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Electrostatic Scattering of Neutrons*

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A weak force of attraction between electrons and thermal neutrons is indicated in recent experiments by Havens, Rabi, and Rainwater, and by Fermi and Marshall. According to meson theory this would be expected, since the neutron is considered as one charge state of a nucleon (the proton is the other state) which is coupled to a meson field. The attraction is then interpreted as an electrostatic scattering of the neutron which exists part of the time as a proton and meson. We perform a third order perturbation calculation in the approximation of weak coupling between mesons and nucleons. Neutrons and protons are here treated as Dirac particles which are coupled to a meson field of spin zero. The results indicate that the observed interaction is suitably described in terms of the meson field.

1. INTRODUCTION

If the nuclear forces are to be accounted for wholly or partly in terms of the exchange of charged mesons between nuclear particles then it will follow that the neutron will be subject to a deflection when passing through an inhomogeneous electric field. This is brought about by the action of the field on the proton and mesons which exist in virtual states in the neighborhood of the neutron. Attempts to detect such interaction were made by Havens, Rabi, and Rainwater[1] and by Fermi and Marshall,[2] who studied the scattering of neutrons in lead and in xenon, respectively. The scattering arises predominantly from three sources:

(a) Scattering by the specific nuclear forces;
(b) Electric scattering of the type in question due to the nuclear charge;
(c) Electric scattering due to the atomic electrons.

The scattering (a) is strongly predominant. Nevertheless, the interference which exists between (a) and the electric scattering makes it not unreasonable to look for experimental effects of the latter—effects which would surely be far below the present limits of experimental sensitivity were not (a) simultaneously present.

Scattering of slow (thermal) neutrons by nuclear forces will be spherically symmetric and will, in general, have no dependence on wave-length. On the other hand, thermal neutrons have a wave-length comparable with atomic dimensions, so that scattering of the type (c) will show a marked wave-length dependence and will not be spherically symmetric. Thus if one can extract from the observed scattering any part which varies with wave-length and scattering angle in a manner consistent with the atomic form factor, then one has a measure of the effect being studied.

A theoretical estimate of the magnitude of the effect has been given by Fermi and Marshall.[2] Their calculation is not based on any specific meson theory, but makes use of qualitative features of meson theories in general. Their numerical result might be expected to be in order-of-magnitude agreement with that obtained from any particular formulation. Since the experiment in question is such a critical test for the validity of the meson field hypothesis it was felt desirable to have at hand theoretical values as precise as can be derived.

The present calculation is based on the assumption that protons and neutrons (nucleons) obey the Dirac equation and that the mesons have spin zero (scalar). The assumption is frequently made that the massiveness of nucleons makes it possible to treat them as undeviated by the acts of virtual emission and reabsorption of mesons. Our results show that the latter assumption would lead to a serious quantitative error in this problem. We also find that a change in the equations of motion of the nucleon would lead to a small but appreciable change in the result.

* The method and results of this calculation were first presented at the November, 1948 meetings of the American Physical Society in Chicago (Phys. Rev. **75**, 341A (1949).

[1] W. W. Havens, I. I. Rabi, and L. J. Rainwater, Phys. Rev. **72**, 634 (1947).

[2] E. Fermi and L. Marshall, Phys. Rev. **72**, 1139 (1947).

The work is divided into the following sections:

2. Choice of interaction function, and quantization of the theory.
3. Display of transition schemes and of scattering matrix element.
4. Calculation of the change in neutron scattering due to the electron-neutron interaction.
5. Comparison with experiment.
6. Calculation of neutron scattering with spin flip due to the electron-neutron interaction.
7. Discussion.

