Straggling of Electrons near the Critical Energy

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The straggling of electrons caused by loss of energy by radiation and collision is discussed. The paper is essentially in three parts. In Section III we have found the "straggling probability" for an electron that loses energy only by radiation, and give curves corresponding to various approximations to the expression for the radiative cross section. In Section IV collision loss is also taken into account and a formal expression for the straggling probability is found in the form of a series whose terms decrease rapidly for energies not too small and not too large thickness. No proof of the convergence is given however and, in fact, for very small energies the terms increase and some higher order terms become infinite at zero energy. In Section V some remarks are made on the small "showers" initiated by a particle of energy near the critical energy.

INTRODUCTION

We shall be mainly concerned in this paper with the behavior of an electron with energy of order of the critical energy, when incident on matter. Such an electron, of energy E_0, cannot itself penetrate a distance t greater than E_0/β, where β is the critical energy.[1] For E_0 less than β, t is automatically less than one. Of course, an electron can make itself felt at larger distances than E_0/β by pair-production by its radiated quanta, i.e., by producing a small "shower," but at small thicknesses and for a considerable range of energy the main contribution to such a "shower" will be due to the straggling of the original electron. It is the main burden of this paper to calculate this straggling, although some remarks on the effect of photons are made in the last section.

It will appear that our solution can be used only for thicknesses which are not too large and energies not too far from the initial energy. The restriction on thickness is not too serious, since we have seen that for an initial energy equal to the critical energy, one is never interested in the original electron for thicknesses greater than unity. Also, the restriction that the energy be close enough to the initial energy has one redeeming feature; for this case we can to some extent take into account the variation of the radiation cross section with energy by using an expression appropriate to some average of the initial energy and the energy considered.

There has been at least one other attempt to calculate the small showers initiated by an electron of order of the critical energy. This is due to Bhabha and Chakrabarty,[2] who claim to have found a solution of the general shower equations, including the effect of collision loss. As a special case of their results they derive an expression for the energy spectrum at small thicknesses, which purports to hold for incident particles with energy of the order of the critical energy. Their expression is in the form of a series in t in which terms of order t^3 and higher are dropped as negligible. For $t=0.1$ and some energy values, however, the term in t^2 is as much as 300 times as large as the linear term;[3] the dropping of higher order terms is thus quite suspect, and the expressions they give for small thickness cannot be considered as a correct solution.

The present paper is essentially in three parts. In Section III we have discussed the "straggling probability" for electrons which lose energy only by radiation, and investigated how this probability depends on the expression for the radiative cross section. In the next section we have extended this to the case where electrons lose energy by constant collision loss as well. Finally, in Section V we have considered the effect of photons and pair production.

II. THE DIFFUSION EQUATION

Let $\pi(E, t)dE$ be the probability that an electron which loses energy by radiation and by collision loss has energy in the range E to $E+dE$ at thickness t. Then $\pi(E, t)$ satisfies the well-known diffusion equation[4]

$$\frac{\partial\pi(E,t)}{\partial t}=-\int_0^1\left[\pi(E,t)-\frac{1}{1-v}\pi\left(\frac{E}{1-v},t\right)\right]\varphi(v)dv-\beta\frac{\partial\pi(E,t)}{\partial E}. \quad (1)$$

Here $\varphi(v)dvdt$ is the probability that in the thickness dt the electron emits a photon which has a fraction between v and $v+dv$ of the electrons energy. The first term on the right-hand side of (1) describes the decrease in $\pi(E, t)$ due to electrons initially in the interval (E, dE) which leave it by radiation, the second describes the increase in $\pi(E, t)$ due to electrons of energy greater than E which enter the interval (E, dE) by radiation, and the last term takes account of collision loss.

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[1] We measure lengths in radiation lengths, as usual, and take β, the critical energy, to be also the (constant) energy loss per radiation length.

[2] H. J. Bhabha and S. K. Chakrabarty, Proc. Roy. Soc. **A181**, 267 (1943) and Phys. Rev. **74**, 1352 (1948).

[3] This was pointed out to me by Professor H. A. Bethe. See Table IX, p. 299 in the first paper of reference 2.

