delta}{k^2} = \frac{1}{k^2 + k^2 \cot^2 \delta} = \frac{1}{k^2 + \alpha^2 - k^2 \alpha r_0} + \frac{1}{4}k^2 r_0^2 + O(k^2 \alpha r_0^3)]^{-1}. \quad (17)$$

Actually, the more complicated Schwinger expression (1) gives a simpler result because several terms combine, *viz.*

$$\sigma = \frac{4\pi}{(k^2 + \gamma^2)} \left[1 - \gamma r_0 + \frac{1}{4}(k^2 + \gamma^2)r_0^2 + O(k^2 \gamma r_0^3) \right]^{-1}. \quad (18a)$$

Thus, in first approximation, the cross section is multiplied by the constant factor $1/(1 - \gamma r_0)$ as already pointed out by Bethe and Peierls. But the following correction term, $\frac{1}{4}(k^2 + \gamma^2)r_0^2$, while in general much smaller than the first, still is unaffected by the omission of higher order corrections in (16) or (1): these neglected terms will only give contributions of relative order $(kr_0)^3$. Schwinger has pointed out that the term in $(kr_0)^2$ is required

in order to keep the cross section everywhere below the permitted maximum, $4\pi/k^2$. This maximum is reached when $\cot \delta = 0$, i.e., $k^2 = 2\alpha/r_0$. For triplet scattering, with the numerical values given in the next section, this corresponds to 20 Mev neutron energy (laboratory system).

III. DETERMINATION OF RANGE FROM PARAHYDROGEN SCATTERING

If the state 2 in (10) is taken to be the ground state, then from (15) we have $\psi_2'(0) = -\gamma$, and we find

$$\alpha = \gamma - \frac{1}{2}\gamma^2 \rho(0, -\epsilon) \approx \gamma(1 - \frac{1}{2}\gamma r_0), \quad (19)$$

which is also contained in the Schwinger relation (1). This provides the most accurate way to determine r_0 for the triplet state^{1,6} because γ is known from the binding energy of the deuteron and α from the parahydrogen scattering.

Since all the relevant experimental data are still subject to correction, it is desirable to indicate the change of the result which would be caused by a given change of the experimental numbers. We put: Binding energy of deuteron

$$\epsilon = 2.21(1 + \epsilon_1) \text{ Mev.} \quad (20a)$$

Scattering cross section of slow neutrons by free

$$\text{protons} \quad \sigma_f = 20.36(1 + \epsilon_2) \text{ barns.} \quad (20b)$$

Scattering of parahydrogen

$$(3a_t + a_s)^2 = 0.624(1 + \epsilon_3) \text{ barns,} \quad (20c)$$

where a_t and a_s are the scattering lengths for triplet and singlet, respectively, so that

$$\sigma_f = \pi(3a_t^2 + a_s^2). \quad (21)$$

The corrections ϵ_1 , ϵ_2 , ϵ_3 may be just as easily positive as negative; the present indicated probable errors are about

$$\epsilon_1 = \pm 0.009 \quad (\pm 20 \text{ kev}), \quad (22a)$$

$$\epsilon_2 = \pm 0.005 \quad (\pm 0.10 \text{ barn}), \quad (22b)$$

$$\epsilon_3 = \pm 0.08 \quad (\pm 0.050 \text{ barn}). \quad (22c)$$

The value for the binding energy of the deuteron is taken¹⁰ mainly from the reaction



whose threshold has been accurately determined by Shoupp and Jennings¹¹ giving $Q = -620 \pm 9$ kev. The energy of the β -rays from C^{14} is extremely well known, e.g., from the measurement of Langer,

¹⁰ I am indebted to Dr. K. Bainbridge for a discussion of the experimental information on the deuteron binding energy at the Conference on Classical Nuclear Physics at Chicago, December 1948. Dr. Bainbridge will publish a more detailed discussion in the near future.

¹¹ Shoupp, Jennings, and Sun, Phys. Rev. **75**, 1 (1949).

Cook, and Price,¹² and is 156.3 ± 1.0 kev. This gives for the difference†

$$n-H = 776 \pm 9 \text{ kev.} \quad (23a)$$

Combining this with the mass spectrograph doublet

$$H_2-D = 1432 \pm 2 \text{ kev,} \quad (23b)$$

we get for the binding energy of the deuteron

$$n+H-D = \epsilon = 2208 \pm 7 \text{ kev.} \quad (23c)$$

Probably the best *direct* measurement of the binding energy is the recent one by Bell and Elliot¹³ at Chalk River who measured the energy of the γ -rays from the capture reaction and found

$$\epsilon = 2237 \pm 5 \text{ kev} \quad (24)$$

referred to the ThC'' γ -ray which was taken as 2620 kev. Since this involves an absolute calibration of γ -ray energies¹⁴ and since (23a) involves only the measurement of small energies, we shall provisionally adopt (23c) until the small discrepancy between it and (24) is resolved. In any case, the lower values (2185 kev) found earlier¹⁵ seem to be discredited.

For the free proton cross section, we take the very accurate value obtained by Melkonian, Rainwater and Havens¹⁶ with the Columbia velocity selector. For the parahydrogen scattering, the value of Sutton *et al.*¹⁷ obtained at Los Alamos was taken; a relatively generous probable error seemed indicated (see the discussion in BJ, Section VI).

The values chosen in (20) give for the "radius of the deuteron"

$$1/\gamma = 4.332(1 + \frac{1}{2}\epsilon_1) \times 10^{-13} \text{ cm} \quad (25)$$

for the two scattering lengths

$$-a_s = (2.375 + 1.11\epsilon_2 + 0.072\epsilon_3) \times 10^{-12} \text{ cm,} \quad (26a)$$

$$a_t = (0.528 + 0.37\epsilon_2 - 0.108\epsilon_3) \times 10^{-12} \text{ cm,} \quad (26b)$$

and, using (19), for the effective range

$$r_0 = (1.56 + 2.7\epsilon_1 + 5.0\epsilon_2 - 1.5\epsilon_3) \times 10^{-13} \text{ cm.} \quad (27)$$

¹² Cook, Langer, and Price, Phys. Rev. **74**, 548 (1948).

† Note added in proof.—Recent measurements by Taschek, Jarvis, Argo, and Hemmendinger (Phys. Rev. **75**, 1268 (1949)) and by Tollestrup, Jenkins, Fowler, and Lauritsen (Phys. Rev. **75**, 1947 (1949) and private communication) on the difference He^3+n-H^3-H give values for $n-H$ of 782 ± 2 kev and 789 ± 6 kev, respectively, which would slightly raise ϵ if (23b) is correct. However, this does not resolve the discrepancy pointed out in footnote 14.

¹³ R. E. Bell and L. G. Elliot, Phys. Rev. **48**, 1552 (1948).

