On the Part Played by Scalar and Longitudinal Photons in Ordinary Electromagnetic Fields

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When electromagnetic interactions are calculated directly by second-order perturbation calculus without preceding separation of the Coulomb field from the transverse photon field by a canonical transformation, it is fundamentally not allowed to take a state without photons as zero-order approximation. A more correct zero-order approximation is determined, in which scalar and longitudinal photons are present in pairs. The probability distribution over the numbers of pairs present is given by the Schrödinger functional Eq. (20); these numbers are not limited. No transverse photons are present. The relativistic invariance of such a zero-order approximation is proved.

This zero-order approximation is then used for the calculation of the self-energy of a free electron at rest, of the Møller matrix element for electron-electron scattering and of the Breit interaction energy between electrons in an atom. Apart from mathematical ambiguities in the derivation of the self-energy, this leads to the same results as can also be obtained by the usual separation of the Coulomb field from the transverse photon field.

INTRODUCTION

IN recent days, much use is being made of a formulation of quantum electrodynamics, in which all four components of the quantized potential four-vector are treated more or less on an equal footing,[1] instead of the usual elimination of all but the transverse part of its spatial components after a preceding separation of the Coulomb field. Some discussion of the properties of the longitudinal and the time-like (so-called "scalar") part of the quantized potential can be found in old papers by Fock and Podolsky.[2] Also Pais used this field.[3] An advantage of this method lies in the fact that interactions between charges can now be calculated by one single second-order perturbation calculation.

In such calculations, one starts out from a zero-order approximation with "no field present." It is usually assumed that this "no field present" should then be interpreted as "no photons present." A close look at the theory, however, shows immediately that a state with no photons present even in the longitudinal and in the "scalar" field can never exist, as it is not only not relativistically invariant, but even in contradiction to the Lorentz condition.[4] Therefore, such state cannot serve as a zero-order approximation. One might then think that this procedure is completely impossible.

The purpose of this paper is to show how this method still leads to correct results, if one simply uses a different zero-order approximation. This zero-order approximation is determined below. In the following chapters it is shown, by examples, how it can be used. It then leads to results familiar from the ordinary theory, in which the Coulomb field and the transverse photon field are dealt with separately. However, when applied to self-energy questions, its results are not free from mathematical ambiguities. (See the small letters underneath Eq. (40).)

We may start with a Lagrangian function

$$L=(4\pi)^{-1}\{\tfrac{1}{4}F^{\nu\rho}F_{\nu\rho}-F^{\nu\rho}\nabla_\nu A_\rho+\tfrac{1}{2}S^2-S\nabla_\nu A^\nu\}$$
$$+A_\nu j^\nu-\hbar c\psi^\dagger(\kappa\beta-i\alpha^\nu\nabla_\nu)\psi, \quad (1)$$

where the notation is the usual one.[5] In the variational principle

$$\delta\int dt\int dxdydzL=0,$$

the variables $\psi^\dagger$, ψ, A_ν, and the antisymmetric tensor $F_{\nu\rho}(\equiv -F_{\rho\nu})$ are all to be varied independently, while j^ν is an abbreviation for $(-e)\psi^\dagger\alpha^\nu\psi$. This gives the usual field equations of Fermi's quantum electrodynamics. Without difficulty one finds the pairs of canonically conjugate variables (the magnetic field is a *derived variable*[6]); the commutation relations can be written down, and the Hamiltonian is found to be, in the usual three-dimensional vector notation,[5]

$$\mathcal{H}=\mathcal{H}_f+\mathcal{H}_m+\mathcal{W}, \quad (2)$$

$$8\pi\mathcal{H}_f=\int\mathbf{dx}\{\mathbf{E}^2+(\mathrm{curl}\mathbf{A})^2-S^2$$
$$+2S\,\mathrm{div}\mathbf{A}-2\Phi\,\mathrm{div}\mathbf{E}\}, \quad (3)$$

$$\mathcal{W}=\int\mathbf{dx}\{\rho\Phi-\mathbf{j}\cdot\mathbf{A}\}, \quad (4)$$

while the total momentum of the field is

$$\mathbf{P}_j=\int\mathbf{dx}\{(4\pi c)^{-1}(\mathbf{E}\cdot\nabla_j\mathbf{A}+S\nabla_j\Phi)+(\hbar/i)\psi^\dagger\nabla_j\psi\}. \quad (5)$$

We now introduce Jordan-Klein matrices for the electromagnetic field in the usual way. This yields (with

[1] J. Schwinger, Phys. Rev. **74**, 1439 (1948), **75**, 651 (1949); also F. J. Dyson, Phys. Rev. **75**, 486 (1949), (see his Eq. (16)).
[2] V. A. Fock and B. Podolsky, Physik. Zeits. Sowjetunion **1**, 801 (1931); **2**, 275 (1932).
[3] A. Pais, Proc. Roy. Acad. Amsterdam (Verhandelingen, 1e Sectie) **19**, 5 (1947), in particular, pp. 44–45.
[4] F. J. Belinfante, Phys. Rev. **75**, 337 (1949).
[5] F. J. Belinfante, Physica **12**, 1 (1946).
[6] F. J. Belinfante, Physica **7**, 765 (1940).

$\|$ and $\perp$ standing for "longitudinal" and "transverse"):

$$\Phi = i\Omega^{-\frac{1}{2}} \sum_{\mathbf{k}} (hc/k)^{\frac{1}{2}} b_{\mathbf{k}}^* \exp(i\mathbf{k}\cdot\mathbf{x}) + \text{conj.}, \tag{6a}$$

$$\mathbf{A}_{\|} = -i\Omega^{-\frac{1}{2}} \sum_{\mathbf{k}} (hc/k)^{\frac{1}{2}} a_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}^{0} \exp(i\mathbf{k}\cdot\mathbf{x}) + \text{conj.}, \tag{6b}$$

$$\mathbf{A}_{\perp} = -\frac{i}{\Omega^{\frac{1}{2}}} \sum_{\mathbf{k}} \sum_{\mu=1,2} \left(\frac{hc}{k}\right)^{\frac{1}{2}} c_{\mathbf{k},\mu} \mathbf{e}_{\mathbf{k}}^{\mu} \exp(i\mathbf{k}\cdot\mathbf{x}) + \text{conj.}, \tag{6c}$$

$$S = \Omega^{-\frac{1}{2}} \sum_{\mathbf{k}} (hck)^{\frac{1}{2}} (a_{\mathbf{k}} + b_{\mathbf{k}}^*) \exp(i\mathbf{k}\cdot\mathbf{x}) + \text{conj.}, \tag{6d}$$

$$\mathbf{E}_{\|} = \Omega^{-\frac{1}{2}} \sum_{\mathbf{k}} (hck)^{\frac{1}{2}} (a_{\mathbf{k}} + b_{\mathbf{k}}^*) \mathbf{e}_{\mathbf{k}}^{0} \exp(i\mathbf{k}\cdot\mathbf{x}) + \text{conj.}, \tag{6e}$$

$$\mathbf{E}_{\perp} = \Omega^{-\frac{1}{2}} \sum_{\mathbf{k}} \sum_{\mu=1,2} (hck)^{\frac{1}{2}} c_{\mathbf{k},\mu} \mathbf{e}_{\mathbf{k}}^{\mu} \exp(i\mathbf{k}\cdot\mathbf{x}) + \text{conj.} \tag{6f}$$