[4] See B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

If we apply the Mellin transform

$$M(s, t)=\int_0^\infty E^s\pi(E, t)dE \tag{2}$$

with the inverse transform

$$\pi(E, t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} E^{-(s+1)}M(s, t)ds \tag{3}$$

to (1), we are led to the equation

$$\partial M(s, t)/\partial t=-A(s)M(s, t)-\beta sM(s-1, t), \tag{4}$$

where

$$A(s)=\int_0^1 [1-(1-v)^s]\varphi(v)dv. \tag{5}$$

The contour in (3) should lie to the right of all singularities of the integrand.

III. STRAGGLING WITHOUT COLLISION LOSS

We will first consider an electron that loses energy only by radiation, and add a subscript zero to the function $\pi(E, t)$ to denote this. Then $\pi_0(E, t)$ satisfies Eq. (1) with β set equal to zero, and the corresponding equation for the transform $M(s, t)$ is

$$\partial M(s, t)/\partial t=-A(s)M(s, t). \tag{6}$$

The solution of (6) which corresponds to the boundary condition of one electron of energy E_0 incident at $t=0$ is

$$M(s, t)=E_0^s e^{-A(s)t}. \tag{7}$$

The straggling function is then given by:

$$\pi_0(E_0, E, t)dE=\frac{dE}{E_0 2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{E_0}{E}\right)^{s+1} e^{-A(s)t}ds. \tag{8}$$

The integral probability $\Pi_0(E, t)$ i.e., the probability that the electron has energy *greater* than E at t is

$$\Pi_0(E_0, E, t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{E_0}{E}\right)^{s}\frac{e^{-A(s)t}}{s}ds. \tag{9}$$

It is convenient to introduce the new variable $y=\ln(E_0/E)$ and write:

$$\pi_0(y, t)e^{-y}dy=\frac{e^{-y}}{2\pi i}dy\int_{\delta-i\infty}^{\delta+i\infty} e^{ys-A(s)t}ds, \tag{10}$$

$$\Pi_0(y, t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{e^{ys-A(s)t}}{s}ds, \tag{11}$$

We will carry out calculations for three different expressions for $\varphi(v)$, which are good approximations in different energy ranges. They are:

$$\varphi_1(v)=1/v, \tag{12a}$$

$$\varphi_2(v)=-b[(1-v)^a/\ln(1-v)], \tag{12b}$$

$$\varphi_3(v)=(4/3)(1-v)/v. \tag{12c}$$

In $\varphi_2(v)$ the constants a and b are to be considered as parameters which can be chosen to fit the correct functions at various energies. In Fig. 1 are plotted the approximate expressions given above, along with the correct expressions for various energies in air.

The expressions for the functions $A(s)$ corresponding to the three cases (12) are:

$$A_1(s)=\Psi(s)+C, \tag{13a}$$

$$A_2(s)=b\ln[1+(s/1+a)], \tag{13b}$$

$$A_3(s)=-(4/3)+(4/3)(\Psi(s+1)+C), \tag{13c}$$

where $\Psi(s)$ is the logarithmic derivative of the factorial functions i.e.,

$$\Psi(s)=d/ds(\ln s!),$$

and $C=0.5772\cdots$ is the Euler-Mascheroni constant.

Bethe and Heitler have shown[5] by other reasoning that the straggling function corresponding to the expression for φ_2 given above, and in particular for the special case $a=0$, is

$$\pi_{02}(E_0, E, t)dE=\frac{dE}{E_0}\frac{[\ln(E_0/E)]^{bt-1}}{(bt-1)!}. \tag{14}$$

It is easy to derive this result. If we use Eqs. (9) and (13b) we get

$$\pi_{02}(E_0, E, t)dE=\frac{(1+a)^{bt}}{2\pi i}\frac{dE}{E_0}\int_{\delta-i\infty}^{\delta+i\infty}\frac{e^{y(s+1)}}{(s+a+1)^{bt}}. \tag{15}$$

Completing the contour in (15) by an infinite semi-circle in the left half plane and evaluating the residue at the singularity $s=-(1+a)$ gives

$$\pi_{02}(E_0, E, t)dE=\frac{dE}{E_0}(1+a)^{bt}\left(\frac{E}{E_0}\right)^a\frac{[\ln(E_0/E)]^{bt-1}}{(bt-1)!} \tag{16}$$

which for $a=0$ is just the Bethe-Heitler result.