2. QUANTIZATION

In the presence of external electromagnetic fields the Lagrangian density is taken as

$$\mathcal{L}=-c^2[(\partial\psi^*/\partial x_\nu+ie\phi_\nu\psi^*)(\partial\psi/\partial x_\nu-ie\phi_\nu\psi)+\mu^2\psi^*\psi] +\hbar ci\Psi_P^+[\gamma^\nu(\partial/\partial x_\nu-ie\phi_\nu)+M]\Psi_P+\hbar ci\Psi_N^+[\gamma^\nu(\partial/\partial x_\nu) +M]\Psi_N+igc(\Psi_P^+\psi\Psi_N+\Psi_N^+\psi^*\Psi_P). \quad (1)$$

Henceforth $\hbar$ and c will be set equal to one. The components of x are (x,y,z,it). ψ is the wave field of a charged scalar meson. Ψ_P and Ψ_N are the spinor wave fields of protons and neutrons. μ and M are the reciprocal Compton wave-lengths of meson and nucleon, respectively. γ^ν are the four Dirac matrices; the first three are $\gamma^k=i\alpha^k\beta$, while $\gamma^4=\beta$. Also $\Psi^+=i\Psi^*\beta$. The constant g determines the strength of interaction, having the dimensions of an electric charge. ϕ_ν is the externally applied 4-vector potential, having components $(\mathbf{A},i\phi)$, where $\mathbf{A}$ and ϕ are vector and scalar potentials, respectively.

Canonically conjugate variables are

$$\pi=\dot\psi^*-ie\phi\psi^*,\quad \Pi_P=i\Psi_P^*,$$
$$\pi^*=\dot\psi+ie\phi\psi,\quad \Pi_N=i\Psi_N^*.$$

The Hamiltonian density is

$$\mathcal{H}=(\pi^*\pi+\mathbf{grad}\psi^*\cdot\mathbf{grad}\psi+\mu^2\psi^*\psi) -i\Psi_P^*(\boldsymbol{\alpha}\cdot\mathbf{grad}+iM\beta)\Psi_P-i\Psi_N^*(\boldsymbol{\alpha}\cdot\mathbf{grad} +iM\beta)\Psi_N+g(\Psi_P^*\beta\Psi_N\psi+\Psi_N^*\beta\Psi_P\psi^*) +ie\phi(\pi^*\psi^*-\pi\psi)+ie\mathbf{A}\cdot(\psi^*\mathbf{grad}\psi-\psi\mathbf{grad}\psi^*) +e^2(\psi^*\psi)\mathbf{A}^2+e\Psi_P^*(-\boldsymbol{\alpha}\cdot\mathbf{A}+\phi)\Psi_P. \quad (2)$$

This expression is conveniently split as follows:

$$\mathcal{H}={}^0\mathcal{H}+{}^g\mathcal{H}+{}^e\mathcal{H}. \quad (3)$$

${}^0\mathcal{H}$ represents the first three terms of (2). ${}^g\mathcal{H}$, the fourth term, gives the meson-nucleon coupling. ${}^e\mathcal{H}$, the last four terms, gives the coupling of the charges with the electromagnetic field.

We wish to quantize the fields so that the meson field satisfies Bose statistics, the nucleon fields satisfy Fermi statistics. To achieve this it is necessary that the following commutation relations be maintained for fields measured simultaneously at two points $\mathbf{r}$ and $\mathbf{r}'$:

$$[\pi(\mathbf{r}),\psi(\mathbf{r}')]=[\pi^*(\mathbf{r}),\psi^*(\mathbf{r}')]=-i\delta(\mathbf{r}-\mathbf{r}'), \quad (4)$$

whereas all other pairs of the meson functions commute. Likewise

$$[\Pi_{P\sigma}(\mathbf{r}),\Psi_{P\tau}(\mathbf{r}')]_+=[\Pi_{N\sigma}(\mathbf{r}),\Psi_{N\tau}(\mathbf{r}')]_+=i\delta(\mathbf{r}-\mathbf{r}')\delta_{\sigma\tau}. \quad (5)$$

The subscripts σ and τ refer to particular components of the Dirac spinors. The (+) subscript indicates the anti-commutator. All other pairs of proton functions have vanishing anti-commutators, as do all other pairs of neutron wave functions.

So far the requirements are the usual ones. But now we also demand that all meson field quantities *commute* with all proton as well as all neutron field quantities; and finally that all proton field quantities *anticommute* with all those for the neutron field. The appearance of the coupling terms in the canonical equations of motion makes these requirements non-trivial.

The possibility of quantizing the theory in a manner consistent with the above assumed commutation relations depends on the possibility of expressing them in relativistically covariant form. That is, the values of the various commutators and anticommutators corresponding to simultaneous events in one Lorentz frame should imply the same values for simultaneous events in another. This covariance property of the commutator relations may be demonstrated by the method of Heisenberg and Pauli.[3] (We indicate this in the appendix).