¹⁴ T. Lauritsen has informed the author that he has re-determined the energy of the ThC'' γ -ray by direct comparison with annihilation radiation, and found 2618 ± 5 kev. This would indicate a definite discrepancy between (23c) and (24).

¹⁵ See the review by W. E. Stephens, Rev. Mod. Phys. **19**, 19 (1947).

¹⁶ Melkonian, Rainwater, and Havens, Phys. Rev. **75**, 1295 (1949).

¹⁷ Sutton, Hall, Anderson, Bridge, DeWire, Lavatelli, Long, Snyder, and Williams, Phys. Rev. **72**, 1147 (1947).

Inserting (22), we find for the contributions to the uncertainty of the effective range:

from the deuteron binding energy

$$0.024 \times 10^{-13}, \quad (27a)$$

from the free-proton scattering

$$0.025 \times 10^{-13}, \quad (27b)$$

from the parascattering

$$0.12 \times 10^{-13}. \quad (27c)$$

As is to be expected, the parascattering gives the greatest uncertainty and more precise experiments are very desirable. An additional uncertainty arises from the slight difference between $\rho(0, -\epsilon)$ and $\rho(0, 0)$ which depends on the shape of the potential; from Fig. 14 of BJ this effect may be estimated to be an increase of r_0 by about 0.01 for the square well, and a decrease of about 0.04 for the Yukawa potential, as compared with the value given in (27).

The singlet scattering range cannot be deduced with any certainty from existing neutron-proton experiments (BJ Fig. 8). The best procedure still seems to be to assume approximate equality of force in the singlet state for proton-proton and neutron-proton. The proton-proton data (Section 6) give $r_0 = 2.7 \times 10^{-13}$ which we shall assume to be true also for neutron-proton singlet scattering. This means a large difference between the effective ranges for triplet and singlet states. Now for square wells the effective range is nearly equal to the actual width (intrinsic range) of the square well and therefore the intrinsic ranges come out very differently for triplet and singlet (1.85 and 2.55×10^{-13} , respectively). But for long-tailed potentials, there is a large difference between intrinsic and effective range, especially for the triplet state which is relatively strongly bound. For such potentials, then, the *same* intrinsic range may easily be compatible with the observed large difference in effective ranges in singlet and triplet states. This is true, in particular, for a Yukawa potential if the intrinsic range is chosen to be

$$b_s = b_t = 2.5 \times 10^{-13} \text{ cm.} \quad (28)$$

That it is possible to use the same intrinsic potential for triplet and singlet, is due to the fact that the "Yukawa" curve for b/r_0 vs. αr_0 , Fig. 6 of BJ, is very steep when αr_0 is large which it is for the triplet state, *viz.* about 0.3 from Eq. (26b) and (27). Therefore almost any value of b_t is compatible with a given r_0 , and among them also b_s . It is therefore better to turn the argument around and assume $b_t = b_s$ to be given by (28), and r_t by (27), then we get $b_t/r_t = 1.6$; and from BJ Fig. 6 we find $\alpha r_t = r_t/a_t = 0.28$ in good agreement with (26b), (27). Thus, if we *knew* (a) that the potential had a Yukawa shape and (b) that αb is rather large, we could predict the effective triplet range quite accurately from the scattering length a_t alone,

without knowing the binding energy of the ground state; or we might even predict r_t and a_t (and thus the parahydrogen scattering) directly from the binding energy of the deuteron, using $\alpha r_t \approx 0.3$ and Eq. (19).

The fact that the small effective triplet range is compatible with a large singlet range for a long-tailed potential, is the reason why Bohm and Richman¹⁸ found it possible to explain the observed neutron-proton scattering up to 6 Mev with the same well for singlet and triplet *provided* they took a long-tailed well.

IV. ENERGY DEPENDENCE OF EFFECTIVE RANGE

We shall now study the dependence on energy of $\rho(0, E)$, as defined by Eq. (13). For this purpose we expand the comparison function ψ up to second order in k ; we get from (7)

$$\psi = \cos kr + \cot \delta \sin kr \\ = 1 - \frac{1}{2}k^2 r^2 + k \cot \delta r - \frac{1}{6}k^3 \cot \delta r^3 + O(k^4 r^4). \quad (29)$$

Now we insert for $k \cot \delta$ the first approximation (16) which is also accurate including terms of order k^2 ; then

$$\psi = 1 - \alpha r + \frac{1}{2}k^2 r(r_0 - r) - \frac{1}{6}k^2 \alpha r^3. \quad (29a)$$

The first two terms are independent of energy, the other two are proportional to the energy. It is often convenient to write ψ in the form

$$\psi = \psi_0 + k^2 \chi_1 + k^4 \chi_2 + \dots \quad (30)$$

Then, according to (29a),

$$\chi_1 = \frac{1}{2}r(r_0 - r) - \frac{1}{6}\alpha r^3. \quad (30a)$$

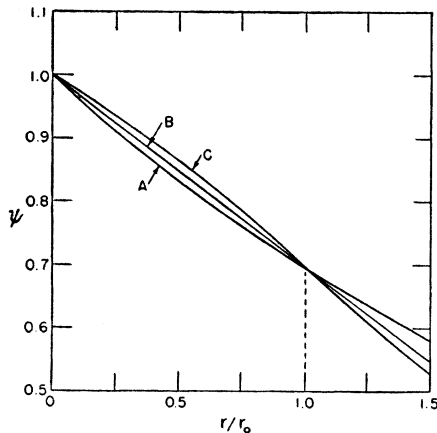


FIG. 1. The comparison functions ψ vs. radius for (A) the ground state of the deuteron, (B) the triplet state of energy zero, and (C) a neutron whose kinetic energy in the laboratory system is twice the binding energy. All curves intersect at the effective range r_0 .

¹⁸ D. Bohm and C. Richman, Phys. Rev. **71**, 567 (1947). This fact was pointed out to the author by Dr. Blatt.

This function is zero at $r=0$ because all ψ 's are normalized to unity at this point; then it becomes positive because $k \cot \delta$ (the slope of ψ) increases with energy; χ_1 has a negative curvature which arises from the term $\cos kr$ in (29); therefore it becomes zero again, at

$$r \equiv r_1 \approx r_0(1 - \frac{1}{3}\alpha r_0) \quad (31)$$

which is rather close to r_0 , and is negative thereafter. In this approximation, then, the functions ψ for all energies will intersect near the effective range. (See Fig. 1.)

The energy dependent term χ_1 is obviously small for any value of r comparable with r_0 , i.e., anywhere within the range of the nuclear forces, whatever their shape. This is because χ_1 has two zeros, one at 0 and one at r_1 and is therefore a small fraction of r_0^2 in the interesting range. Therefore $\psi - \psi_0$ will be a small numerical coefficient multiplied by $(kr_0)^2$ which itself is small for energies up to 10 Mev or so. We shall see that this makes the simple theory of Section II such a good approximation.