Here, Ω is the volume of the fundamental cube; $a_{\mathbf{k}}$, $b_{\mathbf{k}}$, $c_{\mathbf{k},1}$ and $c_{\mathbf{k},2}$ are annihilation operators and their conjugates are creation operators satisfying the usual commutation relations such that

$$N_{\mathbf{k}}{}^{a} = a_{\mathbf{k}}{}^* a_{\mathbf{k}}, \quad N_{\mathbf{k}}{}^{b} = b_{\mathbf{k}}{}^* b_{\mathbf{k}}, \quad \text{and} \quad N_{\mathbf{k},\mu}{}^{c} = c_{\mathbf{k},\mu}{}^* c_{\mathbf{k},\mu}. \tag{7}$$

All have eigenvalues 0, 1, 2, 3, $\cdots$ only. Further, $\mathbf{e}_{\mathbf{k}}^{\mu}$ with $\mu = 0, 1, 2$ are three mutually perpendicular unit vectors, with $\mathbf{e}_{\mathbf{k}}^{0} = \mathbf{k}/k$, where $k = |\mathbf{k}|$. Substituting the expansions (6) into (3), one finds $\mathcal{H}_f = \mathcal{H}_{\perp} + \mathcal{H}_{\|}$, with

$$\mathcal{H}_{\perp} = \sum_{\mathbf{k}} (N_{\mathbf{k},1}{}^{c} + N_{\mathbf{k},2}{}^{c} + 1) hck, \tag{8a}$$

$$\mathcal{H}_{\|} = \sum_{\mathbf{k}} (N_{\mathbf{k}}{}^{a} - N_{\mathbf{k}}{}^{b}) hck, \tag{8b}$$

while the part of the total momentum $\mathbf{P}$ depending on the electromagnetic field can be written in a similar way with hck in Eqs. (8) replaced by $\hbar\mathbf{k}$.

It is evident that the energy q-number (8b) is not positive definite. Only states of positive energy are admitted, however, if one imposes the usual auxiliary conditions

$$S(x, y, z)\Psi = 0, \tag{9}$$

$$\{\text{div}\mathbf{E} - 4\pi\rho\}\Phi = 0 \tag{10}$$

on the Schrödinger state functional. It is well known that these conditions cause difficulties with the normalization of the state functional Ψ itself.[7] We shall meet these difficulties as soon as we try to apply the theory to specific problems; but formally we can overcome them there, although they sometimes lead to ambiguity in the mathematics (compare Application 1).

The conditions (9)–(10) are also valid in interaction representation, if Ψ and S, $\mathbf{E}$ and ρ all are taken at one and the same time.

In the applications, we shall consider $\mathcal{W}$ of Eq. (4) as a perturbation, and e as a small coupling constant. The zero-order approximation Ψ_0 shall be an eigenfunction of $\mathcal{H}_f + \mathcal{H}_m$. (Here, $\mathcal{H}_m$ may or may not include some c-number external electrostatic potential.) *The function Ψ_0 shall also satisfy the conditions* (9)–(10) with this difference, that in zero-order approximation we may omit the term $(-4\pi\rho)$ from Eq. (10). In the first term, we use (6e) or

$$\text{div}\mathbf{E} = i\Omega^{-\frac{1}{2}} \sum_{\mathbf{k}} (hck^3)^{\frac{1}{2}} (a_{\mathbf{k}} + b_{\mathbf{k}}^*) \exp(i\mathbf{k}\cdot\mathbf{x}) + \text{conj.} \tag{11}$$

We multiply (9) and (10) by $\exp(-i\mathbf{k}\cdot\mathbf{x})$ and integrate over $\mathbf{x}$, thus picking out one Fourier component. We thus find, from (9) with (6d),

$$(a_{\mathbf{k}} + b_{\mathbf{k}}^* + a_{-\mathbf{k}}^* + b_{-\mathbf{k}})\Psi_0 = 0 \tag{9a}$$

and, from (10) with (11),

$$(a_{\mathbf{k}} + b_{\mathbf{k}}^* - a_{-\mathbf{k}}^* - b_{-\mathbf{k}})\Psi_0 = 0. \tag{10a}$$

Adding and subtracting, we obtain

$$a_{\mathbf{k}}\Psi_0 = -b_{\mathbf{k}}^*\Psi_0, \tag{12}$$

$$b_{\mathbf{k}}\Psi_0 = -a_{\mathbf{k}}^*\Psi_0. \tag{13}$$

If we take the $N_{\mathbf{k}}{}^{a}$ and $N_{\mathbf{k}}{}^{b}$ to be on diagonal form, the Eqs. (12)–(13) give the dependence of Ψ_0 on these numbers. Actually, Ψ_0 can be considered as a product of factors, each depending on the values of one pair of numbers $N_{\mathbf{k}}{}^{a}$ and $N_{\mathbf{k}}{}^{b}$ belonging to one wave vector $\mathbf{k}$ only. Each such factor in Ψ_0 we shall expand in simultaneous eigenfunctions $\chi_{m,n}$ of $N_{\mathbf{k}}{}^{a}$ and $N_{\mathbf{k}}{}^{b}$:

$$\Psi_0 = \cdots \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} \chi_{m,n} \right) \cdots. \tag{14}$$

Here, m and n stand for the eigenvalues of $N_{\mathbf{k}}{}^{a}$ and $N_{\mathbf{k}}{}^{b}$, to which $\chi_{m,n}$ belongs.

We may choose the arbitrary phase factors in the eigenfunctions $\chi_{m,n}$ in such a way that the Jordan-Klein matrices $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are represented by

$$a_{\mathbf{k}}\chi_{m,n} = m^{\frac{1}{2}}\chi_{m-1,n}; \qquad b_{\mathbf{k}}\chi_{m,n} = n^{\frac{1}{2}}\chi_{m,n-1}; \tag{15a}$$

$$a_{\mathbf{k}}{}^*\chi_{m,n} = (m+1)^{\frac{1}{2}}\chi_{m+1,n}; \quad b_{\mathbf{k}}{}^*\chi_{m,n} = (n+1)^{\frac{1}{2}}\chi_{m,n+1}. \tag{15b}$$

We substitute (14) into (12)–(13), make use of (15), and equate the coefficients of $\chi_{m-1,n}$ on both sides of Eq. (12), and of $\chi_{m,n-1}$ in Eq. (13). This gives the following recursion formulas for $c_{m,n}$,

$$m^{\frac{1}{2}} c_{m,n} = -n^{\frac{1}{2}} c_{m-1,n-1} \quad (\text{for } m \geqslant 1,\ n \geqslant 0), \tag{12a}$$

$$n^{\frac{1}{2}} c_{m,n} = -m^{\frac{1}{2}} c_{m-1,n-1} \quad (\text{for } m \geqslant 0,\ n \geqslant 1). \tag{13a}$$

For $m \geqslant 1$, $n \geqslant 1$, both equations are valid. For $m \neq n$, they have then only the zero solution

$$c_{m,n} = 0, \text{ if } m \neq n, \tag{16}$$

which is also correct, if m **or** n becomes zero. Thus,

$$\Psi_0 = \cdots \left(\sum_{n=0}^{\infty} c_{n,n} \chi_{n,n} \right) \cdots. \tag{17}$$

From (17) we see that Ψ_0 is a superposition of states, in which the numbers $N_{\mathbf{k}}{}^{a}$ and $N_{\mathbf{k}}{}^{b}$ have the same value. Considering (8b), we conclude that there are just as many "longitudinal photons" with momentum $\hbar\mathbf{k}$ and

[7] F. J. Belinfante, Physica **12**, 17 (1946).

energy hck in the field, as there are "scalar photons" with momentum $(-\hbar\mathbf{k})$ and energy $(-\hbar k)$ and energy $(-hck)$. In other words, in field-free space the longitudinal and "scalar" photons occur in "pairs". Thus, the *value* of $\mathfrak{H}_{II}$ is zero, and the *value* of the total electromagnetic energy $\mathfrak{H}_f$ becomes positive definite, even if $\mathfrak{H}_f$ is not positive definite as a q-number.

