assumed above to hold for simultaneous events. We obtain neglecting terms in $(t-t')^2$,

$$[\Pi_{N\sigma}(\mathbf{r},t),\Psi_{P\nu}(\mathbf{r}',t')]_+=g\psi(\mathbf{r})\beta_{\sigma\nu}\{(t-t')\delta(\mathbf{r}-\mathbf{r}')\}. \quad \text{(b)}$$

The quantity in the brace is, to first order, an invariant,† having the value zero for $|t-t'|<|\mathbf{r}-\mathbf{r}'|$. Consider the system in which the events $(\mathbf{r},t)$ and $(\mathbf{r}',t')$ transform into simultaneous events $(\bar{\mathbf{r}},\bar{t})$ and $(\bar{\mathbf{r}}',\bar{t})$. This will represent an infinitesimal Lorentz transformation, since it was assumed that t and t' were nearly equal. The expression on the left of (b) therefore takes on the transformed value

$$[\Pi_{N\sigma}(\bar{\mathbf{r}},\bar{t}),\Psi_{P\nu}(\bar{\mathbf{r}}',\bar{t})]_+=0.$$

The linearity property of the Lorentz transformation then permits us to write in the transformed system

$$[\bar{\Pi}_{N\sigma}(\bar{\mathbf{r}},\bar{t}),\bar{\Psi}_{P\nu}(\bar{\mathbf{r}}',\bar{t})]_+=0,$$

verifying covariance of this relation for infinitesimal Lorentz transformations. Since a finite transformation can be represented as a sequence of infinitesimal ones, the general covariance follows.

† It is the small argument expansion of the invariant D function of Jordan and Pauli.

The proofs of the covariance of the other relations follow similar patterns and will not be given. We remark only that it does not seem possible to quantize using other commutation rules than those assumed in the text to operate between two field quantities, each belonging to a *different* type of field.

Charge and Current

In the absence of external electromagnetic fields we define charge and current densities as follows:

$$\rho^{\mu}=-ie(\psi\pi-\psi^*\pi^*);$$
$$\mathbf{s}^{\mu}=ie(\psi\,\mathbf{grad}\psi^*-\psi^*\,\mathbf{grad}\psi);$$
$$\rho^{M}=-ie(\Pi_P\Psi_P);$$
$$\mathbf{s}^{M}=-ie(\Pi_P\boldsymbol{\alpha}\Psi_P).$$

Then, by virtue of the Hamiltonian density and commutation rules that have been assumed, we obtain the differential conservation law:

$$\partial(\rho^{\mu}+\rho^{M})/\partial t+\mathbf{div}(\mathbf{s}^{\mu}+\mathbf{s}^{M})=0.$$

Here, as always,

$$\partial\rho/\partial t=i[\textstyle\int\mathcal{H}d^3x,\rho].$$

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Correlated Probabilities in Multiple Scattering*

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The correlated probabilities of lateral and angular displacements of cloud-chamber tracks, resulting from multiple small-angle scattering, have been calculated for several cases of interest. The results are applicable to curvatures and other measurements taken in the presence of a magnetic field. The usual Gaussian-type scattering law has been used in the form of the fundamental correlated distribution function derived by Fermi. One direct application of this function is to the effect of scattering on angle measurements in nuclear "stars."

A "three-point formula" is derived, involving a correlated distribution of two successive lateral displacements with the resultant angular displacement. The distribution of scattering-produced curvatures, originally derived by Bethe, is calculated. A "four-point formula" allows a quantitative discussion of the tendency of scattered tracks to appear circular rather than skewed or S-shaped.

Finally, a formula is derived for the distribution of the successive chord angles for a track observed at several points, and used to discuss the best method of averaging the observations to reduce scattering-produced curvature errors. The error produced by scattering is not appreciably diminished by taking the best mean for an observation of the track at a large number of points, instead of a single observation of chord and sagitta (three points).

INTRODUCTION

THE multiple scattering of charged particles is of considerable importance for several types of cloud-chamber experiments, and has been treated by various authors.[1] Several problems of interest involving correlated probabilities of angular and lateral displacements may, in fact, be discussed using a fundamental distribution function due to Fermi.[2] It is the purpose of this paper to derive and discuss some of these results, in particular, those dealing with the measurement of track curvatures in magnetic fields.

* Research carried out at Brookhaven National Laboratory under the auspices of the AEC.

[1] H. A. Bethe, Phys. Rev. **70**, 821 (1946); R. Richard-Foy, J. de phys. et rad. **7**, 370 (1946). Other references are quoted in these papers.

[2] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

I. THE FUNDAMENTAL SOLUTION

We proceed to derive the fundamental distribution function[2,3]

$$W(y,\eta|x)dyd\eta=\frac{\lambda\sqrt{3}}{2\pi x^2}\exp\left\{-\frac{\lambda}{x}\left[\eta^2-\frac{3\eta y}{x}+\frac{3y^2}{x^2}\right]\right\}, \quad (1)$$

which gives the probability that a particle in traversing a distance x in a scattering material suffers a lateral displacement between y and $y+dy$ projected on a plane of observation containing x, and a net change of direction between η and $\eta+d\eta$ projected on the same plane.

The equation satisfied by this function may be derived

[3] We use the vertical bar | to separate given entities on the right from entities whose distribution is under consideration on the left, and shall in this way use the same function symbol W to denote several different distribution functions.

from the general Boltzmann equation,[4] but it is simpler to eliminate consideration of time and velocity and deal only with the deflections resulting from scattering.

We consider the angles involved to be small, i.e., $\sin\theta \cong \theta$ and $\cos\theta \cong 1$. We further ignore any change of energy of the particle during its passage through the material in question. We shall start with a three-dimensional distribution, which is easily reduced to the plane projection in the case of small angles.