The correction to the simple theory is given by

$$\frac{1}{2}\rho(E, 0) - \frac{1}{2}\rho(0, 0) = \int [\psi_0(\psi - \psi_0) - u_0(u - u_0)] dr \\ = k^2 \int_0^\infty (\psi_0 \chi_1 - u_0 v_1) dr, \quad (32)$$

if we set, in analogy to (30),

$$u = u_0 + k^2 v_1 + k^4 v_2 + \dots \quad (32a)$$

The first part of (32), the integral of $\psi_0 \chi_1$, is small because χ_1 has been shown to be small (compared with r_0^2) inside the range of the nuclear forces. The function v_1 must also be small in this region because it must approach χ_1 as soon as the nuclear forces become negligible, and must be zero for $r=0$. Therefore the factor of k^2 in (32) must be small compared with r_0^3 .

BJ have written the phase shift formula in the form

$$k \cot \delta = -\alpha + \frac{1}{2}k^2 r_0 - P k^4 r_0^3 + \dots \quad (33)$$

Our argument then shows that P must be a small numerical coefficient. This is indeed what BJ find by explicit calculation. According to their Fig. 10, P is in most cases less than 0.05 and never exceeds 0.14. It depends, of course, on the shape of the potential and on the quantity αr_0 ; see Eq. (11).

It is very easy to calculate P for a square well, especially if we assume the resonance level to lie exactly at zero energy, i.e., if $\alpha = 0$. Then ψ_0 is equal to unity, α is zero, r_0 is equal to the actual width b of the well, and χ_1 is positive throughout the region contributing to the integral in (32). The first term

in this integral then gives

$$\int_0^{r_0} \psi_0 \chi_1 dr = \frac{1}{2} \int_0^{r_0} r(r_0 - r) dr = \frac{1}{12} r_0^3. \quad (34)$$

The factor here is indeed small. Moreover, the second term in the integral (32) has always the opposite sign and is in general smaller than the first (since $u_0 \leq \psi_0$ everywhere); so the actual result of (32) will be even smaller than that obtained from the first term alone. Explicitly, we have for the square well

$$u_0 = \sin Kr, \quad K = \pi/2b, \quad (35a)$$

$$u = \sin[(K^2 + k^2)^{1/2} r], \quad (35b)$$

$$v_1 = (u - u_0)/k^2 = (r/2K) \cos Kr, \quad (35c)$$

$$\int_0^b u_0 v_1 dr = r_0^3 / 2\pi^2. \quad (35)$$

This is indeed of the same sign, and smaller than, Eq. (34). Combining them we get

$$\rho(0, k^2) = r_0 + 0.0654 k^2 r_0^3 \quad (36)$$

$$P = -0.0327. \quad (36a)$$

The correction to the effective range is very small. For 10 Mev and the observed value of r_0 , Eq. (27), we have $kr_0 = 0.54$ and the second term of (36) is 2 percent of the first. The effect on the phase shift is about 0.3° . This shows the excellence of the simple (Schwinger) approximation given by Eq. (16).

We have seen that the correction function χ_1 is positive for $r < r_1 \approx r_0$, negative for $r > r_1$. Therefore the correction term (32) is positive for the square well where the entire contribution comes from $r < r_1$. On the other hand, for a long-tailed potential such as that of Yukawa, most of the contribution to (32) is likely to come from $r > r_1$, where χ_1 is negative, especially because $|\chi_1|$ is on the whole greater for $r > r_1$. Therefore we should expect that the correction term (32) is negative (P positive) for long-tailed potentials. BJ have shown that this is indeed the case for the Yukawa and the exponential potential.

Another factor which tends to make (32) negative is that in this case v_1 is algebraically larger than χ_1 so that the second term in (32) gives an additional negative contribution. (For the square well, v_1 is algebraically smaller than χ_1 .) This is illustrated in Fig. 2. This argument, as well as the calculations for Fig. 2, are due to Dr. C. Longmire.

The coefficient P is the first parameter in the expansion (33) of $k \cot \delta$ which depends on the shape of the potential. If the scattering experiments could be made accurate enough to deduce P ,

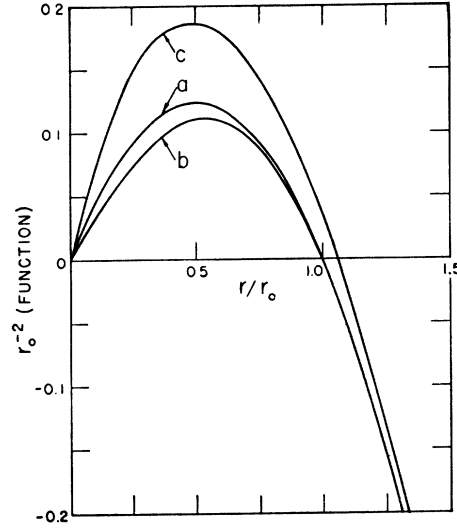


FIG. 2. The energy-dependent corrections to the wave function vs. radius (a) the correction $\chi_1 = d\psi/dk^2$ to the comparison function, (b) the correction $v_1 = du/dk^2$ to the actual wave function for a square well, (c) v_1 for a Yukawa potential. In b and c, the energy of the resonance level was assumed to be zero.

information on the shape would be obtained. If the plot of $k \cot \delta$ vs. k^2 deviates from a straight line upwards we have a concentrated potential (square well or Gaussian); if the deviation is downwards the potential has a long tail (exponential or Yukawa potential). For the present, the experimental accuracy is not sufficient to make any deviations from the straight line significant. The only arguments for the shape of the potential are then of the type presented in the last section, *viz.* that the Yukawa potential permits the intrinsic range to be the same for triplet and singlet, while the square well would require very different intrinsic ranges in the two cases. From the triplet or the singlet scattering separately, each as a function of energy, nothing could be said about the shape. Moreover, to predict the scattering at an energy where it has not been measured, the straight line relation (16) appears to be the simplest as well as a very accurate method.

V. PROTON-PROTON SCATTERING: THEORY

In the case of proton-proton scattering, the Coulomb potential is added to the nuclear interaction. If w_1 is the radial wave function multiplied by r , for wave number k_1 , then

$$w_1'' + k_1^2 w_1 = V w_1 + 2w_1/D r. \quad (37)$$

The last term represents the Coulomb potential, and D is the Bohr radius for two protons,

$$D = 2\hbar^2 / Me^2 = 5.76 \times 10^{-12} \text{ cm} \quad (38)$$

(factor 2 because of reduced mass). Because of the relative smallness of the Coulomb potential, the

Bohr radius D is very large compared with nuclear dimensions. This will be useful in the analysis.