For $m=n$, Eqs. (12a)–(13a) give

$$c_{n,n}=-c_{n-1,n-1} \text{ (for } n\geqslant 1), \tag{18}$$

so that

$$c_{n,n}=(-)^n\cdot c_{0,0}, \tag{19}$$

$$\Psi_0=\cdots\left\{c_{0,0}\sum_{n=0}^{\infty}(-)^n\chi_{n,n}\right\}\cdots. \tag{20}$$

The minus signs are due to the choice of the phase factors in $\chi_{m,n}$ made in (15). The particular form (20) of the Schrödinger functional, which in our zero-order approximation ($e=0$) followed directly from the Lorentz condition (9)–(10), shows that *in field-free space pairs of longitudinal and "scalar" photons certainly are present.*[7a]

For the rest, the wave function (20) *cannot properly be normalized*, as the normalization integral reads

$$\mathfrak{N}=(\Psi_0^*,\Psi_0)=\cdots\left\{\sum_{n=0}^{\infty}|c_{n,n}|^2\right\}\cdots$$

$$=\cdots\left\{|c_{0,0}|^2\sum_{n=0}^{\infty}1\right\}\cdots=\text{divergent}. \tag{21}$$

As stated above, this fact is not surprising.[7] In applications of the theory, however, we shall see that, without explicit use of the value of $|c_{0,0}|^2$, we can always take a factor $\mathfrak{N}$ out of the result of our calculations and then *put this factor formally equal to unity*, as if there were no trouble with the normalization at all.

We thus shall assume that in zero-order approximation, in a space "with no field," the numbers of scalar and longitudinal photons present (as well as the relative phases of the probability amplitudes for the different possible values of these numbers) are given by Eq. (20). *The numbers of transverse photons we shall simply take as zero in such a field-free space.*

Before we proceed to the applications, it should be remarked that the above definition of field-free space is *relativistically invariant in our zero-order approximation* ($e=0$). As far as the scalar and longitudinal photons are concerned, this is obvious, as (20) was derived in this approximation from (9)–(10), which are equivalent with the one invariant condition

$$S(x,y,z,t)\Psi_0=0 \text{ (with } S=\nabla_\nu A^\nu) \tag{22a}$$

in Heisenberg representation, or

$$S(x,y,z,t)\Psi_0[\sigma]=4\pi\int_\sigma D(x-x')j^\mu(x')d\sigma_\mu'\cdot\Psi_0[\sigma] \tag{22b}$$

in interaction representation.[1]

[7a] S. T. Ma in a recent Letter to the Editor comes to similar results. See Phys. Rev. **75**, 535 (1949).

As far as the transverse photons are concerned, we have to prove their absence in a moving coordinate system $\{x'y'z't'\}$, if in $\{xyzt\}$ no transverse photons are present, while at the same time the Lorentz condition holds for the longitudinal and the scalar field. It is sufficient to check this for an infinitesimal Lorentz transformation ($\mathbf{x}'=\mathbf{x}-\mathbf{b}x^0$, $x^{0\prime}=x^0-\mathbf{b}\cdot\mathbf{x}$, with $\mathbf{b}=\mathbf{v}/c$).

We can consider the total $\mathbf{A}$, Φ field as a superposition of plane waves $A^\nu(\mathbf{k})\exp(ik_\mu x^\mu)$, with wave four-vectors $\mathbf{k}$, k^0. The wave equation for A^ν then gives[8] $k^0=\pm k$. For such plane waves, the Lorentz condition (22) imposed on Ψ_0 gives in zero-order approximation,[8]

$$\mathbf{k}\cdot\mathbf{A}^{(\pm)}(\mathbf{k})\Psi_0=k^0\Phi^{(\pm)}(\mathbf{k})\Psi_0=\pm k\Phi^{(\pm)}(\mathbf{k})\Psi_0. \tag{23}$$

Here, we denote by $A_\mu^{(+)}(\mathbf{k})$ the part of $A_\mu(\mathbf{k})$ with "positive frequency" $k^0=+k$ (with a time factor $\exp(ik_0x^0)=\exp(-ikct)$). Then, its *spatial* components $\mathbf{A}^{(+)}(\mathbf{k})$ contain only annihilation operators.[1] Our assumption of absence of transverse photons in the unprimed coordinate system can therefore be expressed by[1,4]

$$\mathbf{A}_\perp^{(+)}(\mathbf{k})\Psi_0=0. \tag{24}$$

By $\mathbf{A}^{(+)}=\mathbf{A}_\perp^{(+)}+\mathbf{A}_{II}^{(+)}$ and by (23), this gives

$$\mathbf{A}^{(+)}(\mathbf{k})\Psi_0=\mathbf{A}_{II}^{(+)}(\mathbf{k})\Psi_0=\mathbf{k}k^{-2}\mathbf{k}\cdot\mathbf{A}^{(+)}(\mathbf{k})\Psi_0=\mathbf{k}k^{-1}\Phi^{(+)}(\mathbf{k})\Psi_0. \tag{25}$$

In the primed coordinate system, by the infinitesimal Lorentz transformations

$$\mathbf{A}'=\mathbf{A}-\mathbf{b}\Phi,\quad \mathbf{k}'=\mathbf{k}-\mathbf{b}k^0, \tag{26}$$

we obtain from (25) for the waves with positive frequencies,

$$\mathbf{A}'^{(+)}(\mathbf{k}')\Psi_0=\mathbf{k}k^{-1}\Phi^{(+)}(\mathbf{k})\Psi_0-\mathbf{b}\Phi^{(+)}(\mathbf{k})\Psi_0=\mathbf{k}'k^{-1}\Phi^{(+)}(\mathbf{k})\Psi_0, \tag{25a}$$

where $\mathbf{A}'^{(+)}(\mathbf{k}')$ is the coefficient of $\exp(ik_\nu'x^{\nu\prime})=\exp(ik_\nu x^\nu)$ in the spatial components of the transformed potential four-vector. As the operator in the right-hand member of (25a) is obviously longitudinal in the new coordinate system, it follows from the transverse part of (25a) that now

$$\mathbf{A}_\perp'^{(+)}(\mathbf{k}')\Psi_0=0, \tag{24a}$$

which means that also in the moving (primed) coordinate system no transverse photons are present.

This obviously means that our simple condition (24) in zero-order approximation ($e=0$) is just as relativistically invariant as Schwinger's condition $\mathfrak{A}_\mu^{(+)}\Psi_0=0$. On the other hand, our condition tells a little more: While $\mathfrak{A}_\mu^{(+)}\Psi_0=0$ is only a covariant way of saying that in *one particular* Lorentz system (with time axis along some given time-like unit four-vector n^μ) no transverse photons are present, we express here by (24)–(24a) that in zero-order approximation *in no Lorentz system at all* there are transverse photons in the field.

