On the Continuous Gamma-Radiation Accompanying the Beta-Decay of Nuclei*

C. S. WANG CHANG† AND D. L. FALKOFF‡
University of Michigan, Ann Arbor, Michigan
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The quantum mechanical calculations of Knipp and Uhlenbeck, and F. Bloch of the continuous γ-spectrum associated with the sudden change of nuclear charge during β-decay have been extended to forbidden β-transitions and to the different types of β-interactions. It is shown that the γ-spectrum has practically the same shape, and the ratio of total γ-intensity to β-intensity is almost the same for forbidden transitions as for allowed transitions, irrespective of the β-interaction. This accounts for the good agreement of the theory for allowed transitions with the experiments of C. S. Wu on ${}_{15}P^{32}$, a "forbidden" β-emitter. A simple classical analogue is given to explain this uniformity and is shown to justify the interpretation that the probabilities for the β- and γ-emission processes may in a very good approximation be taken as independent.

I. INTRODUCTION

THE existence of a weak continuous γ-spectrum accompanying the β-decay of nuclei was first shown experimentally by Aston[1] in his measurements on RaE. Subsequently it has been observed in various β-radioactive elements by many investigators,[2] most recently by C. S. Wu[3] using ${}_{15}P^{32}$.

A satisfactory theory of this inhomogeneous low intensity radiation was given simultaneously by Knipp and Uhlenbeck,[4] and F. Bloch[5] who showed quantum mechanically that continuous radiation of the observed order of magnitude, e.g., roughly $\alpha = 1/137$ quanta per β-particle could be attributed to the sudden change in nuclear charge when the particle is created and leaves the nucleus. This radiation has been called "internal" bremsstrahlung in contradistinction to the "external" bremsstrahlung which is the continuous electromagnetic radiation which is classically associated with the acceleration of a passing electron by the Coulomb field of the nucleus. The need to extend the calculations of Knipp, Uhlenbeck, and Bloch (henceforth denoted by KUB) becomes apparent when one seeks to compare theory and experiment. In the first place the theory was developed only for allowed β-transitions assuming the Fermi[6] polar vector interaction. On the other hand RaE and ${}_{15}P^{32}$ are generally classified[7] as first and second forbidden β-emitters, respectively. Hence any agreement between theory and experiment thus far might appear as fortuitous. Indeed, one knows that, for instance, the internal conversion of γ-radiation is quite different for the higher multipoles which correspond to the successive degrees of forbiddenness of the γ-transition. To check the theory, therefore, it is necessary to extend the calculations of KUB to forbidden β-transitions.

Since the Fermi polar vector interaction is but one of five linearly independent relativistically invariant interactions (see reference 7) which might be used in the β-decay theory, it is also of interest to see whether the calculated internal bremsstrahlung varies appreciably with the choice of interactions, in which case it could serve as a means of distinguishing between them. For this reason we have also extended the calculations of KUB to the different β-interactions for first and second degrees of forbiddenness.

The method of calculation is the same as in KUB. The joint probability per unit time $S(k)dk$ for emission of a quantum of energy k is obtained by a second order perturbation calculation, corresponding to the occurrence of the over-all process in two steps:

(1) The transition from initial to intermediate state consisting of the nuclear transformation accompanied by the creation and emission of a β-particle and a neutrino.

(2) The transition of the electron from its intermediate state to a final state by simultaneous emission of a light quantum of energy k.

The initial state, 0, of the unperturbed system is taken to be the state in which only the parent nucleus is present with an available energy W_0. The transition to the intermediate state, 1, is due to the electron neutrino interaction H_β. In this state of the system some neutron in the nucleus has been transformed into a proton, an electron has been created in a state s' of energy $W_{s'}$ and momentum $\mathbf{p}_{s'}$, and also an anti-neutrino in a state of energy W_σ and momentum $\mathbf{p}_\sigma$. The energy of the intermediate state is W_l. In the final state, f, the nucleus and neutrino remain unchanged, but in virtue of the interaction, H_γ, of the electron with the electromagnetic radiation field, a quantum of energy k and momentum $\mathbf{k}$ has been emitted leaving the electron with energy W_s and momentum $\mathbf{p}_s$.** The energy of the final state is $W_l = W_0 = W_s + W_\sigma + k$. The total prob-

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† Now at the Institute for Advanced Study, Princeton, New Jersey.

‡ Now at the University of Notre Dame, Notre Dame, Indiana.