Let $W(y, z, \eta, \zeta|x)dydzd\eta d\zeta$ denote the probability that at x, the coordinates of the particle are (y, z) and the projections of the tangent to the track in the yx and zx planes make angles η and ζ, respectively, with the x axis. Since the angles are small, $\eta^2+\zeta^2=\theta^2$, where θ is the angle between tangent and x axis.

$W(y, z, \eta, \zeta|x)$ may be derived from $W(y', z', \eta', \zeta'|x-dx)$ by considering that if no scattering occurs in dx, then $y-y'=dy=\eta dx$, $z-z'=dz=\zeta dx$, $\eta=\eta'$ and $\zeta=\zeta'$; if a scattering occurs from (η', ζ') to (η, ζ), the scattering probability is proportional to dx and transport terms ηdx and ζdx are second order and may be neglected; and the probability of more than one scattering in dx may also be neglected. The resulting equation is

$$\frac{\partial W}{\partial x}+\eta\frac{\partial W}{\partial y}+\zeta\frac{\partial W}{\partial z}=\int_0^{2\pi}d\beta\int_0^{\pi}F(\theta)\sin\theta d\theta \times[W(y, z, \eta-\theta\cos\beta, \zeta-\theta\sin\beta|x)-W(y, z, \eta, \zeta|x)], \quad (2)$$

where we have written $\eta-\eta'=\theta\cos\beta$, and $\zeta-\zeta'=\theta\sin\beta$. $F(\theta)$ is the probability per unit track length and unit solid angle of a single scattering through angle θ.

Making the usual Fokker-Planck approximation of expanding the expression in brackets in a Taylor's series and retaining no terms beyond the second derivatives, we find easily (setting $\sin\theta=\theta$)

$$\frac{\partial W}{\partial x}+\eta\frac{\partial W}{\partial y}+\zeta\frac{\partial W}{\partial z}=\frac{1}{\lambda}\left[\frac{\partial^2 W}{\partial \eta^2}+\frac{\partial^2 W}{\partial \zeta^2}\right], \quad (3)$$

where

$$1/\lambda=\tfrac{1}{4}\int_0^{\pi}\theta^3F(\theta)d\theta=\tfrac{1}{4}\langle\theta^2\rangle_{\text{Av}}. \quad (4)$$

λ is a characteristic length describing the amount of scattering, and $\langle\theta^2\rangle_{\text{Av}}$ is the mean square angle of scattering per unit path thickness and is given by[5]

$$\frac{4}{\lambda}=\langle\theta^2\rangle_{\text{Av}}=\frac{8\pi e^4Z^2Z'^2N}{p^2v^2}\ln\frac{150p}{mcZ^{1/3}}\ \text{cm}^{-1}. \quad (5)$$

Ze is the charge of the scattering nucleus, and N is the

[4] Bethe, Rose, and Smith, Proc. Am. Phil. Soc. **78**, 573 (1938).
[5] S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940), Eq. (9); and Phys. Rev. **58**, 36 (1940), Eqs. (2) and (4), for a Thomas-Fermi atom. For a Wentzel potential the coefficient 150 is to be replaced by 166; to agree with Bethe, (reference 1) and Rossi and Greisen, (reference 2), it is to be replaced by $137\theta_{\text{max}}$. Our λ is the w^2 of reference 2, Eq. (6).

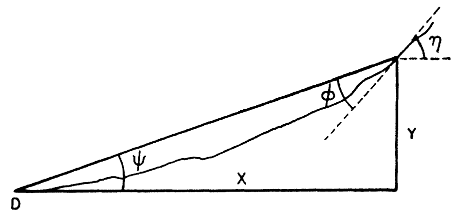

FIG. 1. Illustrating the basic distribution (1) or (10).

number of nuclei per cm^3; $Z'e$ is the charge of the scattered particle, while its mass, momentum, and velocity are denoted, respectively, by m, p, and v.

Equation (3) is easily separable into two equations for the two projections. Writing

$$W(y, z, \eta, \zeta|x)=W(y, \eta|x)W(z, \zeta|x), \quad (6)$$

we find

$$\left(\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)W(y, \eta|x)=\frac{1}{\lambda}\frac{\partial^2 W(y, \eta|x)}{\partial \eta^2}. \quad (7)$$

We take as the boundary condition

$$W(y, \eta|0)=\delta(y)\delta(\eta), \quad (8)$$

corresponding to a track passing through the origin tangent to the x axis.

The solution may be found in a straightforward way by applying a Laplace transform in x, and simultaneously Fourier transforms in y and η to Eq. (7). A first-order ordinary differential equation results that is easily soluble, and all inversion integrals can be carried out. The result is Eq. (1). That (1) is the solution of (6) is evident on substitution. That it is normalized is evident on integration over y and η, each from $-\infty$ to $+\infty$. That it satisfies (8) is evident on writing

$$Wdyd\eta=F(Y, \mu)dYd\mu=\left(\frac{\sqrt{3}}{2\pi}\right)dYd\mu\ \exp\{-\mu^2+3\mu Y-3Y^2\}, \quad (8a)$$

where $Y=y\lambda^{1/2}/x^{3/2}$, and $\mu=\eta\lambda^{1/2}/x^{1/2}$ are dimensionless variables. Since F is independent of x, the distribution in Y and μ is constant, i.e., the distribution has a constant *shape* as well as area. The scale factors vary with x so as to yield a very sharply peaked function of y and η as $x\rightarrow 0$, approaching $\delta(y)\delta(\eta)$.