(Remark that a total absence of *any* kind of photons would have corresponded to conditions $\mathbf{A}^{(+)}\Psi_0=0$ with $\Phi^{(-)}\Psi_0=0$, which obviously would not have been relativistically covariant. Moreover, $\mathbf{A}^{(+)}$ and $\Phi^{(-)}$ do not commute with S, so that such conditions would have been incompatible with (22).)

APPLICATION 1. ELECTRON SELF-ENERGY

As a first application, we shall consider here the calculation of the self-energy of a free electron at rest ($\mathbf{p}=0$) by means of second-order perturbation calculus, using the zero-order approximation of the previous chapter, and using $\mathfrak{W}$ of Eq. (4) as a perturbation. (The self-energy of a moving electron, $\mathbf{p}\neq 0$, could be treated similarly.)

In $\mathfrak{W}$, we expand $\mathbf{A}$ and Φ by Eqs. (6a–c), while ψ and $\psi^\dagger$ in the factors ρ and $\mathbf{j}$ can be expanded similarly,

[8] In Heisenberg representation, the "zero-order" approximation $e=0$ is made in the first step, where we use $\Box A^\nu=0$. In interaction representation, we make the approximation $e=0$ in the next step (23) by neglecting the right-hand member of (22b).

using Jordan-Wigner matrices for them. The matrix elements of $\mathfrak{W}$ for emission or absorption of photons under simultaneous transition of the electron to a state of different momentum then follow directly from (15) and similar equations for the matrices $c_{\mathbf{k},\mu}$ and $c_{\mathbf{k},\mu}^*$ occurring in Eq. (6c). These matrix elements should be used in the numerator of an expression of the type

$$\sum_i \frac{(\Psi_0^*, \mathfrak{W}\Psi_i)(\Psi_i^*, \mathfrak{W}\Psi_0)}{E_0 - E_i}. \tag{27}$$

As by Eq. (20), the number of photon pairs in the initial state Ψ_0 is not definite, substitution of (20) into (27) gives a sum over all possible values of all n's in the factor Ψ_0 in (27) and a sum over all possible values of all n''s in the factor Ψ_0^*. As the numerator of (27) contains only two factors $\mathfrak{W}$, each allowing for a change of only one number $N_{\mathbf{k}}^a$, $N_{\mathbf{k}}^b$, or $N_{\mathbf{k}}^c$, there are only the following nine kinds of terms in (27):

A. First, a transverse photon $(-\hbar\mathbf{k})$ is emitted and the electron takes the recoil $(+\hbar\mathbf{k})$; then this photon is again absorbed and the electron comes to rest.

B. Similarly with longitudinal photons; that is, all numbers N^a but the one $N_{-\mathbf{k}}^a$ remain constant, and for that particular $N_{-\mathbf{k}}^a$ there are transitions $\chi_{n,n} \to \chi_{n+1,n} \to \chi_{n,n}$.

C. A longitudinal photon with momentum $(+\hbar\mathbf{k})$ is first absorbed, then reappears: $\chi_{n,n} \to \chi_{n-1,n} \to \chi_{n,n}$.

D, E. Similarly with negative energy scalar photons: $\chi_{n,n} \to \chi_{n,n\pm1} \to \chi_{n,n}$. Here, it is $N_{\mathbf{k}}^b$ that increases by 1 in case D, and $N_{-\mathbf{k}}^b$ that decreases by 1 in case E, if the recoil is always to be $+\hbar\mathbf{k}$.

F. First, a longitudinal photon with momentum $(-\hbar\mathbf{k})$ is emitted then a scalar photon with momentum $(+\hbar\mathbf{k})$ is emitted while the electron comes again to rest: $\chi_{n,n} \to \chi_{n+1,n} \to \chi_{n+1,n+1}$.

G. Similarly with absorptions of, first, a longitudinal photon $+\hbar\mathbf{k}$ and, later, of a scalar photon $(-\hbar\mathbf{k})$: $\chi_{n,n} \to \chi_{n-1,n} \to \chi_{n-1,n-1}$.

H, I. These processes can also take place in the opposite order of sequence, with the opposite momenta of the photons: $\chi_{n,n} \to \chi_{n,n\pm1} \to \chi_{n\pm1,n\pm1}$.

All these effects are to be taken into account. They all have in common that the number of photons $N_{\mathbf{k}}^a$ and $N_{\mathbf{k}}^b$ for only *one* value of $\mathbf{k}$ are affected; in cases A, B, E, F, I this $\mathbf{k}$ is opposite to the recoil of the electron in units $\hbar$, while in cases C, D, G, H this $\mathbf{k}$ is equal to the momentum of the electron in the intermediate state in units $\hbar$. (These signs are important, as they enter into the matrix elements through the vectors $\mathbf{e}_{\mathbf{k}}^0$ in Eq. (6b).) Therefore, we may consider the summation over all numbers n and n' belonging to *other* $\mathbf{k}$ as having been performed tacitly by putting the corresponding normalization integrals $\mathfrak{N}$ equal to 1 (compare Eq. (21)), and only a summation over *one* set of values of n and of n' is left:

$$\sum_i \sum_n \sum_{n'} c_{n',n'}^* c_{n,n} \frac{\mathfrak{W}_{n'i}\mathfrak{W}_{in}}{E_0 - E_i}. \tag{28}$$

In the terms A, this summation over n and n' reduces to one single term $(n = n' = 0)$, as Ψ_0 is a state without transverse photons. In the other terms, it reduces to a single summation over n, with $n' = n$ in cases B, C, D and E, with $n' = n+1$ in cases F and H, and with $n' = n-1$ in cases G and I.

Not only the electron originally at rest can jump to some intermediate positive energy state, but also a negative energy electron can jump to any positive energy state with exception of the one occupied. (This corresponds to virtual creation of pairs.) According to Weisskopf,[9] we subtract the self-energy of empty space. This overcompensates the jumps from negative energy states, as we now also subtract jumps from negative energy states into the state of an electron at rest. Therefore, we replace (28) by

$$\mathrm{W}_{\mathrm{self}} = \sum_{n'} \sum_n \mathrm{c}_{n'}^* \mathrm{c}_n \left\{ \sum_i \frac{\mathfrak{W}_{(0n')i}\mathfrak{W}_{i(0n)}}{E_0 - E_i} - \sum_{j,l} \frac{\mathfrak{W}_{(jn')l}\mathfrak{W}_{l(jn)}}{E_j - E_l} \right\}. \tag{29}$$

Here, c_n stands now for $c_{n,n}$. In the first sum, $(0n)$ represents a state with an electron with energy $+mc^2(\mathbf{p}=0)$ and with n photons (or pairs of photons) of the type that are emitted or absorbed in the transitions. By i, we understand an intermediate state with an odd number of photons present and with the electron in some positive energy state. We may sum over both spin orientations of the electron in this intermediate positive energy state by Casimir's method,[10] using the operator Λ^+ given by

$$\Lambda^{\pm} = \tfrac{1}{2}\{1 \pm w^{-1}(\kappa\beta + \boldsymbol{\alpha}\cdot\mathbf{k})\}, \tag{30}$$

with

$$w = +(\kappa^2 + k^2)^{\frac{1}{2}}. \tag{31}$$

This Λ^+ is to be written in the numerator between the two factors $\mathfrak{W}$.

In the second sum in (29), the initial state (jn) is one with an electron originally in some negative energy state j, say with momentum $\mathbf{k}$. Also n photons be present of the type that is of interest. This type now has a momentum opposite to what it was in the previous discussion of cases A to I. The intermediate state l is one with the electron now in the positive energy state $\mathbf{p}=0$, but with an odd number of photons present. Again we use Casimir's method, now for the summation over the spins in the states j. This time, Λ^- should operate from the left on the first factor $\mathfrak{W}$ and from the right on the second factor $\mathfrak{W}$. This is equivalent to interchanging the factors $\mathfrak{W}$ *as far as their Dirac matrices are concerned*, and putting Λ^- in between afterwards.