The expressions for π_{01} and π_{03} are got from (10) and (13) and are,

$$\pi_{01}(y, t)e^{-y}dy=\frac{dye^{-Ct}}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{sy-t\Psi(s)}\cdot ds, \tag{17a}$$

$$\pi_{03}(y, t)e^{-y}dy=\frac{dy}{2\pi i}e^{(1-C)t-y}\int e^{sy-(4/3)\Psi(s)t}ds. \tag{17b}$$

We have changed the variable of integration from $s+1$ to s in the last integral. The integrals in these two ex-

[5] H. Bethe and W. Heitler, Proc. Roy. Soc. **A146**, 83 (1934).

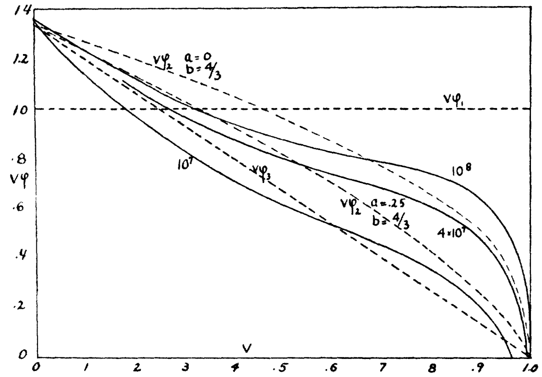

FIG. 1. The approximate expressions for the radiative cross section given by Eqs. (12) are plotted as dotted lines and the correct expressions for various energies in air as solid lines.

pressions are of exactly the same form and since for all $y>0$ and $t>0$, $sy-\Psi(s)t$ has a more or less sharp minimum on the real axis they can be evaluated by the saddle point method. One gets

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{sy-t\Psi(s)}ds \approx \frac{e^{s_0 y - t\Psi(s_0)}}{(-2\pi\Psi''(s_0)t)^{\frac{1}{2}}}, \tag{18}$$

where s_0 is defined by

$$[d\Psi(s)/ds]|_{s=s_0}=y/t.$$

Curves for various t are plotted in Fig. 2.[6] Before discussing them we will discuss briefly the accuracy of the saddle-point method.

There are two indirect checks on this accuracy. If one evaluates $\pi_{02}(y, t)$ by the saddle-point method it is easy to show that the result is equivalent to replacing $(bt-1)!$ in the exact expression (14) by the first term in its Stirling approximation, i.e., by the expression $(2\pi)^{\frac{1}{2}}e^{-bt}(bt)^{\frac{1}{2}-bt}$. For $bt=2$, 1, 0.5, 0.25, the Stirling approximation gives too low a value by 4, 8, 16, 31 percent, respectively. Thus the saddle-point method fails for small t, which is not surprizing, since for $t=0$ there is no saddle-point at all. Since the behavior of the integrand in all three expressions for $\pi_0(y, t)$ is rather similar, although the exact functional form is different, one would suspect that if the saddle-point method is adequate for $\pi_{02}(y, t)$ it will be adequate for the other expressions. Moreover, one should have from the normalization condition

$$\int_0^\infty e^{-y}\pi_0(y, t)dy=1$$

and for the larger values of t this can be seen to be true to a few percent from Fig. 2.

Even though for given y, the saddle-point method for the differential spectrum becomes less accurate for small t, the integral spectrum defined by (11b) is quite accurate for all t. We have

$$\Pi_0(y, t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\exp[sy-A(s)t-\ln s].$$

Evaluation by the saddle point method gives

$$\Pi_0(y, t)\approx\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{\exp[s_0y-A(s_0)t-\ln s_0]}{(-A''(s_0)t+(1/s_0)^2)^{\frac{1}{2}}}, \tag{19}$$

where $y-A'(s_0)t-(1/s_0)=0$.

We should have $\Pi_0(y, 0)=1$. Putting $t=0$ in these equations one gets

$$\Pi_0(y, 0)=e/(2\pi)^{\frac{1}{2}}\approx 1.08. \tag{20}$$

Thus there is only 8 percent inaccuracy even for $t=0$. For larger t the saddle-point method should be even more accurate, for the reasons given above in connection with the differential spectrum.