In case the original direction of the track makes an angle η_0 with the x direction, we must replace η by $\eta-\eta_0$ and y by $y\cos\eta_0-x\sin\eta_0\cong y-\eta_0x$ in (1). This result is also obtained by writing $\delta(\eta-\eta_0)$ for $\delta(\eta)$ in (8). Hence,

$$W(y, \eta|x; 0, 0, \eta_0)=\left(\frac{\lambda\sqrt{3}}{2\pi x^2}\right)\exp\left\{-\frac{\lambda}{x}\left[(\eta-\eta_0)^2-\frac{3}{x}(\eta-\eta_0)(y-\eta_0x)+\frac{3}{x^2}(y-\eta_0x)^2\right]\right\}. \quad (9)$$

A convenient, symmetric way of writing Eq. (1) is to set $y/x=\psi$, the angle between the chord and the original

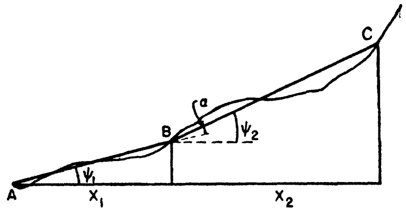

FIG. 2. Illustrating the three-point formula (15).

tangent (see Fig. 1), and $\phi=\eta-\psi$, the angle between the tangent at (x, y) and the chord. The probability of ψ and ϕ is thus,

$$U(\psi,\phi|x)d\psi d\phi=\left(\frac{\lambda\sqrt{3}}{2\pi x}\right)\times\exp\left\{-\frac{\lambda}{x}[\psi^2-\psi\phi+\phi^2]\right\}d\psi d\phi. \quad (10)$$

II. THE EFFECT OF A MAGNETIC FIELD

If a magnetic field H is added perpendicular to the plane of observation, and if ρ denotes the radius of curvature of the given particle in a vacuum, we can readily derive the diffusion equation for the scattered motion in the presence of a gas. In fact, in dx the particle is deflected magnetically by an angle $d\eta=dx\sec\eta/\rho\cong dx/\rho$, which shifts the distribution progressively toward larger values of η. The resulting equation becomes

$$\frac{\partial W_H}{\partial x}+\eta\frac{\partial W_H}{\partial y}+\frac{1}{\rho}\frac{\partial W_H}{\partial \eta}=\frac{1}{\lambda}\frac{\partial^2 W_H}{\partial\eta^2}.$$

Now, if we set $W_H(y,\eta|x)=f(y',\eta'|x)$, where $y'=y-(x^2/2\rho)$ and $\eta'=\eta-(x/\rho)$, we find readily

$$\frac{\partial f}{\partial x}+\eta'\frac{\partial f}{\partial y'}=\frac{1}{\lambda}\frac{\partial^2 f}{\partial\eta'^2}$$

which is just Eq. (7). Further, $W_H(y,\eta|0)=f(y',\eta'|0)=f(y,\eta|0)=\delta(y)\delta(\eta)$, so that $f=W(y,\eta|x)$ given by Eq. (1).

The curve $y_H=x^2/2\rho$, $dy_H/dx=x/\rho$ is a parabola, i.e., a circle within the small angle approximations, of curvature $1/\rho$, tangent to the original direction of the particle. That is, all the results of this paper for $H=0$ may be applied to the case $H\neq0$, if x is measured along the arc of a circle of radius ρ, and y and η are measured with respect to that circle. Since curvatures in the small angle approximations are linear functions of ordinates y, the magnetic curvature $1/\rho$ is simply to be added to the scattering produced curvatures.

III. CONSEQUENCES OF EQ. (1)

Inasmuch as $\eta^2-3\eta\psi+3\psi^2=(\eta-\frac{3}{2}\psi)^2+\frac{3}{4}\psi^2$, we find on integration over η,

$$W(y|x;0,0,0)=\left(\frac{3\lambda}{4\pi x^3}\right)^{\frac{1}{2}}\exp\left\{-\frac{3\lambda y^2}{4x^3}\right\} \quad (11a)$$

or

$$W(\psi|x;0,0,0)=\left(\frac{3\lambda}{4\pi x}\right)^{\frac{1}{2}}\exp\left\{-\frac{3\lambda}{4x}\psi^2\right\}, \quad (11b)$$

and using the relation derived from elementary probability theory that $W(y,\eta|x)=W(\eta|y,x)W(y|x)$,

$$W(\eta|x,y;0,0,0)=\left(\frac{\lambda}{\pi x}\right)^{\frac{1}{2}}\exp\left[-\frac{\lambda}{x}\left(\eta-\frac{3y}{2x}\right)^2\right]. \quad (12)$$

The mean value of η is $\langle\eta\rangle_{Av}=3y/2x=3\psi/2$, which implies that once a displacement of angle ψ has occurred, the particle is likely to have a direction that will increase this displacement, making evident the correlated nature of the distribution (1).

Similarly, we may integrate over y and find

$$W(\eta|x;0,0,0)=(\lambda/4\pi x)^{\frac{1}{2}}\exp(-\lambda\eta^2/4x). \quad (13)$$

This is essentially the Gaussian scattering formula of Williams.[6] It yields the result that $\langle\eta^2\rangle_{Av}=2x/\lambda$. Of course $\langle\zeta^2\rangle_{Av}$ is identical, so that $\langle\eta^2\rangle_{Av}+\langle\zeta^2\rangle_{Av}=4x/\lambda$ in agreement with (4). It is to be noted that the distribution in displacement $\psi=y/x$ is sharper by a factor $\sqrt{3}$ than the distribution in deflection η.

Equation (10) may be interpreted as the combined probability of a direction ψ at (0, 0) and a direction ϕ at $(x, 0)$, if the particle is known to pass through those two points. Integrated over either ψ or ϕ, it yields the form (11b), which may thus be interpreted as the likelihood of a scattering-produced error in measuring the direction of emergence of a nuclear reaction product observed in a cloud chamber or photographic emulsion.* The mean square error is $2x/3\lambda$; if, in addition, there is an experimental root-mean-square error of ϵ in the location of each end of the track, we have by the usual rule to add $2(\epsilon/x)^2$. The minimum value of $(2x/3\lambda)+2(\epsilon^2/x^2)$ occurs for $x^3=6\lambda\epsilon^2$.