In the denominators, $E_0 - E_i = \hbar c\{\kappa - w - \eta k\}$ and $E_j - E_l = -\hbar c\{w + \kappa + \eta k\}$, where $\eta = +1$ in case of creation of a vector photon or absorption of a scalar photon in the first transition (cases A, B, E, F, and I), while $\eta = -1$ in the other cases (D, C, G, and H).

With due care for the plus and minus signs, we thus finally obtain, replacing $\Omega^{-1}\sum_{\mathbf{k}}$ by $(2\pi)^{-3}\int \mathbf{dk}$:

$$\mathrm{W}_{\mathrm{self}} = e^2(2\pi)^{-2} \int \mathbf{dk} k^{-1} f(\mathbf{k}), \tag{32}$$

[9] V. F. Weisskopf, Phys. Rev. **56**, 72 (1939), and A. Pais (reference 3), in particular, p. 25.

[10] H. B. G. Casimir, Helv. Phys. Acta **6**, 287 (1933), or W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1936) in particular, pp. 84–87 and pp. 149–153.

with the following contributions to $f(\mathbf{k})$ from the various processes A-I:

$$f_A(\mathbf{k})=\sum_{\mu=1,2}\left\{\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{\mu})\Lambda^{+}(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{\mu})}{\kappa-w-k}+\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{\mu})\Lambda^{-}(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{\mu})}{\kappa+w+k}\right\}_{Av}, \quad (33A)$$

$$f_B(\mathbf{k})=\sum_{n=0}^{\infty} c_n{}^{*}c_n(n+1)\left\{\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})\Lambda^{+}(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})}{\kappa-w-k}+\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})\Lambda^{-}(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})}{\kappa+w+k}\right\}_{Av}, \quad (33B)$$

$$f_C(\mathbf{k})=\sum_{n=1}^{\infty} c_n{}^{*}c_n n\left\{\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})\Lambda^{+}(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})}{\kappa-w+k}+\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})\Lambda^{-}(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})}{\kappa+w-k}\right\}_{Av}, \quad (33C)$$

$$f_D(\mathbf{k})=\sum_{n=0}^{\infty} c_n{}^{*}c_n(n+1)\left\{\frac{\Lambda^{+}}{\kappa-w+k}+\frac{\Lambda^{-}}{\kappa+w-k}\right\}_{Av}, \quad (33D)$$

$$f_E(\mathbf{k})=\sum_{n=1}^{\infty} c_n{}^{*}c_n n\left\{\frac{\Lambda^{+}}{\kappa-w-k}+\frac{\Lambda^{-}}{\kappa+w+k}\right\}_{Av}, \quad (33E)$$

$$f_F(\mathbf{k})=\sum_{n=0}^{\infty} c_{n+1}{}^{*}c_n(n+1)\left\{\frac{\Lambda^{+}(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})}{\kappa-w-k}+\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})\Lambda^{-}}{\kappa+w+k}\right\}_{Av}, \quad (33F)$$

$$f_G(\mathbf{k})=\sum_{n=1}^{\infty} c_{n-1}{}^{*}c_n n\left\{\frac{\Lambda^{+}(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})}{\kappa-w+k}+\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})\Lambda^{-}}{\kappa+w-k}\right\}_{Av}, \quad (33G)$$

$$f_H(\mathbf{k})=\sum_{n=0}^{\infty} c_{n+1}{}^{*}c_n(n+1)\left\{\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})\Lambda^{+}}{\kappa-w+k}+\frac{\Lambda^{-}(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})}{\kappa+w-k}\right\}_{Av}, \quad (33H)$$

$$f_I(\mathbf{k})=\sum_{n=1}^{\infty} c_{n-1}{}^{*}c_n n\left\{\frac{(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})\Lambda^{+}}{\kappa-w-k}+\frac{\Lambda^{-}(\boldsymbol{\alpha}\cdot\mathbf{e}_{k}{}^{0})}{\kappa+w+k}\right\}_{Av}. \quad (33I)$$

Into these expressions we substitute (30) for $\Lambda^{\pm}$. In the state 0 of the electron at rest, we take the expectation value $\{\ \}_{Av}$ of the products of Dirac matrices thus appearing in the above expressions. We may use here $\{\boldsymbol{\alpha}\}_{Av}=0$, $\{\alpha_i\alpha_j\alpha_k\}_{Av}=0$, $\{\beta\}_{Av}=1$. Thus, by $\mathbf{e}_k{}^{\mu}\cdot\mathbf{e}_k{}^{\mu'}=\delta_{\mu\mu'}$, $[\mathbf{k}\times\mathbf{e}_k{}^{0}]=0$, and omitting terms odd in $[\mathbf{e}_k{}^{\mu}\times\mathbf{e}_k{}^{\mu'}]$, which will vanish anyhow after the later integration over $\mathbf{k}$ in (32), we may replace:

$$\left.\begin{array}{rll}\Lambda^{\pm} & \text{by} & \tfrac{1}{2}\pm\kappa/2w,\\ (\boldsymbol{\alpha}\cdot\mathbf{e})\Lambda^{\pm}(\boldsymbol{\alpha}\cdot\mathbf{e}) & \text{by} & \tfrac{1}{2}\mp\kappa/2w,\\ \Lambda^{\pm}(\boldsymbol{\alpha}\cdot\mathbf{e}_k{}^{0})=-\Lambda^{\pm}(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0}) & \text{by} & \pm k/2w,\\ (\boldsymbol{\alpha}\cdot\mathbf{e}_k{}^{0})\Lambda^{\pm}=-(\boldsymbol{\alpha}\cdot\mathbf{e}_{-k}{}^{0})\Lambda^{\pm} & \text{by} & \pm k/2w.\end{array}\right\} \quad (34)$$

After these substitutions, we put $k=\kappa\sinh z$, $w=\kappa\cosh z$, $w\pm k=\kappa\exp(\pm z)$, and

$$\int \mathbf{dk}/4\pi^2 kw=(\kappa/\pi)\int_0^{\infty}\sinh z\,dz.$$

As the resulting integral over z diverges, we cut it off at some maximum value M for z. By $z=\ln\{(k+w)/\kappa\}$, we may call

$$\mathrm{M}=\ln\frac{\mathrm{P}+\mathrm{P}^0}{mc}, \quad (35)$$

where P is the maximum momentum and $c\mathrm{P}^0$ is the maximum energy of the electron or of the positon in the intermediate state. Thus, one finds easily for the transverse self-energy derived from $f_A(\mathbf{k})$:

$$W_{tr}=\frac{e^2\kappa}{2\pi}(\mathrm{M}-\tfrac{1}{2}+\tfrac{1}{2}e^{-2\mathrm{M}})\rightarrow\frac{mc^2}{2\pi}\cdot\frac{e^2}{hc}\cdot\left\{\ln\frac{\mathrm{P}+\mathrm{P}^0}{mc}-\tfrac{1}{2}\right\}, \quad (36)$$

and for the static self-energy derived from the other terms B to I:

$$W_{st}=\frac{e^2\kappa}{4\pi}\Big\{\sum_{n=0}^{\infty}|c_n|^2(n+1)(\tfrac{1}{2}e^{2\mathrm{M}}+4\mathrm{M}-1+\tfrac{1}{2}e^{-2\mathrm{M}})+\sum_{n=1}^{\infty}|c_n|^2 n(\tfrac{1}{2}e^{2\mathrm{M}}-4\mathrm{M}-1+\tfrac{1}{2}e^{-2\mathrm{M}})+\Big[\sum_{n=0}^{\infty}c_{n+1}{}^{*}c_n(n+1)+\sum_{n=1}^{\infty}c_{n-1}{}^{*}c_n n\Big]\times\Big[\tfrac{1}{2}e^{2\mathrm{M}}-1+\tfrac{1}{2}e^{-2\mathrm{M}}\Big]\Big\}. \quad (37)$$