Symmetric Theory

In our formulation of the interaction between scalar and spinor fields we have thus far limited our discussion to a charge-bearing meson field. Charge independence of nuclear forces requires a symmetric theory of the meson field[4] with both electrically charged and neutral mesons. The development of this section remains essentially unaltered if the meson field is modified in order to satisfy the charge-independence requirement. We introduce a real wave field, φ, for the neutral mesons, which commutes with all other wave fields of the nucleons and charged mesons and which satisfies commutator relation (4) with its own conjugate momentum, $\eta=\dot\varphi$:

$$[\eta(\mathbf{r}),\varphi(\mathbf{r}')]=-i\delta(\mathbf{r}-\mathbf{r}').$$

For a symmetric theory the meson-nucleon coupling term in the Hamiltonian density (2) is modified to

$${}^{g\prime}\mathcal{H}=g'(\sqrt{2}\Psi_P^*\beta\Psi_N\psi+\sqrt{2}\Psi_N^*\beta\Psi_P\psi^* +\Psi_P^*\beta\Psi_P\varphi-\Psi_N^*\beta\Psi_N\varphi). \quad (6)$$

Neutral mesons are not coupled with the electromagnetic field, so that ${}^e\mathcal{H}$ remains unchanged as in (2).

3. TRANSITION SCHEMES

We wish to calculate the cross section for an incident thermal neutron with momentum $\mathbf{K}_0$ to be scattered

[3] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929).
[4] N. Kemmer, Proc. Camb. Phil. Soc. **34**, 354 (1938).

elastically to a final state of momentum $\mathbf{K}_f$ (with $|\mathbf{K}_f|=|\mathbf{K}_0|$) in the field of a static, scalar potential ϕ. The interaction terms of the Hamiltonian density in Eqs. (2) and (3) reduce to

$$\begin{aligned} {}^g\mathcal{H} &= g[\Psi_P^*\beta\Psi_N\psi+\Psi_N^*\beta\Psi_P\psi^*], \\ {}^e\mathcal{H} &= ie\phi(\pi^*\psi^*-\pi\psi-i\Psi_P^*\Psi_P), \end{aligned} \tag{7}$$

for a charged meson theory.

The calculations are carried out using a weak coupling approximation, i.e., with the assumption that ${}^g\mathcal{H}$ may be treated as a perturbation. The validity of such an assumption depends on the smallness of the parameter $(g^2/4\pi)$. Arguments based on the strength of nuclear forces would set this parameter at 0.3 (although it is questionable whether the scalar coupling assumed here could adequately describe nuclear forces). Such a magnitude for the coupling parameter would give rise to errors of the order of 30 percent if just the leading terms in a perturbation theory calculation are kept.

Electrostatic scattering occurs in three steps. The matrix element describing a transition from an initial state i to a final state f is

$$M.E.=\sum_{m,m'\neq i}\frac{{}^eV_{fm'}{}^gV_{m'm}{}^gV_{mi}+{}^gV_{fm'}{}^eV_{m'm}{}^gV_{mi}+{}^gV_{fm'}{}^gV_{m'm}{}^eV_{mi}}{(E_i-E_m)(E_i-E_{m'})}, \tag{8}$$

where $V_{mi}=\int\mathcal{H}_{mi}d^3x$ is the spatial integral of the interaction term in the Hamiltonian density describing a transition from state i to state m, E_i is the total energy of state i, and the double sum is carried over all intermediate states m and m', exclusive of the initial state i. Two steps in this process involve the interaction between nucleon and meson fields, treated in the weak coupling approximation. The total momentum of the mesons and nucleons is conserved in these steps. Momentum $\mathbf{\Delta}=\mathbf{K}_f-\mathbf{K}_0$ is exchanged with the applied field in the third step. Although the momenta of the virtual intermediate states are arbitrarily large, the *momentum exchange* with the static field is quite small, of magnitude $|\mathbf{\Delta}|=2K_0\sin\theta/2$ for scattering angle θ. In calculating the cross section for electric scattering of thermal neutrons by atomic electrons bound to heavy nuclei we are thus justified in treating the neutron as interacting with a fixed static field. The matrix element will be proportional to the square of the coupling parameter, g, times the fine structure constant, α.