We use again a comparison function, φ , which is the solution in the Coulomb field alone, *viz.*

$$\varphi_1'' + k_1^2 \varphi_1 = 2\varphi_1/Dr. \quad (39)$$

It is so chosen that w becomes equal to φ asymptotically, i.e., outside the range of the nuclear forces. The function φ itself is quite complicated but most of its relevant properties, as well as tables, have been given by Yost, Wheeler, and Breit.²⁰ For $kr \gg 1$, φ has the asymptotic form

$$\varphi = C \sin(kr - \eta \log 2kr + 2 \arg \Gamma(1 + i\eta) + \delta_1), \quad (40)$$

where C is a normalizing constant,

$$\eta = e^2/\hbar v = 1/kD = (25.0 \text{ kev}/E)^{1/2} \quad (41)$$

is a very useful parameter, and $\arg \Gamma$ is the complex phase of the Γ -function,

$$\Gamma(x) = |\Gamma(x)| \exp[i \arg \Gamma(x)]. \quad (41a)$$

The phase shift relative to the regular Coulomb function is denoted by δ (Breit's K_0). The function φ , like its counterpart ψ in Section II, shall be normalized to unity at $r=0$. It can then be expressed, at any value of r , by a combination of regular and irregular solutions in the Coulomb potential which are denoted by F and G , respectively, by YWB, thus:

$$\varphi = C_0(G + F \cot \delta), \quad (42)$$

where C_0 is the well-known normalizing constant for Coulomb wave functions,

$$C_0 = \left(\frac{2\pi\eta}{e^{2\pi\eta} - 1} \right)^{1/2}. \quad (43)$$

As before, the normalization of φ determines that of w .

Then, using a proof entirely analogous to Section II, we get

$$\begin{aligned} & \varphi_2 \varphi_1' - \varphi_1 \varphi_2' - w_2 w_1' + w_1 w_2' \Big|_r^R \\ &= (k_2^2 - k_1^2) \int_r^R (\varphi_1 \varphi_2 - w_1 w_2) dr. \end{aligned} \quad (44)$$

The only difference as compared with Section II is that we now take a *finite* lower limit r , which, however, we assume to be very small compared with r_0 . The integrated term in (44) is again zero at the upper limit; at the lower limit, the functions w are zero and the φ 's are 1 (within corrections of order

²⁰ Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1937). Quoted as YWB.

r). Then we get

$$\begin{aligned} \varphi_2'(r) - \varphi_1'(r) &= (k_2^2 - k_1^2) \int_r^\infty (\varphi_1 \varphi_2 - w_1 w_2) dr \\ &= \frac{1}{2} (k_2^2 - k_1^2) \rho(E_1, E_2) \end{aligned} \quad (45)$$

in complete analogy with (10), (13a). Equation (45) defines the "effective range" ρ .

We must now investigate the derivative $\varphi'(r)$. For small but finite r , YWB give for the irregular function

$$\begin{aligned} C_0 G &= 1 + \frac{2r}{D} \left(\ln \frac{2r}{D} - \ln \eta \right. \\ &\quad \left. + \text{R.P. } \Psi(i\eta) + 2C - 1 \right) + O(r^2), \end{aligned} \quad (46)$$

where C is Euler's constant, $0.577 \dots$ and Ψ the logarithmic derivative of the Γ -function.

$$\Psi(z) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} - \frac{1}{z+\nu} \right) - C \quad (46a)$$

so that

$$\text{R.P. } \Psi(i\eta) + C = \eta^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu^2 + \eta^2)}. \quad (46b)$$

Further evaluations are given by YWB. For reasonably high energy (> 0.5 Mev) it is sufficient to set

$$\text{R.P. } \Psi(i\eta) + C = \eta^2 s_3, \quad s_3 = \sum_{\nu=1}^{\infty} \nu^{-3} = 1.202. \quad (46c)$$

We abbreviate

$$C_0 G = 1 + (2r/D) [\ln(2r/D) + \frac{1}{2} g(\eta) + C - 1], \quad (47)$$

where g is a known function of η , *viz.*

$$g = -2 \ln \eta + 2\eta^2 \sum_{\nu=1}^{\infty} \nu^{-1} (\nu^2 + \eta^2)^{-1}. \quad (47a)$$

The regular function is, up to order r , simply

$$F = C_0 k r \quad (48)$$

and therefore

$$\begin{aligned} \varphi &= C_0 (G + F \cot \delta) \\ &= 1 + (2r/D) [\ln(2r/D) + \frac{1}{2} g(\eta) + C - 1 \\ &\quad + \frac{1}{2} C_0^2 k D \cot \delta] \\ &= 1 + \frac{2r}{D} \left[\ln \frac{2r}{D} + \frac{1}{2} g(\eta) + C - 1 + \frac{\pi \cot \delta}{e^{2\pi\eta} - 1} \right], \end{aligned} \quad (49)$$

where the values of C_0 , (42), and of η , (41), have been inserted.

It is seen that φ' is logarithmically infinite for $r=0$. Therefore the direct use of $\varphi'(0)$ is impossible,

and it is for this reason that we had to use a finite lower limit r in (45). But if we take the *difference* of the values of φ' for two different energies, E_2 and E_1 , and at the same r , the troublesome term $\log r$ cancels out since it is energy-independent. We get then from (45), (49):

$$\begin{aligned}\varphi_2' - \varphi_1' &= \frac{1}{D} \left[g(\eta_2) - g(\eta_1) + \frac{2\pi \cot \delta_2}{e^{2\pi \eta_2} - 1} - \frac{2\pi \cot \delta_1}{e^{2\pi \eta_1} - 1} \right] \\ &= \frac{1}{2} (k_2^2 - k_1^2) \rho(E_1, E_2),\end{aligned}\quad (50)$$

or, putting $k_1 = 0$:

$$f \equiv g(\eta) + \frac{2\pi \cot \delta}{e^{2\pi \eta} - 1} = D[-\alpha' + \frac{1}{2} k^2 \rho(0, E)], \quad (51)$$

where α' is a constant, independent of energy, and analogous to the reciprocal scattering length α in (16).

Equation (51) is quite similar to (16). It also was obtained by Schwinger¹ from a variational principle. Previously, the relation had been stated but not proved by Landau and Smorodinsky. The quantity f itself was already defined and found useful by Breit, Condon, and Present²¹ in their first theory of the proton-proton scattering. These authors pointed out that f should be independent of energy if the range of the nuclear forces were zero, and derived the consequences of such an assumption for the behavior of δ as a function of energy.

The left-hand sides of (51) and (50) differ from (16) in two respects: first, the factor of $\cot \delta$ is not

simply k but (in (50))

$$\frac{2\pi}{D(e^{2\pi \eta} - 1)} = k \frac{2\pi \eta}{e^{2\pi \eta} - 1}. \quad (51a)$$

For large energies, η becomes small and this expression goes over into k , as it should. For small energies (large η), however, the denominator becomes very large and consequently $\cot \delta$ approaches infinity ($\delta \rightarrow 0$) much more rapidly than for neutron-proton scattering. Physically this means the elimination of nuclear scattering by the Coulomb potential barrier.

The second difference is the appearance of the given function $g(\eta)$ on the left side of (50) and (51).