The two summations from 1 to ∞ can now be changed into summations from 0 to ∞ because of the factor n. Thus, we may write

$$W_{st}=\frac{e^2\kappa}{\pi}\cdot\mathrm{M}\cdot\sum_{n=0}^{\infty}|c_n|^2+\frac{e^2\kappa}{2\pi}\cdot\sinh^2\mathrm{M}\cdot\sum_{n=0}^{\infty}\{|c_n|^2(2n+1)+c_{n+1}{}^{*}c_n(n+1)+c_{n-1}{}^{*}c_n n\}. \quad (38)$$

In the last expression, we shall now make use of Eq. (18), by putting

$$c_{n+1}{}^{*}c_n=c_{n-1}{}^{*}c_n=-|c_n|^2. \quad (39)$$

Thus, the last sum in Eq. (38) vanishes, and a quadratic divergence (proportional to P^2) is avoided. Also putting the sum in the first term of (38) formally equal to unity (compare Eq. (21)), we find the well-known result[9]

$$W_{self}=W_{tr}+W_{st}=\frac{mc^2}{2\pi}\cdot\frac{e^2}{\hbar c}\cdot\left\{3\ln\frac{P+P^0}{mc}-\tfrac{1}{2}\right\}. \quad (40)$$

This all seems quite satisfactory, but it should be pointed out that the mathematical method used is ambiguous because of the divergence of the summations over n. Indeed, one might well have preferred to call the summation index n in the processes G and I rather $(n+1)$, with the new n summed at once from 0 to ∞ without need of adding a zero term. This would make f_G and f_I nicely the conjugates of f_H and f_F. It would have changed the $c_{n-1}^*c_n n$ in the last term of (38) into $c_n^*c_{n+1}(n+1)$. Then, after application of (39), we would find that a term

$$-\frac{e^2\kappa}{2\pi}\cdot\sinh^2 M\rightarrow\frac{e^2\kappa}{8\pi}\cdot(2-e^{2M})=\frac{mc^2}{8\pi}\cdot\frac{e^2}{\hbar c}\cdot\left\{2-\left(\frac{P+P^0}{mc}\right)^2\right\} \quad (41)$$

would have been added to the self-energy, thus making the latter negative and quadratically divergent.

Similarly, we could have written $(n+1)$ for n in the processes C and E. This would change $|c_n|^2 n$ into $|c_{n+1}|^2(n+1)$ in Eq. (37). For this, we could have written $|c_n|^2(n+1)$ again by Eq. (18), so that we would find that this time a positive and quadratically divergent term

$$\frac{e^2\kappa}{8\pi}\cdot(e^{2M}-8M-2) \quad (42)$$

would have been added to the static self-energy.

If *both* (41) and (42) would be added to (38)–(40), one would find

$$W_{st}'=0; \quad W_{self}'=W_{tr}. \quad (43)$$

All this shows that calculations based on perturbation theory without a preceding separation of Coulomb field and photon field are rather dangerous, if the auxiliary condition is taken seriously as a condition imposed on the state-vector Ψ.

When we compare the above calculation with the usual type of perturbation calculations, then we remark mainly the following differences:

(I) The zero-order approximation is here no longer one with $c_n=\delta_{n0}$.

(II) There are many more intermediate states to be taken into account, and there are processes F, G, H, I, in which a pair of photons is created or annihilated.

(III) The scalar photons in the intermediate states have negative energy, and enter with such energy in the resonance denominators.

If incorrectly Ψ_0 would have been taken to be a state without photons, so that no use would have been made of (I) and (II), then the calculation would lead to an erroneous result, in which the static and the total self-energy would become quadratically divergent.

If at the same time the scalar photons are taken with the wrong energy sign in the resonance denominators, then the static self-energy becomes again logarithmically divergent, but one would wrongly find $(e^2\kappa/2\pi)(-M-\frac{1}{2}+\frac{1}{2}e^{-2M})$ for it. Added to (36), this would give a finite *total* self-energy.

An interesting result is obtained, if *beside* these two mistakes we make a *third* error in the derivation of the self-energy, by taking the contributions from the processes D with the wrong sign. These three *combined* mistakes would change $f(\mathbf{k})$ into

$$F(\mathbf{k})=\left\{\frac{\boldsymbol{\alpha}\cdot\Lambda^+\boldsymbol{\alpha}-\Lambda^+}{\kappa-w-k}+\frac{\boldsymbol{\alpha}\cdot\Lambda^-\boldsymbol{\alpha}-\Lambda^-}{\kappa+w+k}\right\}_{Av}. \quad (44)$$

Thus, the self-energy would become

$$W_{self}=\frac{e^2}{4\pi^2}\int\frac{\mathbf{dk}}{k}\left\{\frac{\alpha^\lambda\Lambda^+\alpha_\lambda}{\kappa-w-k}+\frac{\alpha^\lambda\Lambda^-\alpha_\lambda}{\kappa+w+k}\right\}_{Av}. \quad (45)$$

This expression, which leads again exactly to (40), was guessed in a recent paper by Feynman[11] as a relativistic generalization of the expression for the transverse self-energy. (Compare (44) with (33A).) It was derived by French and Weisskopf[12] directly from a calculation of the self-energy, in which Coulomb field and photon field were treated independently, so that their zero-order approximation (without transverse photons) was correct anyhow. It would be interesting to know, on ground of what symmetry properties of the theory the above three errors here compensate each other completely.

APPLICATION 2. ELECTRON-ELECTRON SCATTERING

As a second application, we shall calculate the matrix element for scattering of free electrons by free electrons. Let $\hbar\mathbf{k}_1$ and $\hbar\mathbf{k}_2$ be the original momenta of two colliding electrons, and $\hbar\mathbf{k}_1'$ and $\hbar\mathbf{k}_2'$ their momenta after scattering. The matrix element for this process is then given by

$$\sum_i\sum_n\sum_{n'}c_{n'}^*c_n\frac{(\mathbf{k}_1'\mathbf{k}_2'n'|\mathfrak{W}|\mathbf{k}_1'\mathbf{k}_2i)(\mathbf{k}_1'\mathbf{k}_2i|\mathfrak{W}|\mathbf{k}_1\mathbf{k}_2n)}{E_0-E_i}$$
$$+\text{Sym}-\text{Exch}. \quad (46)$$

Here, Sym means the terms obtained by complete interchanging of the two electrons, while Exch stands for the exchange terms,[13] in which the $\mathbf{k}_1$ and $\mathbf{k}_2$ are interchanged, but not the $\mathbf{k}_1'$ and $\mathbf{k}_2'$. Because of the conservation of energy, E_0 is the final as well as the initial energy. Denoting the energies of the single electrons by $\hbar c\epsilon$, we have in the terms written out in (46),

$$E_0-E_i=\hbar c(\epsilon_1-\epsilon_1'-\eta k),$$

where η has the same meaning as in the text between Eqs. (31) and (32), while $\hbar k$ is the magnitude of the momentum of the photon involved.