We list below (Table I) transition schemes for this scattering problem in terms of the virtual intermediate states. The schemes divide conveniently into three classes, as indicated in Eq. (8), according as the interaction, ${}^e\mathcal{H}$, with the electric field occurs in the first, second, or third step. We label these classes of transitions as type a, b, and c, respectively.[5] We denote a proton by P, a proton hole by P^*, a neutron by N, a meson of positive electric charge by μ^+, and a meson of negative electric charge by μ^-. The values in parentheses for the propagation vectors are dictated by conservation of momentum for the nucleon-meson coupling and by specification of the final state momentum, $\mathbf{K}_f=\mathbf{K}_0+\mathbf{\Delta}$. A proton, neutron, or positively charged meson, represented by a plane wave with propagation vector $\mathbf{K}$, behaves as a particle with momentum $+\mathbf{K}$; a proton hole, neutron hole, or negatively charged meson, as one with momentum $-\mathbf{K}$. Propagation vector $\mathbf{k}$ assumes all values in the sum over the intermediate particle states.

In order to evaluate matrix element (8), we perform a spatial fourier resolution of the canonical field operators of the meson and Dirac fields. We obtain a discrete momentum spectrum for each field function by quantizing in a box of side L with periodic boundary conditions.

The analysis of the proton field takes the form

$$\begin{aligned} \Psi &= L^{-\frac{3}{2}}\sum_{\mathbf{K},S} f_{\mathbf{K},S}e^{i\mathbf{K}\cdot\mathbf{r}}v_{\mathbf{K},S}, \\ \Psi^* &= L^{-\frac{3}{2}}\sum_{\mathbf{K},S} f_{\mathbf{K},S}^*e^{-i\mathbf{K}\cdot\mathbf{r}}v_{\mathbf{K},S}^*, \end{aligned} \tag{9}$$

where the $v_{\mathbf{K},S}$ and $v_{\mathbf{K},S}^*$ are the four component Dirac plane wave spinors and the sums extend over the entire range of positive and negative momentum values, with two spin orientations, S, and two signs of energy for each value of $\mathbf{K}$. The $f_{\mathbf{K},S}^*$ and $f_{\mathbf{K},S}$ are creation and annihilation operators for the proton field. $f_{\mathbf{K},S}^*$ either creates a proton in a positive energy state, or fills a proton hole in the negative energy range, with the four quantum numbers $(\mathbf{K},S)$.

We write for the meson field

$$\begin{aligned} \psi &= (2L^3)^{-\frac{1}{2}}\sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-\frac{1}{2}}(a_{\mathbf{k}}+b_{\mathbf{k}}^*)e^{i\mathbf{k}\cdot\mathbf{r}}, \\ \psi^* &= (2L^3)^{-\frac{1}{2}}\sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-\frac{1}{2}}(a_{\mathbf{k}}^*+b_{\mathbf{k}})e^{-i\mathbf{k}\cdot\mathbf{r}}, \\ \pi &= i(2L^3)^{-\frac{1}{2}}\sum_{\mathbf{k}}\omega_{\mathbf{k}}^{+\frac{1}{2}}(a_{\mathbf{k}}^*-b_{\mathbf{k}})e^{-i\mathbf{k}\cdot\mathbf{r}}, \\ \pi^* &= -i(2L^3)^{-\frac{1}{2}}\sum_{\mathbf{k}}\omega_{\mathbf{k}}^{+\frac{1}{2}}(a_{\mathbf{k}}-b_{\mathbf{k}}^*)e^{i\mathbf{k}\cdot\mathbf{r}}, \end{aligned} \tag{10}$$

where $a_{\mathbf{k}}^*$ and $a_{\mathbf{k}}$ are creation and annihilation operators, respectively, for positively charged mesons, $b_{\mathbf{k}}^*$ and $b_{\mathbf{k}}$ correspondingly for mesons bearing negative charge, and $\omega_{\mathbf{k}}=(k^2+\mu^2)^{\frac{1}{2}}$. The operators in (9) and (10) obey commutation and anticommutation rules obtained directly from (4) and (5).