On the right-hand side, the definition of ρ is the same as for neutron-proton scattering. Therefore the same arguments hold as there: ρ is very nearly independent of the energy, and may therefore be replaced by $\rho(0, 0)$ which may again be called the effective range. Moreover, the Coulomb potential also has very little influence on the wave functions inside the range of the nuclear forces; therefore we may in good approximation replace φ , w by the functions ψ , u obtained in the same nuclear potential without Coulomb potential (see also Section VII). Then the effective range is defined by

$$\frac{1}{2} r_0 = \int_0^\infty (\varphi_0^2 - w_0^2) dr \approx \int_0^\infty (\psi_0^2 - u_0^2) dr, \quad (52)$$

where the subscript 0 again refers to zero energy, just as for neutron-proton scattering.

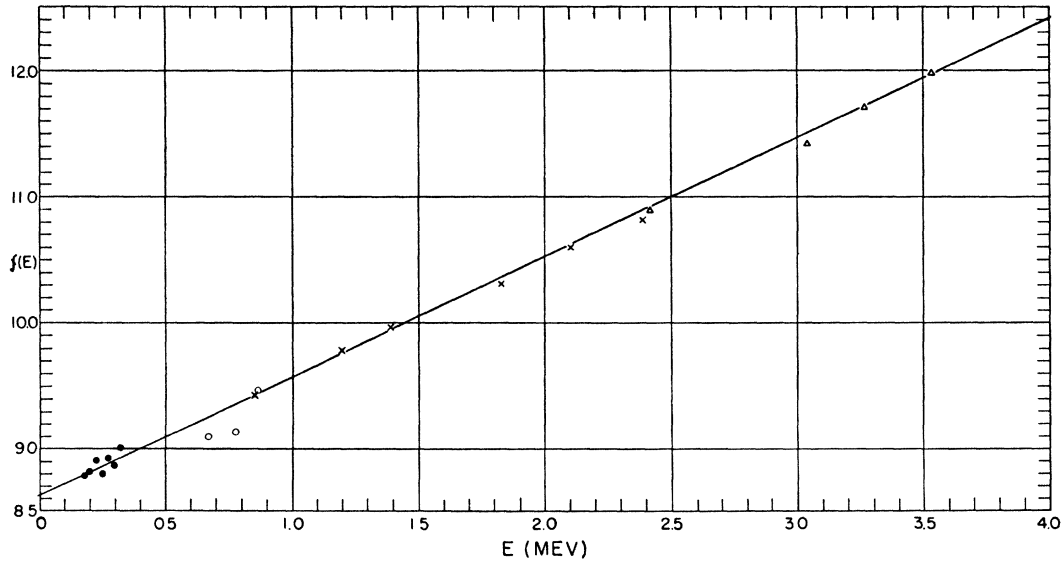


FIG. 3. Experiments on proton-proton scattering up to 4 Mev. The function f , Eq. (51), is plotted vs. energy. The experimental points fall very accurately on a straight line, as the theory predicts.
● Ragan, Kanne, and Taschek; ○ Heydenburg, Hafstad, and Tuve; × Herb, Kerst, Parkinson, and Plain;
△ Blair, Freier, Lampi, Sleator, and Williams.

²¹ Breit, Condon, and Present, Phys. Rev. 50, 825 (1936).

TABLE I. Proton-proton phase shifts.

Energy (kev)	176.5	200.2	225.9	249.5	275.3	298.3	321.4
Reference		22					
R^*	0.544	0.448	0.353	0.256	0.177	0.118	0.070
δ (degrees)	5.77	6.78	7.85	9.03	10.05	10.97	11.83
Energy (Mev)	2.42	3.04	3.27	3.53			
Reference		23					
R	44.1	63.0	69.0	77.5			
δ	48.17	51.17	51.50	52.08			
Energy		8.0		14.5			
Reference		27					
$\sigma_{c.m.}(90^\circ)^{**}$		6.0 ± 0.35		3.34 ± 0.2			
		52.6		52.0			

* R is the ratio of observed scattering at 45° to Mott scattering.

** Differential cross section in center of mass system.

VI. ANALYSIS OF PROTON-PROTON SCATTERING EXPERIMENTS

To analyze the many accurate experiments on proton-proton scattering, we merely need to take the phase shifts of the s -wave from experiment, calculate the left-hand side of Eq. (51), and plot it against the energy. This has been done in Fig. 3.

It is seen that the experimental points fall very accurately on a straight line, as (51) demands. The line was actually drawn through the lowest energy points of Ragan, Kanne, and Taschek²² (at 176 and 200 kev) and through the highest energy points of Blair, Freier, Lampi, Sleator, and Williams²³ (3.27 and 3.53 Mev). Then the points of Herb, Kerst, Parkinson, and Plain²⁴ fall very beautifully on the straight line. Only those of Hafstad, Heydenburg, and Tuve²⁵ are off the line, as was noticed before by Breit *et al.* and as is explainable from the early date at which this work was done, and from the fact that the electrostatic generator was pushed to the limit in energy.

For the two latter sets of experiments, I took the phase shifts from the work of Breit, Thaxton, and Eisenbud.⁴ For the former two, they were evaluated by Longmire, using only the absolute scattering cross section at 45° . We realize that this is not the most accurate method of evaluation but it is simple and quick. The phase shifts obtained by Longmire, and used in constructing Fig. 3, are given in Table I.

The slope of the line in Fig. 3 is

$$0.942 \text{ per Mev} \quad (53a)$$

with an accuracy of 5 percent or better. Using D from (38), and (see (4))

$$E(\text{Mev}) = 0.830k^2(\text{barn}^{-1}), \quad (53b)$$

²² Ragan, Kanne, and Taschek, Phys. Rev. **60**, 628 (1941).

²³ Blair, Freier, Lampi, Sleator, and Williams, Phys. Rev. **74**, 553 (1948).

²⁴ Herb, Kerst, Parkinson, and Plain, Phys. Rev. **55**, 998 (1939).

²⁵ Heydenburg, Hafstad, and Tuve, Phys. Rev. **56**, 1018 (1939).

we get the effective range

$$(2.71 \pm 0.13) \times 10^{-13} \text{ cm.} \quad (53)$$

This is in complete agreement with the analysis by Breit and collaborators. The limits on r_0 can now be set considerably closer than by Breit, Thaxton, and Eisenbud, primarily because of the extension of the experiments to lower and higher energies, and to some extent because the very simple linear relation used here permits a very easy estimate of error by inspection.