IV. THREE-POINT FORMULAS

For consideration of tracks observed at three points, we calculate the probability that the particle pass through (x_1+x_2, y) in direction η if it is known to pass through (0, 0) and (x_1, y_1), using (9) or (11b):

$$\begin{aligned}W(y,\eta|x_1+x_2;x_1,y_1;0,0)&=\int_{-\infty}^{\infty}d\eta' W(\eta'|x_1,y_1;0,0)W(y,\eta|x_1+x_2;x_1,y_1,\eta')\\&=\left\{\frac{3\lambda}{2\pi x_2[x_2(4x_1+3x_2)]^{\frac{1}{2}}}\right\}\exp\left\{-\lambda\left[\frac{3(x_1+x_2)}{x_2(4x_1+3x_2)}\right.\right.\\&\quad\times\left(\eta+\frac{y_1x_2}{2x_1(x_1+x_2)}-\frac{(2x_1+3x_2)(y-y_1)}{2x_2(x_1+x_2)}\right)^2\\&\quad\left.\left.+\frac{3}{4(x_1+x_2)}\left(\frac{y-y_1}{x_2}-\frac{y_1}{x_1}\right)^2\right]\right\}. \quad (14)\end{aligned}$$

[6] E. J. Williams, Proc. Roy. Soc. A168, 531 (1939).

* I.e., $W(\psi|x;0,0,0)$ may be written as $W(\psi|x,0;0\,0,)$. A formula for $W(\psi|x,y;0,0)$ is also easily obtainable.

If we set $y_1/x_1=\eta_0$, and then let x_1 and y_1 approach zero, we get in the limit $W(y_2, \eta | x_2;\ 0,\ 0,\ \eta_0)$. If, however, $x_1=x_2=x_3$,

$$W(y, \eta | 2x;\, x, y_1;\, 0, 0)$$
$$=\left[\frac{3\lambda}{2\pi x^2(7)^{\frac{1}{2}}}\right]\exp\left\{-\frac{\lambda}{x}\left[\frac{6}{7}\left(\eta+\frac{1}{4}\frac{y_1}{x}-\frac{5}{4}\frac{y-y_1}{x}\right)^2 +\frac{3}{8}\left(\frac{y-y_1}{x}-\frac{y_1}{x}\right)^2\right]\right\}. \quad (14\text{a})$$

Integrating (14) over η, we obtain

$$W(y | x_1+x_2;\, x_1, y_1;\, 0, 0)$$
$$=\left[\frac{3\lambda}{4\pi x_2^2(x_1+x_2)}\right]^{\frac{1}{2}}\times\exp\left\{\frac{-3\lambda}{4(x_1+x_2)}\left(\frac{y-y_1}{x_2}-\frac{y_1}{x_1}\right)^2\right\}. \quad (15)$$

If we write $y_1/x_1=\psi_1$; $(y-y_1)/x_2=\psi_2$, the parenthesis in the exponent is $\psi_2-\psi_1$, or α, the angle between the track chords. (See Fig. 2.) Now, the curvature c of a circle through the three points is $\sin\alpha/\overline{AC}$, which in our approximation of small angles is $c=\alpha/(x_1+x_2)$. Hence, from (15) we may find the probability of observing a curvature c to $c+dc$ by measuring the coordinates of three points on a track:

$$P(c)dc=[3\lambda(x_1+x_2)/16\pi]^{\frac{1}{2}} \times\exp\{-3\lambda(x_1+x_2)c^2/16\}dc, \quad (16)$$

which is the result of Bethe,[1] and is independent of the ratio x_2/x_1 of the two parts of the track.

The mean square curvature is thus $\langle c^2\rangle_{\text{Av}}=8/3\lambda(x_1+x_2)$. If, as before, an r.m.s. error ϵ in measurement is made at each point, we may calculate the resulting curvature "error." Using a Gaussian error curve for the measurements and integrating over the "true" ordinates at each point, we get a distribution in c yielding

$$\langle c^2\rangle_{\text{Av}}=\frac{8}{3\lambda(x_1+x_2)}+\frac{8\epsilon^2}{(x_1+x_2)^2}\left(\frac{1}{x_1^2}+\frac{1}{x_1x_2}+\frac{1}{x_2^2}\right). \quad (17)$$

Equation (16) may be written as the probability of finding a sagitta u at x_1, when the chord of the track has length x_1+x_2, since $c=2u/x_1x_2$:

$$P(u)du=\left[\frac{3\lambda(x_1+x_2)}{4\pi x_1^2x_2^2}\right]^{\frac{1}{2}}\exp\left\{\frac{-3\lambda(x_1+x_2)u^2}{4x_1^2x_2^2}\right\}du.^7$$

We may now derive, in a similar manner as for Eq. (12), the distribution of track directions at point C when a given y or curvature c is observed. We set $y_1=0$ for simplicity.

$$W(\eta | x_1+x_2, y;\, x_1, 0;\, 0, 0)=\left[\frac{3\lambda(x_1+x_2)}{\pi x_2(4x_1+3x_2)}\right]^{\frac{1}{2}}$$
$$\times\exp\left\{\frac{-3\lambda(x_1+x_2)}{x_2(4x_1+3x_2)}\cdot\left(\eta-\frac{2x_1+3x_2}{2x_2(x_1+x_2)}y\right)^2\right\}. \quad (18)$$

The mean value is

$$\langle\eta\rangle_{\text{Av}}=\frac{y}{x_2}\left(\frac{2x_1+3x_2}{2x_1+2x_2}\right)=\psi_2\left[1+\frac{x_2}{2(x_1+x_2)}\right]$$
$$=(\tfrac{3}{4}x_2+\tfrac{1}{2}x_1)c. \quad (19)$$

A circular track through the three points would have a slope $\eta_c=(x_2+\frac{1}{2}x_1)c$ at (x_1+x_2, y). We see then that once a track has been "curved" by scattering, it is likely to maintain this same direction of curvature, but of a lesser amount (Fig. 3). Further discussion is postponed to the next section.