Upon introducing (7), (9), and (10) into (8) and summing over all of the intermediate states indicated in Table I, we get the following expression for the matrix element between the specified initial and final

[5] Renormalization terms, which are usually present in a third-order perturbation calculation, will contribute nothing in this one due to the fact that the neutron (core) itself has no interaction with an electrostatic field.

TABLE I. Transition schemes for the electrostatic scattering of neutrons according to a charged meson theory.

Scheme	States i	m	m'	f
I_a	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $\mu^+(\mathbf{k})$, $\mu^-(\mathbf{k}-\mathbf{\Delta})$	$P(\mathbf{K}_0+\mathbf{k})$, $\mu^-(\mathbf{k}-\mathbf{\Delta})$	$N(\mathbf{K}_f)$
II_a	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $P(\mathbf{K}_f+\mathbf{k})$, $P^*(\mathbf{K}_0+\mathbf{k})$	$P(\mathbf{K}_f+\mathbf{k})$, $\mu^-(\mathbf{k})$	$N(\mathbf{K}_f)$
III_a	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $P(\mathbf{K}_f+\mathbf{k})$, $P^*(\mathbf{K}_0+\mathbf{k})$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_0+\mathbf{k})$, $\mu^+(\mathbf{k})$	$N(\mathbf{K}_f)$
IV_a	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $\mu^+(\mathbf{k})$, $\mu^-(\mathbf{k}-\mathbf{\Delta})$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_0+\mathbf{k})$, $\mu^+(\mathbf{k})$	$N(\mathbf{K}_f)$
I_b	$N(\mathbf{K}_0)$	$P(\mathbf{K}_0+\mathbf{k})$, $\mu^-(\mathbf{k})$	$P(\mathbf{K}_f+\mathbf{k})$, $\mu^-(\mathbf{k})$	$N(\mathbf{K}_f)$
II_b	$N(\mathbf{K}_0)$	$P(\mathbf{K}_0+\mathbf{k})$, $\mu^-(\mathbf{k})$	$P(\mathbf{K}_0+\mathbf{k})$, $\mu^-(\mathbf{k}-\mathbf{\Delta})$	$N(\mathbf{K}_f)$
III_b	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_f+\mathbf{k})$, $\mu^+(\mathbf{k})$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_0+\mathbf{k})$, $\mu^+(\mathbf{k})$	$N(\mathbf{K}_f)$
IV_b	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_0+\mathbf{k})$, $\mu^+(\mathbf{k}-\mathbf{\Delta})$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_0+\mathbf{k})$, $\mu^+(\mathbf{k})$	$N(\mathbf{K}_f)$
I_c	$N(\mathbf{K}_0)$	$P(\mathbf{K}_0+\mathbf{k})$, $\mu^-(\mathbf{k})$	$N(\mathbf{K}_f)$, $\mu^-(\mathbf{k})$, $\mu^+(\mathbf{k}-\mathbf{\Delta})$	$N(\mathbf{K}_f)$
II_c	$N(\mathbf{K}_0)$	$P(\mathbf{K}_0+\mathbf{k})$, $\mu^-(\mathbf{k})$	$N(\mathbf{K}_f)$, $P(\mathbf{K}_0+\mathbf{k})$, $P^*(\mathbf{K}_f+\mathbf{k})$	$N(\mathbf{K}_f)$
III_c	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_f+\mathbf{k})$, $\mu^+(\mathbf{k})$	$N(\mathbf{K}_f)$, $P(\mathbf{K}_0+\mathbf{k})$, $P^*(\mathbf{K}_f+\mathbf{k})$	$N(\mathbf{K}_f)$
IV_c	$N(\mathbf{K}_0)$	$N(\mathbf{K}_0)$, $N(\mathbf{K}_f)$, $P^*(\mathbf{K}_0+\mathbf{k})$, $\mu^+(\mathbf{k}-\mathbf{\Delta})$	$N(\mathbf{K}_f)$, $\mu^-(\mathbf{k})$, $\mu^+(\mathbf{k}-\mathbf{\Delta})$	$N(\mathbf{K}_f)$

states:

$$
\begin{aligned}
M.E._{es}=&(g^2e\phi_\Delta/4L^6)\sum_{\mathbf{k}}\Bigg[\left(\frac{\omega_{k-\Delta}-\omega_k}{\omega_k\omega_{k-\Delta}(\omega_k+\omega_{k-\Delta})}\right)\\
&\times\left(\frac{\langle\beta\Lambda^+{}_{k+K_0}\beta\rangle}{E_{k+K_0}+\omega_k-E_{K_0}}-\frac{\langle\beta\Lambda^+{}_{k+K_0}\beta\rangle}{E_{k+K_0}+\omega_{k-\Delta}-E_{K_0}}
+\frac{\langle\beta\Lambda^-{}_{k+K_0}\beta\rangle}{E_{k+K_0}+\omega_k+E_{K_f}}-\frac{\langle\beta\Lambda^-{}_{k+K_0}\beta\rangle}{E_{k+K_0}+\omega_{k-\Delta}+E_{K_f}}\right)\\
&-\left(\frac{2}{\omega_k(E_{k+K_0}+E_{k+K_f})}\right)\left(\frac{\langle\beta\Lambda^-{}_{k+K_f}\Lambda^+{}_{k+K_0}\beta\rangle}{E_{k+K_0}+\omega_k-E_{K_0}}
+\frac{\langle\beta\Lambda^-{}_{k+K_f}\Lambda^+{}_{k+K_0}\beta\rangle}{E_{k+K_f}+\omega_k+E_{K_f}}+\frac{\langle\beta\Lambda^+{}_{k+K_f}\Lambda^-{}_{k+K_0}\beta\rangle}{E_{k+K_f}+\omega_k-E_{K_0}}
+\frac{\langle\beta\Lambda^+{}_{k+K_f}\Lambda^-{}_{k+K_0}\beta\rangle}{E_{k+K_0}+\omega_k+E_{K_f}}\right)\\
&+\left(\frac{1}{\omega_k(E_{k+K_0}+\omega_k-E_{K_0})}\right)
\times\left(\frac{2\langle\beta\Lambda^+{}_{k+K_f}\Lambda^+{}_{k+K_0}\beta\rangle}{E_{k+K_f}+\omega_k-E_{K_0}}
-\frac{(\omega_k+\omega_{k-\Delta})\langle\beta\Lambda^+{}_{k+K_0}\beta\rangle}{\omega_{k-\Delta}(E_{k+K_0}+\omega_{k-\Delta}-E_{K_0})}\right)\\
&+\left(\frac{1}{\omega_k(E_{k+K_0}+\omega_k+E_{K_f})}\right)\left(\frac{2\langle\beta\Lambda^-{}_{k+K_f}\Lambda^-{}_{k+K_0}\beta\rangle}{E_{k+K_f}+\omega_k+E_{K_f}}
-\frac{(\omega_k+\omega_{k-\Delta})\langle\beta\Lambda^-{}_{k+K_0}\beta\rangle}{\omega_{k-\Delta}(E_{k+K_0}+\omega_{k-\Delta}+E_{K_f})}\right)\Bigg], \qquad (11)
\end{aligned}
$$

where $\Lambda_\mathrm{l}^\pm$ are the positive and negative energy projection operators,

$$\Lambda_\mathrm{l}^\pm=\tfrac{1}{2}(1\pm(\boldsymbol{\alpha}\cdot\mathbf{l}+\beta M)/E_\mathrm{l}),$$

$\langle Q\rangle$ denotes the product of a matrix Q with initial and final state spinors,

$$\langle Q\rangle=(v_{\mathbf{K}_f}{}^*|Q|v_{\mathbf{K}_0}),$$

$(\boldsymbol{\alpha},\beta)$ are the usual 4×4 Dirac matrices, $\phi_\Delta=\int e^{-i\Delta\cdot r}\phi d^3x$, the integration being carried over the volume L^3 of the box in which the wave fields are quantized, $E_\mathbf{K}=(K^2+M^2)^{\frac{1}{2}}$ for nucleons of rest mass M, and $\omega_\mathbf{k}=(k^2+\mu^2)^{\frac{1}{2}}$, for mesons of rest mass μ. The first four terms are contributed by transition schemes I_a, I_c, IV_a, IV_c, in which the scalar potential serves as a source for creation or annihilation of a meson pair. The second set of four terms is contributed by schemes II_a, II_c, III_a, III_c, in which the static field serves to create or annihilate a proton pair. The last four terms come from the type b processes in which scattering of charged particles occurs in the electric field.