Let us now consider the differential spectrum plotted in Fig. 2. We see that for small y, i.e., energies close enough to the initial energy, π_{03} is considerably greater than π_{01} or π_{02}. This is simply a reflection of the fact that $\varphi_3(v)$ drops off much more quickly for large v than $\varphi_1(v)$ or $\varphi_2(v)$ and hence gives more weight to small energy losses. For large y, $e^{-y}\pi_0$ is always small, of course; moreover the values given by the three expressions differ widely from one another. This is because large y corresponds to an electron which has low energy and therefore probably emitted a quantum containing an appreciable fraction of its energy. But we see from Fig. 1 that for this case, i.e., for v close to unity, the expressions for the radiation probability differ widely, hence it is not surprising that the expressions in Fig. 2 should also differ widely for large y. The integral straggling function is plotted in Fig. 3.

IV. STRAGGLING WITH COLLISION LOSS

In this section we will consider the straggling of an electron which loses energy both by radiation and ionization. We denote by $\pi_\beta(E, t)$ and $\Pi_\beta(E, t)$ the differential and integral straggling functions. $\pi_\beta(E, t)$, and its Mellin transform, which we call $M_\beta(s, t)$, satisfy Eqs. (1) and (2). The boundary condition for one electron of energy E_0 at $t=0$ is:

$$M_\beta(s, 0)=E_0^s. \tag{21}$$

If we make the substitution

$$M_\beta(s, t)=e^{-A(s)t}N(s, t), \tag{22}$$

Eq. (4) becomes:

$$e^{D(s)t}\frac{\partial N(s, t)}{\partial t}=-\beta sN(s-1, t), \tag{23}$$

where $D(s)=A(s-1)-A(s)$. The expressions for the

[6] For making these calculations a useful table of the factorial function and its derivatives is: Tracts for Computers No. 1, Eleanor Pairman, *Tables of the Digamma and Trigamma Functions* (Cambridge University Press, London, 1919).

functions $D(s)$ corresponding to the expressions (13) for the functions $A(s)$ are:

$$D_1(s)=-1/s, \tag{24a}$$

$$D_2(s)=b\ln(s+a/s+a+1), \tag{24b}$$

$$D_3(s)=-4/3(s+1). \tag{24c}$$

We will be interested mainly in not very large t, as was stated in the introduction. Now if we put $t=0$ in the exponent of (23), the solution of the resulting equation, remembering the boundary condition, which now reads $N(s,0)=E_0{}^s$, is just $(E_0-\beta t)^s$. For non-vanishing t we therefore look for a solution of (23) in the form

$$N(s,t)=(E_0-\beta t)^s\cdot P(s,t), \tag{25}$$

where $P(s,t)$ is a power series in t with coefficients functions of s, i.e.,

$$P(s,t)=\sum_{n=0}^{\infty}c_n(s)t^n, \tag{26}$$

in which we must have $c_0(s)=1$ to satisfy the boundary condition. If we put (25) into (23) we get for $P(s,t)$

$$\left(\frac{E_0}{\beta}-t\right)\frac{\partial P(s,t)}{\partial t}-sP(s,t)=-sP(s-1,t)e^{-D(s)t}. \tag{27}$$

If we now put (26) into (27) and successively equate the coefficients of powers of t on either side of (27), we get for coefficients up to $c_4(s)$

$$c_1(s)=0, \tag{28a}$$

$$c_2(s)=(\beta/2E_0)sD(s), \tag{28b}$$

$$c_3(s)=(\beta^2/E_0{}^2 3!)[(s+2)D(s)-(s-1)D(s-1)] -(\beta/E_0 3!)sD^2(s), \tag{28c}$$

$$c_4(s)=\frac{\beta}{E_0}\frac{s+3}{4}c_3(s)+\frac{\beta}{E_0}\frac{s}{4}\left(\frac{D^3(s)}{3!}+D(s)c_2(s-1) -c_3(s-1)\right), \tag{28d}$$

In the last equation we have, for simplicity, expressed $c_4(s)$ in terms of $c_3(s)$ instead of expressing it directly in terms of $D(s)$.

If we reassemble $M_\beta(s,t)$ using (22) and (25) we get the general expression for π_β

$$\pi_\beta(E_0,E,t)dE=\frac{dE}{E_0-\beta t}\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}e^{-A(s)t} \times\left(\frac{E_0-\beta t}{E}\right)^{s+1}\left(\sum_{n=0}^{\infty}c_n(s)t^n\right)ds. \tag{29}$$

We will see later that the series under the integral sign converges quite well for small t and E close enough to E_0; under these conditions the term in t^2 is small. If we now compare (29) with (8) and remember that $c_0(s)=1$, we see that the first and *dominant term for $\pi_\beta(E,t)$ is just the spectrum for straggling without collision loss in which E_0 is replaced by $E_0-\beta t$.* Also, the next term is of order t^2 since $c_1(s)$ is zero. This result holds for any $\varphi(v)$.