V. FOUR-POINT FORMULAS

In a similar fashion to the derivation of Eq. (14), that expression along with (1) may be combined to find a distribution in y and η at $x=x_1+x_2+x_3$, when the track is known to pass through $(0, 0)$, $(x_1, 0)$, and (x_2, y'). For simplicity, write $y'/x_2=\psi_2$ and $(y-y')/x_3=\psi_3$, and let Δ signify $4(x_1+x_2)(x_2+x_3)-x_2^2$. The result is

$$W(y, \eta | x_1+x_2+x_3;\, x_1+x_2, y';\, x_1, 0;\, 0, 0)$$
$$=\left[\frac{9\lambda^2(x_1+x_2)}{4\pi^2x_3^3(\Delta-x_1x_3-x_2x_3)}\right]^{\frac{1}{2}}\cdot\exp\left\{\frac{-3\lambda\Delta}{4x_3(\Delta-x_1x_3-x_2x_3)}\right.$$
$$\times\left[\eta-\frac{\psi_3}{\Delta}(\Delta+2x_1x_3+2x_2x_3)+\frac{\psi_2}{\Delta}(2x_1x_3+3x_2x_3)\right]^2$$
$$\left.-\frac{3\lambda(x_1+x_2)}{\Delta}\left[\psi_3-\psi_2\frac{2x_1+3x_2}{2(x_1+x_2)}\right]^2\right\}. \quad (20)$$

We then find readily

$$W(y | x_1+x_2+x_3;\, x_1+x_2, y';\, x_1, 0;\, 0, 0)$$
$$=\left[\frac{3\lambda(x_1+x_2)}{\pi x_3^2\Delta}\right]^{\frac{1}{2}}\exp\left\{\frac{-3\lambda(x_1+x_2)}{\Delta}\times\left[\psi_3-\frac{\psi_2(2x_1+3x_2)}{2(x_1+x_2)}\right]^2\right\}. \quad (21)$$

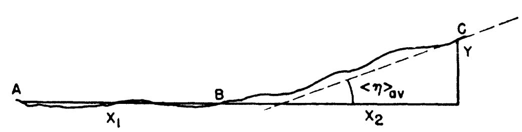

FIG. 3. Illustrating the continuation of a "curvature" once started.

[7] See R. Richard-Foy (reference 1).

This formula yields Eq. (14) when $\psi_3=\eta$ and $x_3\to 0$. One sees immediately that $\langle\psi_3\rangle_{\text{Av}}$ is the same as $\langle\eta\rangle_{\text{Av}}$ for the three-point formula (Eq. (19)), as it evidently should be. An expression for the distribution in η when the track is observed to pass through four points is given by the ratio of (20) to (21).

Equation (21) is more useful if curvatures are used instead of ψ_3 and ψ_2. Let $c_a=2\psi_2/(x_1+x_2)$ be the curvature for points ABC (Fig. 4) and $c_b=2(\psi_3-\psi_2)/(x_2+x_3)$ be the curvature for points BCD. Then we have the distribution in c_b when c_a is given,

$$P(c_b|c_a;x_1,x_2,x_3)dc_b=(x_2+x_3)\left[\frac{3\lambda(x_1+x_2)}{\pi\Delta}\right]^{\frac{1}{2}}\exp\left\{\frac{-3\lambda(x_1+x_2)(x_2+x_3)^2}{4\Delta}\times\left[c_b-\frac{x_2c_a}{2(x_2+x_3)}\right]^2\right\}dc_b. \quad (22)$$

We see that $\langle c_b\rangle_{\text{Av}}=x_2c_a/2(x_2+x_3)$; if $x_2=x_3$, $\langle c_b\rangle_{\text{Av}}=c_a/4$, which bears out the assertion at the end of Section III. If, further, $x_2=x_3=x_1=x$, $\langle[c_b-(c_a/4)]^2\rangle_{\text{Av}}=5\lambda x/4$, or just slightly less than the value $4\lambda x/3$ for Eq. (16).

We may calculate the probability that $c_b>0$, and so obtain the probability that the track be of C-type rather than S-type[1] for a given value of c_a. For $x_1=x_2=x_3=x$, we have

$$P_c=\left(\frac{2\lambda x}{5\pi}\right)^{\frac{1}{2}}\int_{-c_a/4}^{\infty}\exp\left(\frac{-2\lambda xc^2}{5}\right)dc$$
$$=\tfrac{1}{2}+erf\left(\frac{c_a}{(30)^{\frac{1}{2}}c_{\text{r.m.s.}}}\right). \quad (23)$$

The result is most easily expressed as a function of $c_a/c_{\text{r.m.s.}}$ where $c_{\text{r.m.s.}}=4\lambda x/3$ is the root-mean-square curvature for three points, from Eq. (16), and is generally readily available. Figure 5 gives the resulting graph.