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** The relativistic units will be used throughout, $\hbar = m = c = 1$.

ability $S(k)dk$ for the emission of a γ-quantum with energy between k and $k+dk$ is

$$S(k)dk=\frac{1}{(2\pi)^8}dkk^2\int d\Omega_s\int d\Omega_\sigma\int d\Omega_k\int dW_e$$
$$\times(W_0-W_e)^2(W_e-k)[(W_e-k)^2-1]^{\frac{1}{2}}$$
$$\times\left|\sum_l\frac{(f|H_\gamma|l)(l|H_\beta|0)}{W_l-W_0}\right|^2, \quad (1)$$

where $W_e=W_0-W_\sigma$ is the energy with which the electron is "born," and where the integrations over Ω_s, Ω_σ and Ω_k are integrations over the directions of the momenta of the electron, anti-neutrino and the light quantum and summations over the polarizations of these particles respectively. The particular interaction H_β used by KUB was the "polar vector allowed." If one wanted only the probability per unit time for the non-radiative emission with energy between W and $W+dW$, this would be given by

$$P(W)dW=1/(2\pi)^5pW(W_0-W)^2|(0|H_\beta|l)|^2dW. \quad (2)$$

On comparing (1) and (2) there appears to be no good reason to suppose that the probability for the over-all process should be simply the product of the probability for each. Nevertheless, Knipp and Uhlenbeck[4] have given an alternative method for this calculation in which the two steps in the radiative β-decay are assumed to be independent. They first obtain the probability $P(W)dW$ for the emission of a β-particle with energy W using the conventional β-theory as indicated by Eq. (2), and then with a first order perturbation calculation they obtain the conditional probability per unit time $\Phi(W_e, k)$ that an electron coming from the nucleus with energy W_e will emit a quantum of energy k. (In this calculation the initial electron wave function is taken to be an outgoing wave solution of the Dirac equation divergent at the origin corresponding to a source of electrons at the nucleus. The final electron state is that for a free particle.) Then in virtue of the assumed independence of these two processes, they are able to write

$$S(k)=\int dW_eS(W_e, k)=\int dW_eP(W_e)\Phi(W_e, k). \quad (3)$$

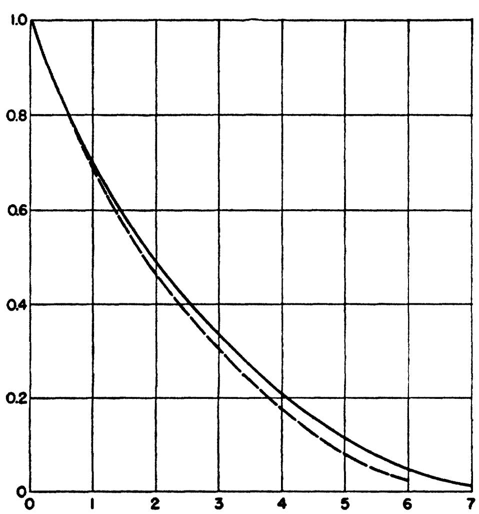

FIG. 1. Energy distribution of the γ-rays following allowed β-transitions, scalar and vector theory, for $W_0=10$ in relativistic units. Ordinate $kS(k)$ normalized to unity for $k=0$. Abscissa, k measured in relativistic units.
——— Exact, from Eq. (5).
– – – – Approximate W_0 large, from Eq. (5a).

This expression, if it were valid, has the advantage that for any given β-spectrum one can insert for $P(W)$ the experimentally observed energy distribution of the outgoing β-electrons thereby eliminating from the calculation of the radiative effects any lack of uniqueness inherent in the β-theory itself. For example, the question of which interaction to use is avoided, as is also any mention of the unobserved neutrino.

The sole justification of Knipp and Uhlenbeck for introducing this method of calculation is the *a posteriori* one that for allowed transitions it yields exactly the same result as the more rigorous second order perturbation method. But Morrison and Schiff[8] argue that it is only for allowed β-transitions that these two methods of calculation will agree, and that for forbidden β-transition in which the electron-neutrino coupling depends explicitly on the momenta of these particles these two methods should no longer agree.

Our calculations with the forbidden β-transition show that although the two different methods of evaluating the internal bremsstrahlung do not yield exactly the same results, they are nevertheless in good enough agreement to warrant the use of the simpler form (3) in practice for comparison with experiment. In addition, by way of making physically plausible the interpretation of the β-emission and radiation as independent processes, we show in Section III that the function $\Phi(W_e, k)$ may also be obtained classically.