In calculating higher order terms the coefficients $c_n(s)$ depend of course on the expression $\varphi(v)$ chosen for the radiation cross section. Because of the rather simpler form of $D(s)$ for the functions $\varphi_1(v)$ and $\varphi_3(v)$, the coefficients $c_n(s)$ for these two functions are also simpler than for the function $\varphi_2(v)$, and we shall confine our attention from now on to the two former functions. For thicknesses and energies for which the higher order terms are not only small, but also negligible, the straggling function corresponding to $\varphi_2(v)$ may be quite useful, however, since it is an explicit formula, and the two adjustable coefficients in $\varphi_2(v)$ enable one to fit the correct cross sections fairly well over a considerable range of energies.

From Fig. 1 we see that $\varphi_1(v)$ is a rough approximation to the correct expressions for the radiation cross section near the critical energy in the lightest elements, i.e., around 10^8 ev and $\varphi_3(v)$ a fair approximation near the critical energy in the heaviest elements, i.e., around 10^7 ev. The first few coefficients are, remembering $c_0(s)=1$ and $c_1(s)=0$

$\varphi_1(v)$:

$$c_2(s)=-\beta/2E_0, \tag{30a}$$

$$c_3(s)=-(\beta^2/3E_0{}^2)-(\beta/3!E_0s), \tag{30b}$$

$$c_4(s)=-\frac{\beta^3}{4E_0{}^3}+\frac{\beta^2}{12E_0{}^2} +\frac{\beta^2}{4!E_0{}^2}\left(\frac{s}{s-1}-\frac{3}{s}\right)-\frac{\beta}{4!E_0s^2}. \tag{30c}$$

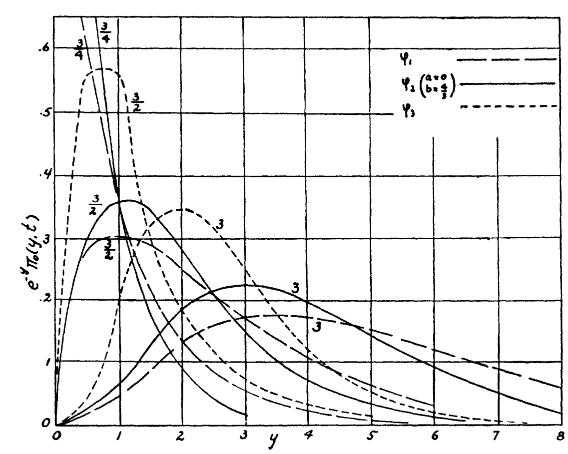

FIG. 2. Differential straggling functions for an electron that loses energy by radiation only. The numbers attached to the curves are thickness in radiation units. $y=\ln(E_0/E)$.

$\varphi_3(v)$:

$$c_2(s) = -(2\beta/3E_0)(s/s+1), \tag{31a}$$

$$c_3(s) = -\frac{2}{9}\frac{\beta^2}{E_0^2}\left(1+\frac{s}{s+1}\right)-\frac{8}{27}\frac{\beta}{E_0}\frac{s}{(s+1)^2}. \tag{31b}$$

Let us consider $\varphi_1(v)$. Denoting the *integral* straggling function for this case by $\Pi_{\beta 1}(E_0, E, t)$, introducing the new variable z,

$$z = \ln(E_0 - \beta t/E), \tag{32}$$

and using the expressions (30) we see that up to terms in t^3 we have:

$$\Pi_{\beta 1}(z, E_0, t) = \Pi_{01}(z, t)\left(1-\frac{\beta t^2}{2E_0}-\frac{\beta^2 t^3}{3E_0^2}\right) - \frac{t^3\beta}{3!E_0}\int_0^z \Pi_{01}(z, t)dz. \tag{33}$$

The integral in the last term in (33) is just an alternate way of writing

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\exp[zs-A(s)t]}{s^2}ds.$$

Since a particle can be found at thickness t only if $E_0 > \beta t$, we see that the terms in parentheses in (33) decrease quite rapidly for t not too large. It is the integral in the last term in (33) (and similar integrals in the expressions for the higher order $c_n(s)$) that, as stated previously, limit our solution to energies not too far from the initial energy. This can be seen if we remember that for large z, i.e., small E, $\Pi_{01}(z, t)$ is constant, so that the integral diverges. For values of E for which the integral is small compared with the dominant term, we assume that (33) is a correct approximate expression.