Using (22) and (16), we can find the combined probability of c_a and c_b,

$$P(c_a,c_b|x_1,x_2,x_3)=P(c_b|c_a;x_1,x_2,x_3)P(c_a|x_1,x_2)$$
$$=\left[\frac{3\lambda(x_1+x_2)(x_2+x_3)}{8\pi\Delta^{\frac{1}{2}}}\right]\exp\left\{\frac{-3\lambda(x_1+x_2)(x_2+x_3)}{4\Delta}\times[(x_1+x_2)c_a^2-x_2c_ac_b+(x_2+x_3)c_b^2]\right\}. \quad (24)$$

The symmetry of this expression shows that large curvatures are equally likely to occur at either end of the track. Let us, however, study the distribution in $r=c_b/c_a$. If $r>0$, the curve is C-type: $r<0$ yields an S-type curve. The curvature increases or decreases accordingly as $r>1$ or $r<1$. Using c_a and r as independent variables, we can integrate over c_a and find

$$p(r)dr=\frac{dr}{\pi}\frac{d}{dr}\tan^{-1}\left[\frac{2(x_2+x_3)}{\Delta^{\frac{1}{2}}}\left(r-\frac{x_2}{2(x_1+x_2)}\right)\right]. \quad (25)$$

$\langle r\rangle_{\text{Av}}=x_2/2(x_1+x_2)$ which becomes $\frac{1}{4}$ if $x_2=x_3$. The probability that $r>0$ is

$$p_c=\tfrac{1}{2}+(1/\pi)\tan^{-1}(x_2/\Delta^{\frac{1}{2}}); \quad (26)$$

if $x_1=x_2=x_3$, $p_c=0.58$. The probability that $r>1$ is $\frac{1}{2}-(1/\pi)\tan^{-1}(3/\sqrt{15})=\frac{1}{2}p_c=0.29$. That is, 58 percent of tracks curved by scattering are of C-type, and are equally divided between increasing and decreasing curvatures.

Another symmetric way of writing (24) is to use $c_m=(c_a+c_b)/2$ and $D=c_b-c_a$ as independent variables. The bracket in the exponent becomes

$$(x_1+x_2+x_3)c_m^2+Dc_m(x_3-x_1)+\tfrac{1}{4}(x_1+3x_2+x_3)D^2,$$

and it is evident that if $x_1=x_3$, or the track is divided symmetrically, the distributions in c_m and D are mutually independent. In fact, (24) may be written, setting $x_1=x_3$, as the product of two normalized probabilities:

$$P(c_m,D)dc_mdD=P_1(c_m)dc_mP_2(D)dD,$$

$$P_1(c_m)=\tfrac{1}{2}(x_1+x_2)\left[\frac{3\lambda}{\pi(2x_1+3x_2)}\right]^{\frac{1}{2}}\times\exp\left\{\frac{-3\lambda(x_1+x_2)^2c_m^2}{4(2x_1+3x_2)}\right\}, \quad (27)$$

$$P_2(D)=\tfrac{1}{4}(x_1+x_2)\left[\frac{3\lambda}{\pi(2x_1+x_2)}\right]^{\frac{1}{2}}\times\exp\left\{\frac{-3\lambda(x_1+x_2)^2D^2}{16(2x_1+x_2)}\right\}.$$

An important consequence is that *selection of tracks according to a criterion of symmetry, with other factors being equal*—i.e., with D smaller than some limit determined by the errors of measurements—*results in no narrowing of the distribution of scattering-produced curvatures over that obtained in including unsymmetrical tracks.* This does not preclude the use of a symmetry criterion

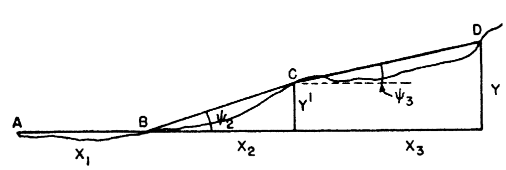

FIG. 4. Illustrating the four-point formula (20).

as a test for freedom from *turbulence* or *convection* errors in a cloud chamber, nor of the use of a mean skewness of a whole set of tracks as a measure of the amount of scattering (see below).

$P_1(c_m)$ in (27) yields the distribution of scattering-produced curvature when the track is observed at four points and the two curvatures are weighted equally. If $x_2=x_1=x$, we find

$$\langle c_m^2\rangle_{\mathrm{Av}}=5/6\lambda x=5/2\lambda l, \tag{28}$$

where $l=3x$ is the total track length. The three-point formula (16) yields $\langle c^2\rangle_{\mathrm{Av}}=8/3\lambda l$, so that the mean square curvature for four points is smaller by only $\frac{15}{16}$. This result partially justifies the common experimental practice of measuring curvatures by means of a chord and sagitta, rather than by observation of the coordinates of more than three points. At least, that part of the error due to multiple scattering is not materially reduced by more measurements. (See also Section VI below.)

$P_2(D)$ yields a distribution in c_b-c_a, which is independent of any magnetic curvature (Section II). Hence, measurements of D in the presence of a magnetic field for a sufficient number of monoenergetic particles should yield a useful check on the mean scattering (i.e., on the value of λ) and may serve to distinguish scattering-produced skewness from turbulence and convection effects.

$P_1(c_m)$ and $P_2(D)$ may be used jointly to decide the likelihood of error in meson mass-measurements from curvature determinations of apparently perfectly circular tracks of knock-on-electrons.[8]

A modification of the four-point formula is the "internal" probability of observing y' at x_1+x_2 when the particle is known to pass through $(0, 0)$, $(x_1, 0)$, and $(x_1+x_2+x_3, y)$. The result is best expressed in terms of the curvature c' for points B, C, D (Fig. 4) when c is the (known) curvature for A, B, D.

$$P_{int}(c')dc'=\tfrac{1}{2}(x_2+x_3)\left[\frac{3\lambda(x_1+x_2+x_3)}{\pi\Delta}\right]^{\frac{1}{2}} \exp\left\{\frac{-3\lambda(x_1+x_2+x_3)}{16\Delta}\times\left[2(x_2+x_3)c'-(x_2+2x_3)c\right]^2\right\}. \tag{29}$$

We have $\langle c'\rangle_{\mathrm{Av}}=(x_2+2x_3)c/(2x_2+2x_3)=3c/4$ if $x_2=x_3$. This is another indication of the extent to which tracks will appear circular as a result of multiple scattering.