II. RESULTS

The electromagnetic interaction for emission of a γ-quantum of momentum $\mathbf{k}$ and polarization $\mathbf{e}_k$ is

$$H_\gamma=(2\pi\alpha/k)^{\frac{1}{2}}(\boldsymbol{\alpha}\cdot\mathbf{e}_k)e^{-i\mathbf{k}\cdot\mathbf{r}},$$

where α is the fine structure constant, $\boldsymbol{\alpha}$ the Dirac matrix operator. In the Born approximation one neglects the effect of the nuclear charge on the electron so that both the electron and anti-neutrino wave functions are the 4-component plane wave solutions of the Dirac equation

$$\psi_s=\frac{A_s}{(\Omega)^{\frac{1}{2}}}\exp(i\mathbf{p}_s\cdot\mathbf{r}), \text{ for the electron}$$

$$\varphi_\sigma=\frac{B_\sigma}{(\Omega)^{\frac{1}{2}}}\exp(-i\mathbf{p}_\sigma\cdot\mathbf{r}), \text{ for the anti-neutrino}$$

[8] P. Morrison and L. I. Schiff, Phys. Rev. 58, 24 (1940).

normalized to unity in the volume Ω. For H_β we take the different interactions for the β-decay.

A. The Allowed Transitions. Scalar and Vector Interactions

For the polar vector interaction which KUB used, the matrix element of H_β is given by

$$(l|H_\beta|0)=G\int d\tau[(V^*QU)(\psi_s^*\varphi_\sigma)-(V^*\boldsymbol{\alpha}QU)\cdot(\psi_s^*\boldsymbol{\alpha}\varphi_\sigma)],$$

where G is the constant determining the strength of the coupling of the electron neutrino field to the nucleons, Q is an operator which transforms a neutron state into a proton state and U and V are the nuclear wave functions for the neutrons and protons respectively. For an allowed transition the second term is dropped. Writing M for the matrix element involving the unknown nuclear wave functions

$$M=\int d\tau V^*QU.$$

KUB obtained

$$S(k)=\frac{\alpha G^2|M|^2}{2\pi^4}\frac{1}{k}\int_{1+k}^{W_0}dW_e(W_0-W_e)^2 \times[(W_e^2+W_s^2)\ln(W_s+p_s)-2W_ep_s], \quad (4)$$

where $W_s=W_e-k$. After integration this becomes

$$\begin{aligned}S(k)=\frac{\alpha G^2|M|^2}{4\pi^4}\frac{1}{k}\Bigg\{\Bigg[&W_0^2(\tfrac{2}{3}x^3+x)-W_0(x^4+x^2-\tfrac{1}{8})\\&+\frac{7}{15}x^5-\frac{3}{8}x\Bigg]\ln(x+(x^2-1)^{\frac{1}{2}})\\&-\Bigg[W_0^2\left(\frac{11}{9}x^2+\frac{4}{9}\right)-W_0\left(\frac{7}{4}x^3+\frac{1}{8}x\right)\\&+\frac{689}{900}x^4-\frac{1021}{1800}x^2-\frac{8}{75}\Bigg](x^2-1)^{\frac{1}{2}}\Bigg\},\end{aligned} \quad (5)$$

with $x=W_0-k$. For W_0 large compared to unity one can simplify the above formula by keeping only the terms of the highest order in W_0 and x, and replacing $(x^2-1)^{\frac{1}{2}}$ by x, then one obtains

$$S(k)\cong\frac{\alpha G^2|M|^2}{4\pi^4}\frac{1}{k}\left\{\left[\frac{2}{3}W_0^2x^3-W_0x^4+\frac{7}{15}x^5\right]\ln 2x-\left(\frac{11}{9}W_0^2x^3-\frac{7}{4}W_0x^4+\frac{689}{900}x^5\right)\right\}. \quad (5a)$$

The comparison of this equation with the exact one (5) is given in Fig. 1.

The total intensity of the γ radiation is

$$\begin{aligned}I_\gamma&=\int_0^{W_0-1}dk\,kS(k)\\&\cong\frac{\alpha G^2|M|^2}{4\pi^4}\left\{\left[\frac{2}{45}W_0^6+\frac{1}{6}W_0^4-\frac{1}{144}\right]\ln(W_0+(W_0^2-1)^{\frac{1}{2}})\right.\\&\left.+W_0(W_0^2-1)^{\frac{1}{2}}\left[-\frac{22}{225}W_0^4-\frac{27}{200}W_0^2+\frac{103}{3600}\right]\right\}.\end{aligned}$$

For large W_0, this simplifies to

$$I_\gamma\cong\frac{\alpha G^2|M|^2}{2\pi^4}\frac{1}{45}W_0^6(\ln 2W_0-2.22). \quad (6)$$