This limitation on the solution can be seen in another way. From Eq. (2) we see that $M(s, t)$ is the s'th energy moment of the straggling function, i.e., $M(1, t)$ is the mean energy, $M(2, t)$ the mean square energy, etc. $M(0, t)$ is the number of particles, i.e., the total probability for a particle of any energy to be present at t. Even for $t < E_0/\beta$ this will be less than one since a particle can e.g., lose a large fraction of its energy in emitting a hard quantum, and the remaining small fraction by collision loss after a short distance, and so disappear.

If we put $s=0$ into the expression for $M(s, t)$, we get for terms up to t^3

$$M(0, t) = 1 - \frac{\beta t^2}{2E_0} - \frac{t^3}{3}\frac{\beta^2}{E_0^2} - \frac{\beta t^3}{6E_0 s}\bigg|_{s=0}.$$

The last term is infinite so that one can attach no meaning to this expression.* Similarly, $c_4(s)$ contains a term $1/(s-1)$ so that $M(1, t)$ is also infinite, and in general the higher moments will diverge due to singularities in the higher coefficients $c_n(s)$. From the argument above we see, however, that these divergences really come from energies near zero, so that if we use our distribution function for energies sufficiently greater than zero, it converges and is probably correct. We would like to emphasize, however, that we have not provided a formal proof of this.

To illustrate the above points we have plotted in Fig. 4 the effect of contributions of various powers of t to the total function for $t=1$ and $E_0 = 2\beta$. One can see that for large z the higher order terms become increasingly important. There is however a considerable range of energy of physical interest for which the terms up to t^3 or even t^2 give a good approximation to the total result.

The remarks made above concerning the function $\Pi_{\beta 1}$ apply equally to the function $\Pi_{\beta 3}$ corresponding to the radiation cross section $\varphi_3(v)$. The integrals involved can all be evaluated by the saddle-point method. Since one can see directly the essential difference in the numerical results for $\Pi_{\beta 1}$ and $\Pi_{\beta 3}$ by comparing their

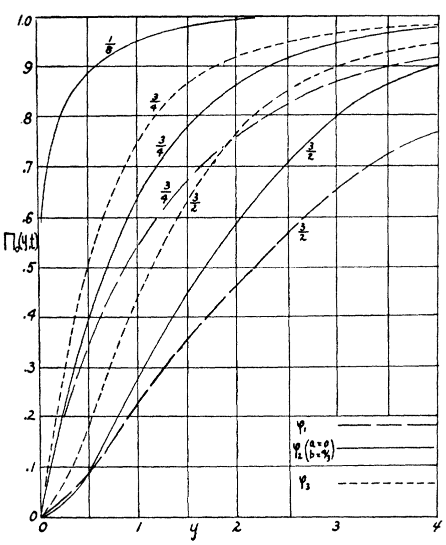

FIG. 3. Integral straggling functions for an electron that loses energy by radiation only. The numbers attached to the curves are thicknesses in radiation units. $y = \ln(E_0/E)$.

* It may be however that the quadratic term is correct, since one can derive it independently in the following way, due to Professor Peierls. If we integrate Eq. (19) with respect to E from 0 to E_0 one gets (using $\varphi_1(v)$),

$$\partial\Pi_{\beta 1}(0, t)/\partial t = -\beta\pi_{\beta 1}(0, t).$$

Now for small t we can get $\pi_{\beta 1}(0, t)$ by putting the value for $t=0$, i.e., $\delta(E_0 - E)$, in the right-hand side of (19). This gives

$$\frac{\partial\pi_{\beta 1}(E, t)}{\partial t} = \int_0^1 \frac{1}{1-v}\delta\left(E_0 - \frac{E}{1-v}\right)\frac{dv}{v} = \frac{1}{E_0 - E}.$$

Therefore, $\pi_{\beta 1}(0, t) = t/E_0$. Thus, using the first equation

$$\Pi_{\beta 1}(0, t) \approx 1 - (\beta t^2/2E_0).$$

dominant terms (see Figs. 2 and 3) we have not computed $\Pi_{\beta 3}$ numerically any further.