VI. A FORMULA FOR ANY NUMBER OF POINTS

For more than four points on a track, the probability formulas are quite unwieldy. However, if the track is divided into N equal segments x, and only the successive chord angles $\alpha_1, \alpha_2, \cdots \alpha_{N-1}$ are considered (see Fig. 6), an explicit probability distribution may be written down.

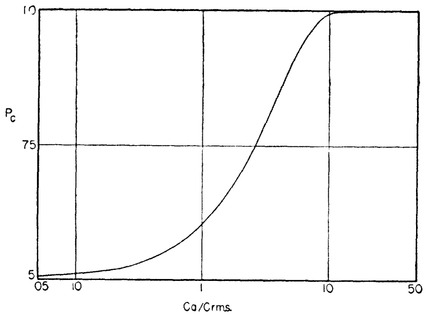

FIG. 5. The probability of C-type tracks as a function of the curvature c_a for the first two-thirds of the track. $c_{\mathrm{r.m.s.}}$ is the root-mean-square curvature for the same track length.

We shall start with Eq. (10) for $U(\psi, \phi|x)$, written as a double Fourier transform,

$$U(\psi,\phi|x)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}ds \times\exp\left\{-it\psi-is\phi-\frac{x}{3\lambda}(t^2+st+s^2)\right\}. \tag{30}$$

This formula is then applied to each of the N sections. We set

$$\phi_1=\alpha_1-\psi_2;\quad \phi_2=\alpha_2-\psi_3;\quad \cdots\phi_{N-1}=\alpha_{N-1}-\psi_N,$$

and integrate the product of N U's over $\psi_1, \psi_2\cdots\psi_N$, and ϕ_N. All these integrations may be readily carried out by use of the fundamental Fourier integral theorem, yielding

$$W(\alpha_1,\alpha_2,\cdots\alpha_N)=\int_{-\infty}^{\infty}\frac{ds_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{ds_{N-1}}{2\pi} \times\exp\left\{-i(s_1\alpha_1+\cdots+s_{N-1}\alpha_{N-1})-\frac{x}{3\lambda}(2s_1^2+s_1s_2 +2s_2^2+s_2s_3+2s_3^2+\cdots+s_{N-2}s_{N-1}+2s_{N-1}^2)\right\}.$$

A general and well-known theorem on generalized Gaussian functions[9] states that

$$\int_{-\infty}^{\infty}ds_1\cdots\int_{-\infty}^{\infty}ds_n\exp[-\mu A_{jk}s_js_k-is_j\alpha_j] =\left(\frac{\pi}{\mu}\right)^{n/2}[\det|A_{jk}|]^{-\frac{1}{2}}\exp\left\{\frac{-A_{lm}^{-1}\alpha_l\alpha_m}{4\mu}\right\},$$

where $\det|A_{jk}|$ is the determinant of the matrix $\|A_{jk}\|$;

[8] Leprince-Ringuet, Gorodetzky, Nageotte, and Richard-Foy, Phys. Rev. **59**, 460 (1941); L. Leprince-Ringuet and M. Lhéritier, J. de phys. et rad. **7**, 66 (1946).

[9] See, e.g., M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945), Section 6a and Appendix I.

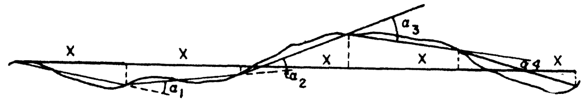

FIG. 6. Illustrating the many-point formula (31).

$\|A_{lm}^{-1}\|$ is the matrix reciprocal to $\|A_{jk}\|$, and the summation convention for double indices is used.

We find, thus,

$$W(\alpha_1\cdots\alpha_{N-1})=\left(\frac{3\lambda}{2\pi x}\right)^{(N-1)/2}\Delta_{N-1}^{-\frac{1}{2}}\times\exp\left\{\frac{-3\lambda}{2x}A_{lm}^{-1}\alpha_l\alpha_m\right\},\quad(31)$$

where $\|A_{jk}\|$ is the $(N-1)$th order matrix:

$$\begin{Vmatrix} 4 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 4 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 4 & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 4 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdot & 1 & 4 \end{Vmatrix}\quad(32)$$

and Δ_{N-1} is its determinant. To evaluate Δ_N, note first that

$$\Delta_N=4\Delta_{N-1}-\Delta_{N-2};\quad(33)$$

second that

$$\sinh Nu=2\cosh u\sinh(N-1)u-\sinh(N-2)u;$$

and third that if $\cosh u=2$,

$$\Delta_1=\sinh 2u/\sinh u\quad\text{and}\quad\Delta_2=\sinh 3u/\sinh u.$$

Hence, it follows that

$$\Delta_N=\sinh(N+1)u/\sinh u\quad\text{with}\quad u=\cosh^{-1}2.\quad(34)$$

The recursion relation (33) for Δ_N may be used to show readily that

$$A_{lm}^{-1}=A_{ml}^{-1}=(-1)^{l+m}\Delta_{l-1}\Delta_{N-m-1}/\Delta_{N-1},\quad l\leq m.\quad(35)$$

A_{lm}^{-1} is, of course, a rational fraction, and may be readily evaluated for any N, using $\Delta_1=4$, $\Delta_2=15$, and the recursion formula (33).