The β-energy distribution is given by

$$P(W_e)=(G^2|M|^2/2\pi^3)(W_0-W_e)^2W_ep_e, \quad (7)$$

whence in the approximation $W_0\gg1$, the total β-intensity becomes

$$I_\beta=\int_1^{W_0}dW_e(W_e-1)P(W_e)\cong\frac{G^2|M|^2}{2\pi^3}\frac{1}{60}W_0^6, \quad (8)$$

so that ratio of total γ-intensity to total β-intensity for this allowed transition becomes

$$I_\gamma/I_\beta=4\alpha/3\pi(\ln 2W_0-2.22). \quad (9)$$

Knipp and Uhlenbeck[4] rewrite $S(k)$ as given by (8) in the form

$$S(k)=\int_{1+k}^{W_0}dW_eP(W_e)\Phi(W_e,k) \quad (10)$$

where $P(W_e)$ is given by (11) and

$$\Phi(W_e,k)=\frac{\alpha p_s}{\pi p_ek}\left\{\frac{W_e^2+W_s^2}{W_ep_s}\ln(W_s+p_s)-2\right\}. \quad (11)$$

$\Phi(W_e,k)$ is interpreted as the conditional probability that a β-electron emitted with energy W_e will radiate a γ-quantum of energy k. In fact they show, further, that if one does not average over the direction of the quantum: $\int d\Omega_k$ in (5), then one may factor out a function depending on the angle θ between $\mathbf{k}$ and $\mathbf{p}_s$. This differential conditional probability is given by

$$d\Phi(W_e,k,\theta)=\frac{\alpha p_s}{p_ek}\left[\frac{W_e^2+W_s^2}{W_e(W_s-p_s\cos\theta)}-\frac{1}{(W_s-p_s\cos\theta)^2}-1\right]d\Omega_k \quad (12)$$

and integrates to (11).

Both the differential and integrated probabilities (11) and (12) are precisely the same as they obtained by the alternative first order perturbation method mentioned in Section I.

By comparing the scalar β-interaction with the vector interaction one can easily convince oneself that one gets the same formulas (4)–(12) for an allowed scalar transition if one redefined M by

$$M=\int d\tau V^*\beta QU.$$

B. First Forbidden Transition; Scalar Interaction

For this case the matrix element depends explicitly on the electron and neutrino momenta

$$(l|H_\beta|0)=G/\Omega(A_{s'}{}^*\beta B_\sigma)[-i(\mathbf{p}_{s'}+\mathbf{p}_\sigma)\cdot\mathbf{M}]$$

with

$$\mathbf{M}=\int d\tau V^*\beta U\mathbf{r}.$$

Substituting this matrix element into (1) and performing the summation over the spins and the two directions of the polarization one finds

$$\begin{aligned}S(k)=&\frac{2\alpha G^2}{(2\pi)^7}k\int dp_\sigma p_\sigma{}^2W_sp_s\int d\Omega_\sigma\int d\Omega_s\int d\Omega_k\\&\times\frac{1}{W_\sigma W_s(W_e{}^2-W_{s'}{}^2)^2}\times[(\mathbf{p}_{s'}+\mathbf{p}_\sigma)\cdot\mathbf{M}]^2\\&\times\Bigg\{2W_eW_\sigma\left[W_eW_s-1-\frac{(\mathbf{k}\cdot\mathbf{p}_s)(\mathbf{k}\cdot\mathbf{p}_{s'})}{k^2}\right]\\&-W_\sigma W_s(W_e{}^2-W_{s'}{}^2)+\frac{(\mathbf{k}\cdot\mathbf{p}_\sigma)(\mathbf{k}\cdot\mathbf{p}_s)}{k^2}(W_e{}^2-W_{s'}{}^2)\\&+2(\mathbf{p}_\sigma\cdot\mathbf{p}_{s'})\left[1+\frac{(\mathbf{k}\cdot\mathbf{p}_s)(\mathbf{k}\cdot\mathbf{p}_{s'})}{k^2}-W_eW_s\right]\Bigg\},\end{aligned}\tag{13}$$

where $W_{s'}$ is related to k, p_s, W_e and θ by

$$W_e{}^2-W_{s'}{}^2=2k(W_s-p_s\cos\theta).$$

Remembering also the conservation of momentum: $\mathbf{p}_{s'}=\mathbf{p}_s+\mathbf{k}$, one sees that in the square bracket k enters either scalarly or through the scalar products with $\mathbf{p}_s$ and $\mathbf{p}_\sigma$. We can thus integrate first over-all directions of $\mathbf{k}$ keeping the angles $\widehat{kp_s}$ and $\widehat{kp_\sigma}$ fixed. After this integration and the integrations over all directions of $\mathbf{p}_s$ and $\mathbf{p}_\sigma$, and changing finally the last variable of integration from p_σ to W_e in order to compare the present result with the probability for the beta-decay, one finds