4. CALCULATION FOR NO SPIN-FLIP

A. Dirac Nucleons

It is between the part of the neutron wave scattered by the atomic electrons without spin flip and the wave isotropically scattered by the specific nuclear forces that interference exists. We compute here the magnitude of this effect.

For neutrons of thermal velocity, (K_0/μ) and (K_0/M) are both extremely small relative to one, so that we expand (11) in a power series of terms $(\Delta/\omega_\mathbf{k})$ and $(\Delta/E_\mathbf{k})$. The contribution from terms proportional to the zeroth and first powers of Δ vanishes upon taking the sum over intermediate momenta $\mathbf{k}$. The lowest nonvanishing contribution is of order Δ^2:

$$
\begin{aligned}
M.E.=&(g^2e\phi_\Delta\Delta^2/8L^6)\sum_\mathbf{k}\Bigg[\Bigg\{\left(\frac{1+M/E_\mathbf{k}}{\omega_\mathbf{k}D_1{}^2}\right)\\
&\times(1/2\omega_\mathbf{k}{}^2-1/\omega_\mathbf{k}D_1-k^2/3\omega_\mathbf{k}{}^4+2k^2/3\omega_\mathbf{k}{}^3D_1-2k^2/3\omega_\mathbf{k}{}^2D_1{}^2\\
&+k^2/3E_\mathbf{k}\omega_\mathbf{k}{}^2D_1-k^2/E_\mathbf{k}\omega_\mathbf{k}D_1{}^2-k^2/3E_\mathbf{k}{}^2D_1{}^2-k^2/6E_\mathbf{k}{}^4)\\
&+(-1/E_\mathbf{k}{}^2\omega_\mathbf{k}D_1{}^2+1/E_\mathbf{k}{}^3\omega_\mathbf{k}D_1+k^2/3E_\mathbf{k}{}^4\omega_\mathbf{k}D_1{}^2\\
&-k^2/6E_\mathbf{k}{}^5\omega_\mathbf{k}D_1)+(Mk^2/E_\mathbf{k}{}^3\omega_\mathbf{k}D_1{}^2)(1/3\omega_\mathbf{k}D_1-1/6\omega_\mathbf{k}{}^2\\
&+1/3E_\mathbf{k}D_1)\Bigg\}+\left\{\begin{matrix}\text{Same term with } M \text{ replaced}\\ \text{by } -M \text{ and } D_1 \text{ by } D_2\end{matrix}\right\}\Bigg], \qquad (12)
\end{aligned}
$$

where

$$D_1=M-E_{\mathbf{k}}-\omega_{\mathbf{k}};\quad D_2=-M-E_{\mathbf{k}}-\omega_{\mathbf{k}}.$$

The **k** sum in (12) can be performed by taking the sum to an integral,

$$\sum_{\mathbf{k}}\rightarrow(L^3/2\pi^2)\int_0^\infty k^2dk.$$

We insert $M=1835m_e$ for the nucleon mass and $\mu=300m_e$ for the meson mass, introduce a new independent variable $x=\tan^{-1}(k/\mu)$, and integrate numerically over the finite range $(0,\pi/2)$ of x. The integrand is reproduced as the solid line in Fig. 1. We see that most of the contribution to the matrix element comes from low momentum values, $k\lesssim\mu$, for the intermediate particles.

The value obtained is

$$M.E._{es}=(1/25.3)(g^2/4\pi)(e\phi_\Delta/\pi L^3)(\Delta^2/\mu^2). \quad (13)$$

This is the matrix element for electrostatic scattering of neutrons which have the same spin orientation before and after scattering. The numerical factor is a slowly varying function of the meson and nucleon masses, here assumed to be 300 and 1835 electron masses, respectively.