In the terms written out in (46), we shall put

$$\mathbf{k}=\mathbf{k}_1'-\mathbf{k}_1=\mathbf{k}_2-\mathbf{k}_2'.$$

(In the exchange terms, therefore, $\mathbf{k}=\mathbf{k}_1'-\mathbf{k}_2$.) The contributions of the various processes A–I to (46) give now:

$$\text{(A)}\quad\frac{2\pi e^2}{\Omega k}\sum_{\mu=1,2}\left\{\frac{(\mathbf{e}_{-k^\mu}\cdot\boldsymbol{\alpha}_2)(\mathbf{e}_{-k^\mu}\cdot\boldsymbol{\alpha}_1)}{\epsilon_1-\epsilon_1'-k}+\text{Sym}\right\}$$
$$-\text{Exch}. \quad (47)$$

Here, $\boldsymbol{\alpha}_1$ means the matrix element of $\boldsymbol{\alpha}$ between the spin factors of the plane waves corresponding to $\mathbf{k}_1$ and $\mathbf{k}_1'$, respectively. (In the exchange terms this would be between $\mathbf{k}_2$ and $\mathbf{k}_1'$.)

[11] R. P. Feynman, Phys. Rev. **74**, 1430 (1948); Eq. (3).

[12] J. B. French and V. F. Weisskopf, Phys. Rev. **75**, 1240 (1949).

[13] Compare J. R. Oppenheimer, Phys. Rev. **32**, 361 (1928) and N. F. Mott, Proc. Roy. Soc. **A125**, 222 (1929). These exchange terms result from the fact that the two steps of the process may be just as well transitions $\mathbf{k}_2\rightarrow\mathbf{k}_1'$ and $\mathbf{k}_1\rightarrow\mathbf{k}_2'$. The minus sign comes in through the rules of second quantization (meaning of the Jordan-Wigner matrices as operators).

(B) and (C)

$$\frac{2\pi e^2}{\Omega k}\sum_n c_n^* c_n\left\{\frac{(n+1)(\mathbf{e}_{-\mathbf{k}}^0\cdot\boldsymbol{\alpha}_2)(\mathbf{e}_{-\mathbf{k}}^0\cdot\boldsymbol{\alpha}_1)}{\epsilon_1-\epsilon_1'-k}+\frac{n(\mathbf{e}_{\mathbf{k}}^0\cdot\boldsymbol{\alpha}_2)(\mathbf{e}_{\mathbf{k}}^0\cdot\boldsymbol{\alpha}_1)}{\epsilon_1-\epsilon_1'+k}+\text{Sym}\right\}-\text{Exch.} \quad (48)$$

(D) and (E)

$$\frac{2\pi e^2}{\Omega k}\sum_n c_n^* c_n\left\{\frac{(n+1)I_2I_1}{\epsilon_1-\epsilon_1'+k}+\frac{nI_2I_1}{\epsilon_1-\epsilon_1'-k}+\text{Sym}\right\}-\text{Exch.} \quad (49)$$

Here, I_1 stands for the matrix elements of the unit operator between the spin functions.

The processes F, G, H, I here lead to terms odd in the vector $\mathbf{k}$, and therefore vanish by integration of $\mathbf{k}$ over angles.

The terms "Sym" are different from the ones written down by the sign of the energy difference $\epsilon-\epsilon'$ in the denominators, as by the conservation of energy

$$\epsilon_2-\epsilon_2'=\epsilon_1'-\epsilon_1=\text{say } \epsilon. \quad (50)$$

(In the exchange terms, therefore, $\epsilon=\epsilon_1'-\epsilon_2$.) Taking the "Sym"-terms together with the others, we obtain

$$\frac{4\pi e^2}{\Omega}\cdot\frac{\left[\sum_{\mu=1,2}(\mathbf{e}_{-\mathbf{k}}^\mu\cdot\boldsymbol{\alpha}_2)(\mathbf{e}_{-\mathbf{k}}^\mu\cdot\boldsymbol{\alpha}_1)+\sum_n|c_n|^2\{(\mathbf{e}_{-\mathbf{k}}^0\cdot\boldsymbol{\alpha}_2)(\mathbf{e}_{-\mathbf{k}}^0\cdot\boldsymbol{\alpha}_1)-I_2I_1\}\right]}{\epsilon^2-k^2}-\text{Exch.} \quad (51)$$

For the dependency of the situation function on the numbers of photons, we have used up to here Eq. (17) only. By the normalization prescription $\sum_n|c_n|^2=1$ alone, and without any use of Eqs. (18)–(20), this time we can conclude that the matrix element for scattering is given by

$$\frac{4\pi e^2}{\Omega}\cdot\frac{\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_1-I_2I_1}{\epsilon^2-k^2}-\text{Exch}, \quad (52)$$

which is exactly Møller's expression for this matrix element.[14]

The above considerations are easily generalized to the case of a collision between one free electron and one electron bound in an atom. From this matrix element, Møller[15] and Bethe[16] have calculated cross sections for electron scattering, which were verified experimentally by Champion.[17]

APPLICATION 3. BREIT INTERACTION ENERGY

As a third application, we shall calculate the interaction energy of two electrons in two stationary states a and b in an atom. For this purpose we perform a second-order perturbation calculation with $\mathfrak{W}$ from Eq. (4), and subtract first the second-order perturbation energy for empty space. Indicating by $(i|\alpha|j)$ a contribution to the perturbation energy for a transition, in which an electron jumps from a state j into a state i, we obtain in this way, in symbolic notation (omitting resonance denominators and summation signs and not yet specifying numbers of photons),

$$(a|\alpha|a)(b|\alpha|b)+(b|\alpha|b)(a|\alpha|a)+(a|\alpha|\mathbf{P}_b)(\mathbf{P}_b|\alpha|a)+(b|\alpha|\mathbf{P}_a)(\mathbf{P}_a|\alpha|b)+(\mathbf{N}|\alpha|\mathbf{P}_{ab})(\mathbf{P}_{ab}|\alpha|\mathbf{N})-(\mathbf{N}|\alpha|\mathbf{P})(\mathbf{P}|\alpha|\mathbf{N}). \quad (53)$$

Here, the second factor of each product refers to the first transition; $\mathbf{P}$ and $\mathbf{N}$ denote arbitrary states of positive and negative energy, respectively, over which we later will have to sum; subscripts a or b mean states to be excluded from such summations.

From (53) we must subtract the self-energies of the two single electrons in states a and b, respectively (compare Eq. (29)):

$$(a|\alpha|\mathbf{P})(\mathbf{P}|\alpha|a)-(\mathbf{N}|\alpha|a)(a|\alpha|\mathbf{N})+(b|\alpha|\mathbf{P})(\mathbf{P}|\alpha|b)-(\mathbf{N}|\alpha|b)(b|\alpha|\mathbf{N}). \quad (54)$$

The difference between (53) and (54) gives the actual *interaction* energy:

$$(a|\alpha|a)(b|\alpha|b)+(b|\alpha|b)(a|\alpha|a)-(a|\alpha|b)(b|\alpha|a)-(b|\alpha|a)(a|\alpha|b), \quad (55)$$

which includes ordinary as well as exchange interaction[18] and can be written as the expectation value in the unperturbed state

$$\Psi_{ab}(1,2)=2^{-\frac{1}{2}}\{\phi_a(1)\phi_b(2)-\phi_b(1)\phi_a(2)\} \quad (56)$$

of an operator of the type

$$\mathrm{W}(1,2)=\alpha(1)\alpha(2)+\alpha(2)\alpha(1) \quad (57)$$

in the symbolic notation used in (53)–(55). In order to calculate here the exact meaning of the symbol $\alpha(1)\alpha(2)$, we have to make use of the proper matrix elements for the various photon processes A to I accompanying the transitions of the electron, and to write the corresponding energy differences in the denominators.