$$\begin{aligned}S(k)=&\frac{\alpha G^2}{4\pi^3}\frac{|M|^2}{3}\frac{1}{k}\int_{1+k}^{W_0}dW_ep_\sigma\\&\times\{[2(p_\sigma{}^2+p_e{}^2)W_\sigma(W_e{}^2+W_s{}^2)\\&-(4/3)p_\sigma{}^2p_e{}^2(W_e+W_s)+4kW_eW_\sigma\\&-(4/3)kp_\sigma{}^2(2-W_\sigma W_s)]\ln(W_s+p_s)\\&+p_s[-4W_eW_\sigma(p_\sigma{}^2+p_e{}^2)+(8/3)p_\sigma{}^2p_e{}^2\\&-4kW_\sigma(W_e{}^2-W_eW_s+W_s{}^2)+(4/3)kW_ep_\sigma{}^2]\}.\end{aligned}\tag{14}$$

The last integral is again elementary but the final expression is lengthy. For large W_0 one has the approximate expression,

$$\begin{aligned}S(k)=&\frac{\alpha G^2|\mathbf{M}|^2}{9\pi^4}\frac{1}{k}\Bigg\{\left[\frac{64}{105}x^7-\frac{31}{15}W_0x^6+\frac{11}{4}W_0{}^2x^5\right.\\&\left.-\frac{7}{4}W_0{}^3x^4+\frac{1}{2}W_0{}^4x^3\right]\ln 2x\\&-\left[\frac{8081}{7350}x^7-\frac{281}{75}W_0x^6+\frac{1189}{240}W_0{}^2x^5\right.\\&\left.-\frac{151}{48}W_0{}^3x^4+\frac{11}{12}W_0{}^4x^3\right]\Bigg\}.\end{aligned}\tag{15}$$

The total γ-energy emitted is to this approximation

$$I_\gamma=\frac{\alpha G^2|\mathbf{M}|^2}{2\pi^4}\frac{1}{315}W_0{}^8(\ln 2W_0-2.30).\tag{16}$$

The probability of β-emission for this case is

$$P_1(W_e)=\frac{G^2|\mathbf{M}|^2}{2\pi^3}\left[\frac{p_\sigma{}^2+p_e{}^2}{3}-\frac{2}{9}\frac{p_\sigma{}^2p_e{}^2}{W_eW_\sigma}\right]\tag{17}$$

from which it follows, to the same approximation as (16), the β-intensity:

$$I_\beta=(G^2|\mathbf{M}|^2/2\pi^3)(1/420)W_0{}^8.$$

The ratio of γ-intensity to β-intensity is

$$I_\gamma/I_\beta=4\alpha/3\pi(\ln 2W_0-2.30).\tag{18}$$

We see that for large W_0, this ratio is almost exactly the same as the ratio (9) calculated for the allowed β-transition.

The exact factorization of the integrand of (14) into a product of $P(W_e)$ and $\Phi(W_e, k)$ is not possible. But if we assume that k is small, so that $W_e\cong W_s$ and all the terms in the curly bracket involving k can be neglected, we can again write,

$$S(k)=\int_{1+k}^{W_0}dW_eP_1(W_e)\Phi(W_e,k),$$

where $\Phi(W_e, k)$ is precisely the same expression as (11), as obtained for the allowed β-transition. One can even show that for small k the factorization holds for the differential probability with the same $d\Phi$ as given by (12).

C. Second Forbidden, Scalar Theory

The matrix element for this case is

$$(l|H_\beta|0)=-(G/\Omega)A_{s'}{}^*\beta B_\sigma[\tfrac{1}{4}(p_{s'}+p_\sigma)_\alpha(p_{s'}+p_\sigma)_\beta M_{\alpha\beta}],$$

where now M is a symmetrical tensor of the second rank given by

$$M_{ij}=\int d\tau V^*\beta QUr_ir_j.$$

M_{ij} can be separated into two invariant parts,

$$M_{ij}=\tfrac{1}{3}M_{\alpha\alpha}\delta_{ij}+(M_{ij}-\tfrac{1}{3}M_{\alpha\alpha}\delta_{ij}).$$