We shall use Eq. (31) here only to calculate the best method of measuring a curvature when N segments of track are taken, and to find the resulting mean square scattering-produced curvature.[10] Any method of curvature measurement is equivalent to a weighted mean value of the $N-1$ curvatures for adjacent segments. Let us calculate the probability distribution of

$$c_m=\sum_{j=1}^{N-1}b_jc_j=-\frac{1}{x}\sum_{j=1}^{N}b_j\alpha_j,\quad\text{with}\quad\sum_j b_j=1.$$

We shall use the Fourier representation of the Dirac δ-function $\delta[c_m-\sum b_j\alpha_j/x]$, and integrate $W(\alpha_1\cdots\alpha_{N-1})$ over all the α's.

$$\begin{aligned}P_N(c_m)&=\frac{1}{2\pi(\Delta_{N-1})^{\frac{1}{2}}}\left(\frac{3\lambda}{2\pi x}\right)^{(N-1)/2}\int_{-\infty}^{\infty}dt\\&\quad\times\exp(itc_m)\int_{-\infty}^{\infty}d\alpha_1\cdots\int_{-\infty}^{\infty}d\alpha_{N-1}\\&\quad\times\exp\{-(3\lambda A_{kj}^{-1}\alpha_k\alpha_j+2itb_j\alpha_j)/2x\}\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\exp\left\{itc_m-\frac{t^2}{6\lambda x}A_{jk}b_jb_k\right\}\\&=\left[\frac{3\lambda x}{2\pi A_{jk}b_jb_k}\right]^{\frac{1}{2}}\exp\left\{-\frac{3\lambda xc_m^2}{2A_{jk}b_jb_k}\right\},\end{aligned}\quad(36)$$

so that

$$\langle c_m^2\rangle_{\mathrm{Av}}=\frac{1}{3\lambda x}A_{jk}b_jb_k=\frac{NA_{jk}b_jb_k}{3\lambda l}$$

if l is the total track length. The best choice of the b's is thus that which will minimize $S=A_{jk}b_jb_k$ subject to $\sum_k b_k=1$.

We shall thus minimize $S'=A_{jk}b_jb_k-\mu\sum_k b_k$, where μ is an undetermined multiplier.

$$\partial S'/\partial b_k=0=2A_{jk}b_j-\mu.$$

Multiplying by b_k and summing, we find that $\mu=2S$.

TABLE I. The quantities $S=\sum_{jk}A_{jk}b_jb_k$ and b_j needed for the "best" way of measuring a mean scattering-produced curvature from N track segments (see Section VI), and the ratio of the resulting mean square curvature $\langle c_m^2\rangle_{\mathrm{Av}}$ for N segments to the value $8/3\lambda l$ for 2 segments.

N	2	3	4	5	6	10
S	4	5/2	7/4	19/14	52/47	724/1137
b_j: $j=1$	1	1/2	3/8	4/14	11/47	153/1137
$j=2$	—	1/2	2/8	3/14	8/47	112/1137
3	—	—	3/8	3/14	9/47	123/1137
4	—	—	—	4/14	8/47	120/1137
5	—	—	—	—	11/47	121/1137
6	—	—	—	—	—	120/1137 etc.
$3\lambda l/8\langle c_m^2\rangle_{\mathrm{Av}}=NS/8$	1.00	0.938	0.875	0.848	0.830	0.796

[10] Eq. (31) is used by the author for the distribution of $\Sigma\alpha_k^2$, in Phys. Rev. **75**, 1763 (1949).

Hence, $S=A_{jk}b_j$. This system of equations may be solved, yielding

$$b_k=S\sum_m A_{km}^{-1}. \tag{37}$$

Summing over k, we find also

$$1/S=\sum_{k,\,m} A_{km}^{-1}. \tag{38}$$

The sums in (37) and (38) may be readily carried out with the aid of two formulas derived from (33)

$$\Delta_N=\Delta_j\Delta_{N-j}-\Delta_{j-1}\Delta_{N-j-1};\quad 0\leq j\leq N \tag{39}$$

$$\sum_0^n(-1)^m\Delta_m=\tfrac{1}{6}(-1)^n[5\Delta_n-\Delta_{n-1}+(-1)^n]. \tag{40}$$

Equation (39) reduces to (33) when $j=1$ or $N-1$, and is easily proved for other values of j by induction. Equation (40) results from summing (33).

We find finally from (38) and (37), respectively,

$$S=\frac{3\lambda l}{N}\langle c_m{}^2\rangle_{\text{Av}}=\frac{18\Delta_{N-1}}{[(3N+2)\Delta_{N-1}-\Delta_N+(-1)^N]}, \tag{41}$$

$$b_k=\frac{3[(-1)^{N-1}\Delta_{N-1}-(-1)^{N-k-1}\Delta_{N-k-1}-(-1)^{k-1}\Delta_{k-1}]}{(3N+2)(-1)^{N-1}\Delta_{N-1}+(-1)^N\Delta_N-1}. \tag{42}$$

Table I gives some values of S and b_k for a few values of N. For large N,

$$1/S\simeq N/6-\sqrt{3}/18=N/6-0.0962. \tag{43}$$

We have, therefore,

$$\langle c_m{}^2\rangle_{\text{Av}}\simeq 2/\lambda l[1-(0.577/N)]^{-1}. \tag{44}$$

Comparing with (17), we see that the greatest improvement possible in reduction of scattering-produced curvature errors by taking more measurements on a single track is a reduction of 25 percent in $\langle c_m{}^2\rangle_{\text{Av}}$. For $N=10$, the improvement over (17) is by a factor 0.796. Table I includes some values of $3\lambda l/8\langle c_m{}^2\rangle_{\text{Av}}$.

Another method of reducing $\langle c_m{}^2\rangle_{\text{Av}}$ would be to use $P_1(c_m)$ in (27), with the most favorable ratio of x to x_2. This turns out to be zero, for which case (27) yields $\langle c_m{}^2\rangle_{\text{Av}}=2/\lambda l$, the same as for (44) with $N\rightarrow\infty$. We have the surprising result that the same r.m.s. error as for a large number of track divisions would be obtained if the directions of the tangents at the two ends of the track could be accurately measured.

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