The intrinsic range is slightly smaller than the effective range since the "level" of the proton-proton system is virtual. Using the scattering lengths given in Section VII, Eqs. (65) and (70), together with Fig. 6 of BJ, one obtains

$$b = 2.55 \times 10^{-13} \text{ for square well,} \quad (54a)$$

$$b = 2.45 \times 10^{-13} \text{ for Yukawa potential.} \quad (54b)$$

Experiments on proton-proton scattering have also been done at higher energy, by Dearnley, Oxley, and Perry²⁶ and by R. R. Wilson and collaborators.²⁷ These are not as accurate as those at lower energy. However, we have plotted them in Fig. 4, together with the continuation of the straight line of Fig. 3. The agreement is not very good. In the case of the experiments of Wilson *et al.* the deviation is easily within the given experimental error. However, the point of Dearnley *et al.* is definitely off the straight line. The deviation is in the direction expected for a square well, while all other arguments favor something similar to a Yukawa potential. Moreover, the deviation is 4 times as great as expected for a square well. In fact, it was already noticed by Dearnley *et al.* that their cross section was lower (therefore $k \cot \delta$ larger) than would be expected from the square well used by Breit *et al.* to fit the

²⁶ Dearnley, Oxley, and Perry, Phys. Rev. **73**, 1290 (1948); (7 Mev).

²⁷ R. R. Wilson and E. Creutz, Phys. Rev. **71**, 339 (1947); (8 Mev). Robert R. Wilson, Phys. Rev. **71**, 384 (1947); (10 Mev). Wilson, Lofgren, Richardson, Wright, and Shankland, Phys. Rev. **72**, 1131 (1947); (14.5 Mev).

low energy data. The direction of the deviation precludes its explanation as a tensor force effect on the P waves. Because of the excellent fit to the straight line at lower energies, we find this deviation hard to understand.

VII. COMPARISON OF PROTON-PROTON AND NEUTRON-PROTON SCATTERING

Breit, Condon, and Present²¹ first pointed out the approximate equality of the proton-proton potential and the neutron-proton potential in the singlet state. It is the purpose of this section to compare proton-proton and neutron-proton interactions without calculating the nuclear potential explicitly.

In addition to the functions w , φ used in Section V to describe the proton-proton scattering, we shall use also the function u which would be obtained in the same nuclear potential *without* the Coulomb potential, and its asymptotic function ψ . Then u satisfies Eq. (5) and ψ has the form (7).

The problem is greatly simplified by considering all these functions for energy zero. It is true that direct experiments on proton-proton scattering at this energy would not give any information about the nuclear force between the protons. However, the value of the function f , Eq. (51) and Fig. 3, for $E=0$ will give us all the necessary information about the wave functions w and φ for zero energy, in particular their behavior for small r . The intercept of the straight line of Fig. 3 gives

$$f(0) = 8.62. \quad (55)$$

We shall first investigate the behavior of the Coulomb comparison function φ . Inserting (51) in (49) we find that for very small r , φ' is given by

$$\varphi_0'(r) = (2/D)[\ln(2r/D) + C + \frac{1}{2}f(0)], \quad (56)$$

and that

$$\varphi = 1 + r[\varphi_0'(r) - 2/D]. \quad (56a)$$

In φ , terms of order r^2 have been neglected which will be permissible for our purposes. In φ' , however, we shall need the next term which may be obtained either from YWB or by integration of (39), using (56a):

$$\varphi' = \varphi_0' + (2r/D)(\varphi_0' - 4/D). \quad (56b)$$

Now let us obtain a relation between wave functions with and without Coulomb field in the same nuclear potential V just as we previously compared wave functions of different energy. The wave equations for the two functions to be compared are (37) and (5), and since V is supposed to be the same, we have, dropping the subscript 1:

$$uw' - wu' \Big|_0^R = (2/D) \int_0^R uw dr/r. \quad (57)$$

A similar relation holds for the "asymptotic func-

tions" ψ and φ whose wave equations also differ by the Coulomb potential:

$$\psi\varphi' - \varphi\psi' \Big|_{r_1}^R = (2/D) \int_{r_1}^R \psi\varphi dr/r. \quad (57a)$$

In this case, we do not integrate from $r=0$ because φ' would be infinite at this limit, and the integral on the right-hand side would diverge. We have therefore used an arbitrary lower limit r_1 , and we shall now *define* r_1 by requiring that the right-hand sides of (57) and (57a) be equal, in other words by the equation

$$\int_{r_1}^{\infty} (\psi\varphi - uw) dr/r = \int_0^{r_1} uw dr/r. \quad (58)$$

An evaluation of r_1 will be given below.

Now, since the right-hand sides of (57) and (57a) have been made equal, the left-hand sides are also equal. At the upper limits, the integrated terms are equal by the definition of ψ , φ as the asymptotic expressions for u , w . At the lower limit, the integrated term in (57) is zero, therefore that in (57a) must also be zero. In other words, at the point r_1 the logarithmic derivatives of ψ and φ are equal,

$$(\psi'/\psi)_{r_1} = (\varphi'/\varphi)_{r_1}. \quad (59)$$

But φ'/φ can, at *any* point, be determined from (56a), (56b). On the other hand, ψ must have the form

$$\psi = 1 - \alpha_P r \quad (60)$$

since it is the asymptotic solution for energy zero in the absence of nuclear and Coulomb potential. The constant α_P can thus be determined from (59) in terms of the "observed" intercept $f(0)$, Eq. (55).

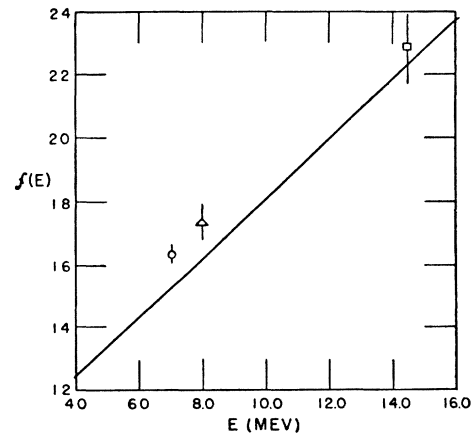


FIG. 4. Experiments on proton-proton scattering from 7 to 15 Mev, f vs. E . The straight line is the same as in Fig. 3, the experiments do not agree very well with it.
○ Dearnley, Oxley, and Perry; △ Wilson and Creutz;
□ Wilson.

Then α_P can be interpreted as an equivalent reciprocal scattering length, i.e., as the reciprocal scattering length which *would* be obtained for neutron-proton scattering if the nuclear potential were the same as for proton-proton scattering. This can then be compared with the measured $\alpha_N = 1/a_s$ for singlet neutron-proton scattering.

Inserting (56a), (56b), and (60) into (59) gives, after some simple algebra and neglect of second-order terms in r_1/D :

$$-\alpha_P = \varphi'_0(r_1) + (4r_1/D)(\varphi'_0 - 2/D), \quad (61)$$

where φ'_0 is still given by (56). Greater accuracy could be obtained using the representation of φ in terms of Bessel functions.²⁰

That it is possible to find a point r_1 at which the logarithmic slopes of the Coulomb function φ and the straight line ψ are equal, is due to the peculiar behavior of the Coulomb function (see Fig. 5). As shown by (56a), the function φ begins at $r=0$ with the value 1 and logarithmically infinite negative slope. It reaches a minimum and then rises. At larger r , it shows the typical exponential rise expected inside the Coulomb potential barrier. There must therefore be a point where the function φ has the same logarithmic slope as any given straight line.