The first term gives merely a small correction to the allowed transition. Only the second part gives a contribution to the second forbidden transition. We shall denote this spur zero tensor by R_{ij}. The calculation goes as before, and one obtains

$$\begin{aligned}S(k)=&\frac{\alpha G^2}{8\pi^4}\frac{2}{15}|R_{\alpha\beta}|^2\int dW_e p_\sigma\\ &\times\{[W_\sigma(W_e{}^2+W_s{}^2)(p_\sigma{}^2+p_e{}^2)^2\\ &+(4/3)W_\sigma p_\sigma{}^2p_e{}^2(W_e{}^2+W_s{}^2)\\ &-(4/3)p_\sigma{}^2p_e{}^2(p_\sigma{}^2+p_e{}^2)(W_e+W_s)\\ &+4kW_eW_\sigma((5/3)p_\sigma{}^2+p_e{}^2)\\ &+(4/3)kp_\sigma{}^2W_eW_s(p_\sigma{}^2+p_e{}^2)\\ &-(8/3)kp_\sigma{}^2(p_\sigma{}^2+2p_e{}^2)]\ln(W_s+p_s)\\ &+p_s[-2W_eW_\sigma(p_\sigma{}^2+p_e{}^2)^2-(8/3)W_eW_\sigma p_\sigma{}^2p_e{}^2\\ &+(8/3)p_\sigma{}^2p_e{}^2(p_\sigma{}^2+p_e{}^2)\\ &+(4/3)kp_\sigma{}^2(p_\sigma{}^2+3p_e{}^2)W_e\\ &-4kW_\sigma(W_e{}^2-W_eW_s+W_s{}^2)((5/3)p_\sigma{}^2+p_e{}^2)\\ &-(8/3)k^2W_eW_\sigma-(16/3)k^2p_\sigma{}^2W_eW_s\\ &+(40/9)k^2p_\sigma{}^2+4k^2W_\sigma W_s(W_e{}^2-W_eW_s+W_s{}^2)\\ &-(4k^2/3)W_eW_\sigma W_s{}^2+(8/9)k^2p_\sigma{}^2W_s{}^2]\}.\end{aligned}\tag{19}$$

Making the same approximation $W_0>1$, and integrating

$$\begin{aligned}S(k)=&\frac{\alpha G^2}{8\pi^4}\frac{2}{15}|R_{\alpha\beta}|^2\frac{1}{k}\bigg\{\bigg[\frac{58}{63}x^9-\frac{86}{21}W_0x^8+\frac{488}{63}W_0{}^2x^7\\ &-\frac{364}{45}W_0{}^3x^6+\frac{151}{30}W_0{}^4x^5-\frac{11}{6}W_0{}^5x^4\\ &+\frac{1}{3}W_0{}^6x^3\bigg]\ln 2x-\bigg[\frac{151553}{79380}x^9-\frac{72661}{8820}W_0x^8\\ &+\frac{33142}{2205}W_0{}^2x^7-\frac{1138}{75}W_0{}^3x^6+\frac{16577}{1800}W_0{}^4x^5\\ &-\frac{239}{72}W_0{}^5x^4+\frac{11}{18}W_0{}^6x^3\bigg]\bigg\}.\end{aligned}\tag{20}$$

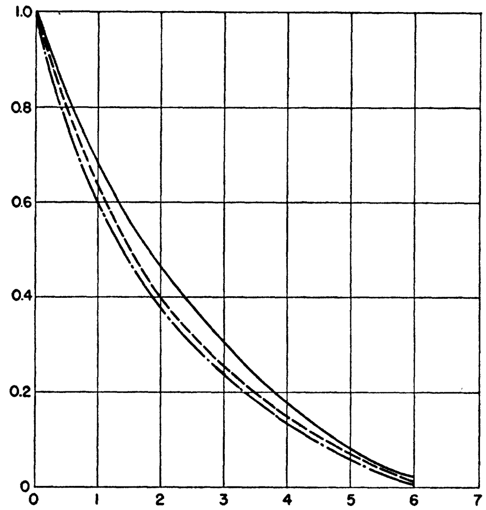

FIG. 2. Energy distribution of the γ-rays following β-transitions of different degree of forbiddenness, scalar theory. $W_0=10$. Ordinate $kS(k)$ normalized to unity for $k=0$, abscissa k measured in relativistic units.
——— Allowed.
— — — — First forbidden.
–·–·– Second forbidden.

The total gamma-intensity is to this approximation

$$I_\gamma=\frac{\alpha G^2|R_{\alpha\beta}|^2}{2\pi^4}\frac{1}{90\cdot 63}W_0{}^{10}(\ln 2W_0-2.35).\tag{21}$$

The corresponding β-emission probability and the total β-intensity are given respectively by

$$P_2(W_e)=\frac{G^2|R_{\alpha\beta}|^2}{2\pi^3}p_eW_ep_\sigma{}^2\bigg[\frac{1}{30}(p_e{}^4+p_\sigma{}^4)+\frac{1}{9}p_e{}^2p_\sigma{}^2-\frac{2}{45}(p_e{}^2+p_\sigma{}^2)\frac{p_e{}^2p_\sigma{}^2}{W_eW_\sigma}\bigg]\tag{22}$$

and

$$I_\beta=\frac{G^2}{2\pi^3}|R_{\alpha\beta}|^2\cdot\frac{1}{90\cdot 84}W_0{}^{10}.\tag{23}$$

The ratio

$$I_\gamma/I_\beta=4\alpha/3\pi(\ln 2W_0-2.35)\tag{24}$$

is again very nearly the same as (9) and (18).