Whatever these photon processes are, we shall first write out here only contributions from electron processes that in the above symbolic notation would have been written as

$$(f|\alpha|a)(g|\alpha|b)+(g|\alpha|b)(f|\alpha|a),$$

[14] C. Møller, Zeits. f. Physik **70**, 786 (1931).
[15] C. Møller, Ann. d. Physik **14**, 531 (1932). See also W. Heisenberg, Ann. d. Physik **13**, 430 (1932) for a correction to Møller's first paper (reference 14).
[16] H. Bethe, Zeits. f. Physik **76**, 293 (1932).
[17] F. C. Champion, Proc. Roy. Soc. **A137**, 688 (1932).
[18] The above derivation of the interaction between two electrons was also suggested by Feynman (reference 11). It is difficult to obtain the exchange interaction by a procedure similar to that used by us in the previous chapter, as intermediate states with both electrons in the same state are naturally excluded.

where f and g stand for a and b or for b and a. By $\boldsymbol{\alpha}_1$ we shall denote here the matrix element

$$(g|\,\boldsymbol{\alpha}e^{i\mathbf{k}\cdot\mathbf{x}}|\,b)=\int \mathbf{dx}\phi_g^{\dagger}e^{i\mathbf{k}\cdot\mathbf{x}}\boldsymbol{\alpha}\phi_b,$$

and by $\boldsymbol{\alpha}_2$ we denote $(f|\,\boldsymbol{\alpha}e^{-i\mathbf{k}\cdot\mathbf{x}}|\,a)$. Similarly, I_1 stands for $(g|\,e^{i\mathbf{k}\cdot\mathbf{x}}|\,b)$, etc. Further, we write $\epsilon_1'-\epsilon_1$ or ϵ for the energy difference $E_g-E_b=E_a-E_f$ in units $\hbar c$. With this notation, the contributions from the various photon processes with one given $\mathbf{k}$ are given exactly by (47)–(49), where Sym interchanges the subscripts 1 and 2 (with $\epsilon_2-\epsilon_2'=\epsilon$ according to Eq. (50)), and where "−Exch" reminds of the fact that the terms with $f=b$, $g=a$ are to be subtracted from the terms with $f=a$, $g=b$. (The processes F, G, H, I again give contributions odd in $\mathbf{k}$ and therefore may be omitted.) Hence, we may proceed at once to Eqs. (51)–(52) with the new meaning of the sepmlos. We still have to sum this over $\mathbf{k}$.

From here we follow the usual procedure.[19] The Dirac equations for ϕ_b and ϕ_g give $-i\,\mathrm{div}(\phi_g^{\dagger}\boldsymbol{\alpha}\phi_b)=\epsilon\phi_g^{\dagger}\phi_b$; similarly for ϕ_a and ϕ_f. Thence,

$$-\mathbf{k}\cdot\boldsymbol{\alpha}_1=\epsilon I_1;\quad -\mathbf{k}\cdot\boldsymbol{\alpha}_2=\epsilon I_2. \tag{58}$$

Substituting $\mathbf{e}_{-\mathbf{k}}{}^0=-\mathbf{k}/k$ and (58) into (51), equating the "normalization integral" (21) to unity, and summing over $\mathbf{k}$, we get for the interaction energy

$$\mathrm{W}=\frac{e^2}{2\pi^2}\int\mathbf{dk}\left\{\frac{\sum_{\mu=1,2}(\mathbf{e}_{\mathbf{k}}{}^{\mu}\cdot\boldsymbol{\alpha}_2)(\mathbf{e}_{\mathbf{k}}{}^{\mu}\cdot\boldsymbol{\alpha}_1)}{\epsilon^2-k^2}+\frac{I_2I_1}{k^2}\right\}-\mathrm{Exch}. \tag{59}$$

This corresponds to the expression (55), and therefore is equal to the expectation value in the state (56) of the operator

$$W(1,2)=\frac{e^2}{2\pi^2}\int\frac{\mathbf{dk}}{k^2}e^{i\mathbf{k}\cdot\{\mathbf{x}(1)-\mathbf{x}(2)\}}\times\{1-\sum_{\mu=1,2}(\mathbf{e}_{\mathbf{k}}{}^{\mu}\cdot\boldsymbol{\alpha}(2))(\mathbf{e}_{\mathbf{k}}{}^{\mu}\cdot\boldsymbol{\alpha}(1))\} \tag{60}$$

corresponding to (57). Here, we made use of the definitions of $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, I_1, I_2. We also have replaced here ϵ^2-k^2 in the first denominator of (59) by the slightly different $(-k^2)$, thus neglecting the retardation in the transverse interaction. (In the other terms, it was properly taken into account.) We shall further write $\mathbf{x}_1$ for $\mathbf{x}(1)$, $\boldsymbol{\alpha}_2$ for $\boldsymbol{\alpha}(2)$, etc. Also, we put $\mathbf{x}_1-\mathbf{x}_2=\mathbf{r}$. By

$$\int\mathbf{dk}k^{-2}\exp(i\mathbf{k}\cdot\mathbf{r})=2\pi^2/r \tag{61}$$

and

$$\int\mathbf{dk}k^{-4}\exp(i\mathbf{k}\cdot\mathbf{r})(\boldsymbol{\alpha}_1\cdot\mathbf{k})(\boldsymbol{\alpha}_2\cdot\mathbf{k})=-(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla})(-\pi^2r), \tag{62}$$

we can write (60) then in the form

$$\mathrm{W}(1,2)=\{(e^2/r)(1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)+(e^2/2)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla})r\}=(e^2/r)\{1-\tfrac{1}{2}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-\tfrac{1}{2}r^{-2}(\boldsymbol{\alpha}_1\cdot\mathbf{r})(\boldsymbol{\alpha}_2\cdot\mathbf{r})\}, \tag{63}$$

which is familiar as Breit's expression for the electron interaction energy.[20]

DISCUSSION

We notice that in Applications 2 and 3, where the terms corresponding to creation or annihilation of photon pairs dropped out for reasons of symmetry, we finally made use of only the properties (16)–(17) of Ψ_0. This explains how one can get the correct Møller and Breit interactions even when wrongly assuming that in zero-order approximation no longitudinal or scalar photons would be present.

In the first application, however, we found that a wrong zero-order approximation definitely leads to an incorrect result, unless one makes two more "mistakes" in order to compensate this first one. On the other hand, with use of the "correct" zero-order approximation consistent with this kind of quantum electrodynamics, the "correct" infinite result for the electromagnetic self-energy of an electron is obtained only as one possible result among many other possibilities, since the procedure of imposing an auxiliary condition on the situation functional leads, through the method of "normalization" of such situation functional, to mathematical ambiguities.

[19] H. Bethe and E. Fermi, Zeits. f. Physik **77**, 296 (1932).

[20] G. Breit, Phys. Rev. **34**, 553 (1929); **36**, 383 (1930); **39**, 616 (1932).