We shall now determine r_1 . Since φ'/φ depends only slightly on r_1 (Eq. (56)), an approximate value of r_1 will be sufficient, and since the Coulomb potential is unimportant inside the nuclear potential, it is permissible to replace the Coulomb functions φ, w in the definition (58) of r_1 by the "neutral" functions ψ, u . We shall do the calculation explicitly for the specially simple case $\alpha=0$ which

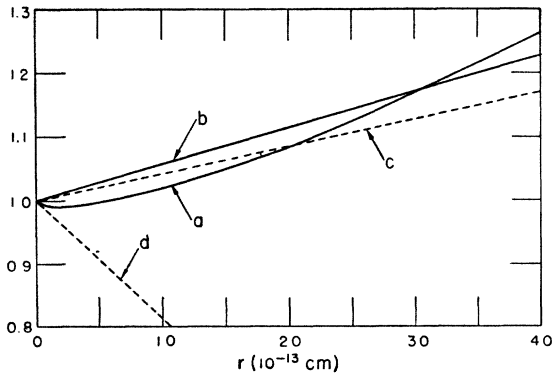


FIG. 5. *a*) The wave function φ (comparison function) in the Coulomb field, *vs.* radius, for zero energy with the constant ζ (Eq. 73a) as given by experiment, *b*) the function ψ corresponding to neutron-proton scattering by the same nuclear potential (for a square well), *c*) ψ for the actual neutron-proton scattering in the singlet state, *d*) ψ for the triplet neutron-proton state. Note the small difference between *a*, *b*, and *c*, the large difference between these and *d*. Note also the greatly enlarged scale of ordinate: All functions are near unity inside the range of nuclear forces, and *b* and *a* agree within a few percent.

corresponds to a resonance level at zero energy. This is a good approximation because α is actually small; a result for finite α will be given below. If we take $\alpha=0$ and a square well, u is given by u_0 in (35a). Since $u=\psi$ for $r>b$, (58) becomes

$$\begin{aligned} \int_{r_1}^b \psi^2 dr/r &= \ln(b/r_1) = \int_0^b u^2 dr/r \\ &= \int_0^b \sin^2(\pi r/2b) dr/r = \frac{1}{2}(C + \ln \pi - \text{Ci} \pi), \end{aligned} \quad (62)$$

where Ci is the integral cosine,

$$\text{Ci } x = - \int_x^\infty \cos y dy/y. \quad (62a)$$

Therefore

$$\ln(r_1/b) = -0.8245, \quad r_1 = 0.4384b \quad (63)$$

so that r_1 turns out considerably smaller than the effective range which, with our assumptions, is equal to b . With (54a), we get

$$r_1 = 1.12 \times 10^{-13} \text{ cm}. \quad (63a)$$

We shall show below that (63) remains a very good approximation, even for finite scattering length, but that the approximation would be considerably less good if we replaced b by r_0 in (63). This makes the calculation slightly involved: one needs α to obtain b from r_0 using the curves of BJ, but one needs b to calculate α . In practice, a very rough estimate of α suffices to get an accurate value for b so that there is no real difficulty.

Using b from (54a), and using (63), (56), (56a), (61), (38), and (55) we get:

$$\begin{aligned} \varphi'_0 s'(r_1) &= (2/D)[\ln(2b/D) + 4.06] \\ &= 0.568 \times 10^{12} \text{ cm}^{-1}, \end{aligned} \quad (64)$$

$$\varphi_s(r_1) = 1.024, \quad (64a)$$

$$-\alpha_{PS} = 0.585 \times 10^{12} \text{ cm}^{-1}, \quad (65)$$

where the subscripts P and S stand for proton and square well, respectively. This result may be compared with the neutron-proton scattering in the singlet state which gives, according to (26a)

$$-\alpha_{NS} = -1/a_{NS} = 0.421 \times 10^{12} \text{ cm}^{-1}. \quad (66)$$

For a Yukawa potential we shall not do the exact calculation but choose an approximate wave function of the form (again for zero binding energy)

$$u_0 = 1 - e^{-\beta r}. \quad (67)$$

As Schwinger¹ has shown, such a wave function when used in the variational method, will give the depth of the potential correctly within 1 part in 10,000. Chew²⁸ has pointed out that u_0 itself agrees

²⁸ Geoffrey F. Chew, Phys. Rev. **74**, 809 (1948).

with the exact wave function within 3 percent. The parameter β is related to the parameter in the meson potential and also directly to the effective range which is

$$r_0 = 2 \int_0^\infty (1 - u_0^2) dr = 3/\beta. \quad (67a)$$

For the "effective Coulomb radius" we get then the equation

$$\begin{aligned} \int_{r_1}^R \psi^2 dr/r &= \ln(R/r_1) = \int_0^R u_0^2 dr/r \\ &= \ln(\beta R/2) + C, \end{aligned} \quad (67b)$$

so that

$$\begin{aligned} r_1 &= 2/(\beta\gamma) = 2r_0/3\gamma = 0.374r_0 \\ &= 0.374b = 0.91 \times 10^{-13} \text{ cm}, \end{aligned} \quad (68)$$

using b from (54b) and with $\gamma = e^C$. This gives instead of (64)–(65)

$$\varphi_{0Y}' = (2/D)[\ln(2b/D) + 3.90_5] = 0.509, \quad (69)$$

$$\varphi_Y = 1.014_5, \quad (69a)$$

$$-\alpha_{PY} = 0.517 \times 10^{12} \text{ cm}^{-1}, \quad (70)$$

where the subscript Y stands for Yukawa potential.

It is seen that the Yukawa potential gives a result somewhat closer to the neutron-proton result (66). This is in agreement with the result of Breit, Hoisington, Share, and Thaxton^{29,4} obtained by explicit calculation of the nuclear potentials. However, the difference between the Yukawa and the square well results is not large, and, judging from Blatt and Jackson's results for other phenomena, the other potentials commonly used (exponential and Gaussian) are almost certain to lie in between. This means, then, that there is a *definite difference* between the nuclear forces in the singlet state for neutron-proton and for proton-proton. This result follows also from the explicit calculations of Breit *et al.*, if the modern value of the neutron-proton cross section, Eq. (20b), is used.

On the other hand, the difference between neutron-proton and proton-proton force is obviously quite small, *viz.* about 0.1 – $0.16 \times 10^{12} \text{ cm}^{-1}$. This figure might properly be compared with the reciprocal range of the forces which is about $4 \times 10^{12} \text{ cm}^{-1}$. In agreement with this, Breit *et al.*²⁹ found that the difference between the depths of the two potentials (for the same range and shape) need only be 1.5–3 percent. Thus the hypothesis of the charge-independence of nuclear forces is still very good, provided, of course, that range and shape are the same for which, unfortunately, there is still no experimental proof.