As in the previous section the factorization of the integrand in Eq. (19) is not possible. By making the same assumption of k small, one gets exactly $P_s(W_e)\Phi(W_e,k)$ with the same function Φ as before.

D. Other Interactions

To investigate the effect of different interactions we have also made the calculations with the tensor interaction. For allowed transitions one gets exactly the same result as in A. The results for first forbidden transitions are much more complicated, since more than one nuclear matrix element enters in the final expression, and we will not reproduce them here. We have shown, that for small k the final result can again be factorized as in Eq. (3) where the $P(W_e)$ is the complete first forbidden energy distribution containing *all* the nucleon matrix elements and Φ is again given by (11). It seems therefore certain that the same factorization will hold in all other cases.

III. DISCUSSION

In Fig. 1 are plotted the γ intensity $kS(k)$ *vs.* k for the allowed scalar interaction using both the exact expression (5) and the approximate one (5a) for $W_0=10$, corresponding to an upper limit of 5 Mev for the β-spectrum. It is seen that the difference is negligible over the whole energy range. Accordingly, we confine our discussion of the behavior of the γ- and β-intensities, I_γ and I_β, to this simplifying approximation.

To compare the shapes of the γ-intensity curves for the different degrees of forbiddenness we have plotted $kS(k)$ *vs.* k for the scalar interaction: allowed, first, and second forbidden in Fig. 2. The ordinates have been adjusted to coincide for $k=0$. It is clear that one could not hope to distinguish degrees of forbiddenness on account of the almost identical shapes of these curves.

The absolute γ-intensities do increase with successive degrees of forbiddenness. However, a striking feature of our results is that the ratio of total γ-intensity to total β-intensity is essentially independent of the degree of forbiddenness of the β-transition and is given by

$$I_\gamma/I_\beta=4\alpha/3\pi(\ln 2W_0-2.3). \tag{25}$$

This is in spite of the fact that the β-intensities $(W-1)P_L(W)$ vary strongly both in shape and magnitude for these same transitions. ($P_L(W)$ denotes the β-distribution for Lth forbidden transition.)

This behavior is not difficult to explain if one grants the factorization of $S(k)$ as given in Eq. (3) to be valid for forbidden as well as allowed β-transitions. For then the logarithmic energy dependence of $\Phi(W_e, k)$ will smooth out the variations in $P_L(W_e)$, and one can show easily that one will get for the ratio I_γ/I_β for large W_0 the value $(4\alpha/3\pi)\log(2W_0)$ for *all* values of L.

It is important to note that while the assumption of factorizability of $S(k)$ in the above form is sufficient to explain the uniformity of the ratio I_γ/I_β, this assumption has not been made in obtaining the formulae for I_γ/I_β. These were all obtained from the second order perturbation method (and evaluated for W_0 large). However, we have shown in Section II that the factorization is valid provided only that k is small compared to W_0 and W_e. But by inspection of the curves of Fig. 2, one sees that this restriction is *a posteriori* satisfied since most of the radiation intensity occurs at the low energy end of the spectrum and goes rapidly to zero as k approaches W_0. Hence the alternative method of Knipp and Uhlenbeck appears justified whatever the β-interaction, and it is clear why the experiments have been in such good agreement with the allowed theory of KUB.

The fact that the rather complicated calculations with different interactions all lead to the same result (25) suggests a simple underlying physical as well as formal reason. We now show that in the same approximation in which the method of Knipp and Uhlenbeck is valid, the function $\Phi(W_ek)$ may be obtained purely classically thus further corroborating their interpretation as to the independence of the β-decay and γ-emission processes.

We consider the following model for the γ-emission: An electron, having been created during the nuclear transformation, is ejected instantaneously with velocity v at time $t=0$, and since $Z=0$, moves with uniform velocity v ever after. Classically, of course, one cannot account for the creation of the electron or how it got its energy. However, once given this kinematical description of the electron's motion following the nuclear β-decay, one may ask: What is the spectral distribution of the energy radiated by the electron? This is a straightforward problem for classical electromagnetic theory, and we sketch its solution.