V. DISCUSSION

The straggling function derived in the last section does not describe completely what happens when a low energy electron falls on matter, since it neglects the electron pairs produced by the radiated photons, as well as the Compton electrons. We shall not treat the latter here, but discuss briefly the effect of pair-production.

One can, of course, supplement the diffusion equation (1) by terms that describe pair-production, thus getting the ordinary shower equations, and again apply Mellin transforms, but then a solution for the transformed equations of the form found in the last section does not seem possible. This form is

$$M(s,t)=e^{-A(s)t}(E_0-\beta t)^s P(s,t), \qquad (34)$$

where $P(s,t)$ is a power series in t. It is a mathematical consequence of (34) that at thickness t a particle cannot have energy greater than $E_0-\beta t$, and that there can be no electrons at thicknesses greater than E_0/β. These statements are obviously correct physically when one does not consider pair production, but not otherwise. With pair production, electrons of any energy can in principle be created at any thickness, since photons, of course, do not suffer collision loss.

It seems clear that for small thicknesses, and for energies E less than $E_0-\beta t$, where there is an appreciable probability for the original electron to be present, the relative effect of pair production will be small, except perhaps for very low energies where the straggling function of the preceding section breaks down anyway. This is made plausible, e.g., by the results of Bhabha and Heitler.[7] These authors consider the solution of the general cascade equations when collision loss is neglected. Their solution is written in the form of a series, the terms of which contain integrals over the straggling function. The nth term of their series has the physical meaning that it corresponds to an electron produced by n photon intermediaries. If one looks at their graphs one sees, e.g., that for $t=0.7$, $E=E_0e^{-3}$ the effect of photons is to increase the probability that a particle of energy greater than E is present by about 30 percent. It is also true that the main effect from photons in Bhabha and Heitler's theory comes from $n=1$, a result we would also expect when collision loss is taken into account.

Finally, some remarks on the evaluation of the integrals in the higher order terms of the straggling function may be helpful in practical computations.

[7] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. **159**, 432 (1937).

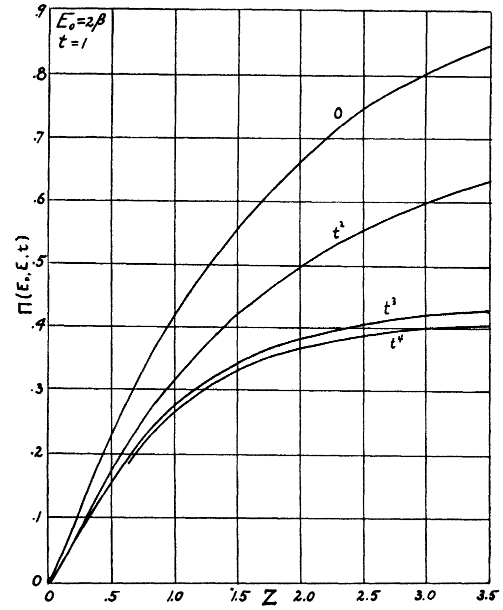

FIG. 4. Illustrating the effect of including higher powers of t in the integral straggling function $\Pi_\beta(E_0, E, t)$, for $E_0=2\beta$, $t=1$ and $\varphi(v)=\varphi_1(v)$. The curve marked 0 corresponds to taking only the constant term $c_0(s)$ in the series (26), that marked t^2 corresponds to taking $c_0(s)+c_1(s)t^2$, etc. $Z\equiv\ln(E_0-\beta t/E)$. For $E_0=2\beta$, $t=1$, $Z=\ln\beta/E$.

Integrals of the form

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\exp[ys-A(s)t]}{s^n(1+s)^m}ds\equiv I(n,m,y,t) \qquad (35)$$

occur, where n and m are small integers. These can be evaluated by the saddle-point method directly, but it is often convenient to use the relations

$$I(n+1,m,y,t)=\int_0^y I(n,m,y',t)dy',$$

$$I(n,m+1,y,t)=\int_0^y e^{y'}I(n,m,y',t)dy'.$$

It is also possible to break the integrand in (36) into partial fractions and evaluate each of the resulting integrals separately. For a considerable range of values of y and t however, it will be found possible to consider $s^n(1+s)^m$ slowly varying and take it out from under the integral sign, particularly since great accuracy is not required in the higher order terms. With a few simple considerations like these the evaluation of the higher order terms can be effected quite quickly.

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