²⁹ Breit, Hoisington, Share, and Thaxton, Phys. Rev. **55**, 1103 (1939).

In (63), (68) we have used wave functions for infinite scattering length. For finite scattering length, the case of a square well can easily be treated and gives

$$r_1 = 0.4384b(1 + 0.114\alpha b + 0.471\alpha^2 b^2 + \dots), \quad (71)$$

where b is the intrinsic range and α the reciprocal scattering length. Using α from (65) and b from (54a), gives for the parenthesis in (71) the value $1 - 0.0065$; an exact calculation gives $1 - 0.0072$. Thus r_1/b is almost unchanged. However,

$$r_0/b \approx 1 - 4\pi^{-2}\alpha b = 1.0640 \quad (71a)$$

differs appreciably from unity: This shows that it is indeed a much better approximation to use b in (63) than to use r_0 . Now a slight change in r_1 , by the relative amount ϵ , will give a change of $-\alpha$ by

$$-\delta\alpha = (2\epsilon/D)(1 + 2r_1\varphi_0') = 0.39\epsilon = -0.0028 \quad (71b)$$

with the values of φ_0' , r_1 , etc. given above. This is practically negligible and changes (65) to

$$-\alpha_{PS} = 0.582 \times 10^{12} \text{ cm}^{-1}. \quad (65a)$$

For the Yukawa well, the approximate wave function

$$u = 1 + \alpha r - e^{-\beta r} \quad (72)$$

gives

$$r_1 = (2/\beta\gamma)[1 - 0.245\alpha/\beta - 0.051\alpha^2/\beta^2 + \dots], \quad (72a)$$

and β is related to the effective range by

$$\beta r_0 = 3 + 4\alpha/\beta. \quad (72b)$$

In our case, we get

$$\begin{aligned} 3r_1\gamma/2r_0 &= 1 - 0.067, \\ b/r_0 &= 1 - 0.097, \\ 3r_1\gamma/2b &= 1.033. \end{aligned} \quad (72c)$$

The change of r_1 with α , for given b , is therefore slightly greater than for the square well, and we get

$$\begin{aligned} -\delta\alpha &= 0.0112, \\ -\alpha &= 0.529 \times 10^{12} \text{ cm}^{-1}. \end{aligned} \quad (70a)$$

This makes the difference between Yukawa and square well somewhat smaller, the difference between the proton-proton and neutron-proton scattering somewhat larger.

Finally, we shall compare the asymptotic wave functions with and without Coulomb potential, φ and ψ (see Fig. 5). The Coulomb function looks fairly complicated, mainly because of its logarithmically infinite slope at $r=0$; but all this complication makes very little difference for the value of the function, owing to the very large value of the "proton Bohr radius" D which is about 20 times the effective range. With the constants as observed

(Eq. (55)), the function φ has a minimum at

$$r_{\min} = \frac{1}{2}De^{-\zeta} = 0.216 \times 10^{-13} \text{ cm}, \quad (73)$$

where

$$\zeta = C + \frac{1}{2}f(0). \quad (73a)$$

The result (73) is about the Compton wave-length of the proton, $\hbar/Mc$. At its minimum, φ has the value

$$\varphi_{\min} = 1 - e^{-\zeta} = 0.9925, \quad (73b)$$

which is very close to unity.

The two functions φ and ψ are very close together for all values of r inside the range of the nuclear forces. Neglecting higher terms,

$$\psi = 1 + (2r/D)(\zeta + \ln 2r_1/D), \quad (74)$$

$$\varphi = 1 + (2r/D)(\zeta + \ln 2r/D - 1), \quad (74a)$$

and the ratio is

$$\psi/\varphi = 1 + (2r/D)(\ln r_1/r + 1). \quad (75)$$

As is clear from the definition of r_1 in (59), the ratio

ψ/φ has a maximum at $r=r_1$ whose value is

$$\begin{aligned} \psi(r_1)/\varphi(r_1) &= 1 + 2r_1/D \\ &= 1.040 \text{ for square well} \\ &= 1.032 \text{ for Yukawa potential.} \end{aligned} \quad (75a)$$

At other values of r , the functions are closer together. This shows that it must be a very good approximation to replace φ and w by ψ and u , respectively, in calculating r_1 . Also it shows that for a given nuclear potential, r_0 and the correction P (see Section IV) must have substantially the same values as in the absence of the Coulomb potential.

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Bands from Doubly Excited Levels of the Hydrogen Molecule*

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A band system of H_2 is described which has as upper state $(2p\sigma)^2\ ^1\Sigma$. Although this state has two electrons excited, its energy is about the same as that of the states with only one electron excited to a two-quantum state. The position of the lines and their intensities are very irregular. These irregularities, however, can be satisfactorily explained as being due to the interaction with the $(1s\sigma)(2s\sigma)\ ^1\Sigma$ state.

I. INTRODUCTION

IN a previous paper¹ the system $2s\ ^1\Sigma \rightarrow 2p\ ^1\Sigma$ of the hydrogen molecule was discussed. The analysis of the upper state $2s\ ^1\Sigma$, or more completely $(1s\sigma)(2s\sigma)\ ^1\Sigma$, revealed strong perturbations which could only be accounted for by the presence of another electronic state slightly above the $2s\ ^1\Sigma$ state. The nature of the perturbations showed that the perturbing state must be a $^1\Sigma_g^+$ state, with the same symmetry properties as the $2s\ ^1\Sigma$ state. As all the low lying states of the molecule with one electron excited were already accounted for, this had to be a state with both electrons excited, and some arguments were given which made it probable that this hypothetical state was $(2p\sigma)^2\ ^1\Sigma$.

The existence of a doubly excited stable state with such low energy seems at first sight rather sur-

prising, and it is, therefore, highly desirable to obtain further information concerning this state. So far the only evidence for its existence has been furnished by the perturbations in the $2s\ ^1\Sigma$ state. Naturally the situation would be strengthened a great deal if bands could be found coming from this $(2p\sigma)^2\ ^1\Sigma$ state. Such bands have been available since 1937, but inconsistencies in their structure made some of them rather uncertain, so that they were not published. In the meantime, quantitative intensity measurements² in the infra-red have become available which make it possible in many doubtful cases to judge whether the bands are genuine or not. In this way the regularities previously found were sifted and amplified and they are now believed to be correct at least in their essential points, and they form the basis of the present paper. The evidence is derived entirely from the H_2 spectrum. The analysis of the analogous bands of the HD and

* This paper was written in 1941, but its publication delayed because of the war. It is presented in the original form with only minor revisions.

¹ G. H. Dieke, Phys. Rev. **50**, 797 (1936).

² N. Ginsburg and G. H. Dieke, Phys. Rev. **59**, 632 (1941).