Starting with an arbitrary current distribution $\mathbf{J}(\mathbf{r}, t)$ having Fourier transform:

$$\mathbf{J}_\omega(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\mathbf{J}(\mathbf{r}, t)e^{-i\omega t}$$

the transform of the vector potential in the radiation field will be given (in Gaussian units) by

$$\mathbf{A}_\omega(\mathbf{r})=\frac{e^{-(i\omega/c)r}}{cr}\int d\mathbf{r}'\mathbf{J}_\omega(\mathbf{r}')\exp[-(i\omega/c)\mathbf{n}\cdot\mathbf{r}'],$$

where the integral is taken over the coordinates $\mathbf{r}'$ of the current distribution and r is the distance from the source to the observation point in the wave zone; $\mathbf{n}$ is a unit vector in the direction $\mathbf{r}$.

For the total energy radiated in the angular frequency range $d\omega$ and in solid angle $d\Omega$ in the direction $\mathbf{n}$, one readily obtains

$$dW_\omega(\theta,\varphi)=\frac{\omega^2}{c^3}\left[\mathbf{n}\times\int\mathbf{J}_\omega(\mathbf{r}')\exp[(i\omega/c)\mathbf{n}\cdot\mathbf{r}']d\mathbf{r}'\right]^2 \tag{26}$$

For our case, the current is given by[9]

$$\begin{aligned}\mathbf{J}(\mathbf{r}, t)&=\mathbf{i}_z ev\delta(x)\delta(y)\delta(z-vt) && \text{for } t>0\\ &=0 && \text{for } t\leqslant 0,\end{aligned}$$

where $\mathbf{i}_z$ is a unit vector in the z-direction, taken as the direction of emission of the β-electron, and the δ's are

[9] I. Tamm, J. of Phys. (U.S.S.R.) 1, 439 (1939).

Dirac δ-functions. Then

$$\mathbf{J}_\omega(\mathbf{r})=\mathbf{i}_z\frac{e}{2\pi}\delta(x)\delta(y)e^{-(i\omega z/v)},\quad z>0$$

Inserting this in the bracket of Eq. (26), one gets a factor $\sin\theta$ from $\mathbf{n}\times\mathbf{i}_z$, and the integral becomes

$$\begin{aligned}\int d\mathbf{r}'\mathbf{J}_\omega(\mathbf{r}')\exp[(i\omega/c)\mathbf{n}\cdot\mathbf{r}']&=\mathbf{i}_z\frac{e}{2\pi}\int_0^\infty dze^{-(i\omega/v)(1-\beta\cos\theta)z}\\&=\mathbf{i}_z\frac{e}{2\pi}\frac{v}{i\omega(1-\beta\cos\theta)}\end{aligned}$$

so that the spectral distribution, differential in angle is

$$dW_\omega(\theta,\varphi)=\frac{e^2v^2}{4\pi^2c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^2}d\Omega.$$

Dividing by $\hbar\omega$, we may interpret

$$d\Phi(v,\theta,\omega)\equiv dW_\omega(\theta,\phi)/\hbar\omega$$

as the probability for the electron with velocity v to emit a quantum of energy of frequency ω at an angle θ. Thus

$$\begin{aligned}d\Phi(v,\theta,\omega)&=\frac{e^2v^2}{2\pi\hbar\omega c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^2}\sin\theta d\theta\\&=\frac{\alpha}{2\pi\omega}\frac{\beta^2\sin^3\theta}{(1-\beta\cos\theta)^2}d\theta\end{aligned}\qquad(27)$$

where $\alpha=e^2/\hbar c$ and $\beta=v/c$. Integrating over θ, one gets

$$\Phi(v,\omega)=\frac{\alpha}{\pi\omega}\left[\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}-2\right].\qquad(28)$$

To see that the expressions (27) and (28) are the proper classical analogues of the quantum mechanically derived formulas (11) and (12), we need only remark that if one sets $W_e=W_s$, $P_e=P_s$ in (11) and (12), which is required by the assumption k small, and take into account the relations for a relativistic electron in relativistic units

$$P/W=\beta\quad\text{and}\quad W^2=1/(1-\beta^2)$$

then they become *exactly* the classical formulas (27) and (28) above.

The total energy radiated, gotten by integrating (28) over all frequencies will be infinite. But this is due to having assumed an instantaneous change in velocity from 0 to v. In reality, it will take a finite time τ for the acceleration of the electron to its final velocity v, and then the spectrum will go to zero for $\omega>2\pi/\tau$. This time must be of the order $\hbar/mc^2$ in order to correspond to the observed γ-spectrum and, of course, has nothing to do with the β-lifetime.

We remark finally on the angular distribution of the emitted γ-quanta. From (27) it is seen that for the fast electrons, $\beta\sim1$, almost all the radiation is in the forward direction in which the β-particle was emitted, while for small electron velocities, (27) reduces to a $\sin^2\theta$ distribution, as for a classical dipole. One can therefore expect an angular correlation between the β-rays and γ-quanta.

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