B. Schrödinger-Pauli Nucleons

Most of the contribution to the matrix element (12) comes from the non-relativistic energy range, $k<M$, of the intermediate nucleons. We may thus hope to obtain a good approximation to it if we neglect the small components of the nucleon wave functions—that is, if we treat neutrons and protons as Schrödinger-Pauli particles. Since we no longer have the filled negative energy levels of the hole theory, only transition schemes I_a, I_b, II_b, and I_c, will contribute. This means that all spin terms in Eq. (11) that contain negative energy projection operators, Λ^-, vanish, whereas Λ^+ is replaced by unity. The result is plotted as the dashed line in Fig. 1. The matrix element for $S-P$ nucleons is greater than (13) calculated for Dirac particles by 29 percent. Most of the discrepancy arises from the upper end of the momentum spectrum where small components of the intermediate virtual nucleons are significant.

Only a minor contribution is lost from the matrix element by neglecting, say, scheme III_b relative to I_b. This is because the energy denominator of III_b, $(E_{\mathbf{k}+\mathbf{K}_f}+\omega_{\mathbf{k}}+E_{\mathbf{K}_f})(E_{\mathbf{k}+\mathbf{K}_0}+\omega_{\mathbf{k}}+E_{\mathbf{K}_f})$, is considerably larger than that of I_b, $(E_{\mathbf{k}+\mathbf{K}_f}+\omega_{\mathbf{k}}-E_{\mathbf{K}_0})(E_{\mathbf{k}+\mathbf{K}_0}+\omega_{\mathbf{k}}-E_{\mathbf{K}_0})$, in the important **k** region. The disagreement will increase as the pair terms become more important. This will happen for a type of coupling between the nucleon-meson fields which stresses larger momentum values, or, in other words, binds the meson cloud more compactly about the nucleons.

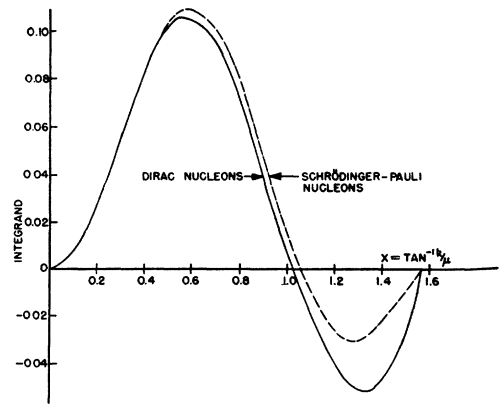

FIG. 1. Graph of the integrand in the matrix element for electrostatic scattering of neutrons without spin flip. The solid line is the result for the case in which nucleons obey Dirac's equation; the dashed line is for the case in which they obey the non-relativistic wave equation.

C. No Recoil

We can evaluate the effect of neglecting recoil of the nucleons by going to the limit $M\rightarrow\infty$ in Eq. (12). Thus $D_1\rightarrow-\omega_{\mathbf{k}}$, $E_{\mathbf{k}}\rightarrow M$, and

$$M.E._{es}=(g^2e\phi_\Delta\Delta^2/8L^6)\sum_{\mathbf{k}}(3-10k^2/3\omega_{\mathbf{k}}^2)/\omega_{\mathbf{k}}^5$$
$$=(\tfrac{1}{12})(g^2/4\pi)(e\phi_\Delta/\pi L^3)(\Delta^2/\mu^2). \quad (14)$$

This approximation is seen to increase the matrix element by a factor two relative to its correct value (12).

D. Symmetric Theory

The presence of the neutral meson wave field in the coupling term (6) of the Hamiltonian density introduces six new transition schemes for the scattering $N(\mathbf{K}_0)\rightarrow N(\mathbf{K}_f)$. Each of these involves a pair creation or annihilation of protons in the potential field. We find that these additional processes do not contribute to the electrostatic neutron scattering cross section. Coupling parameters g and g' in (6) and (7) both have the same value to account for the deuteron binding energy. This equality follows from calculation of the neutron-proton interaction in the approximation of no nucleon recoil. We see then from (6) and (7) that matrix elements (13) and (14) will be multiplied by a factor of two for a calculation based on a symmetric meson theory.

5. COMPARISON WITH EXPERIMENT

The total matrix element for the scattering of thermal neutrons by an atom with zero magnetic moment can be written as the sum of two parts. The dominant part, $M.E._n$, gives the contribution to spherically symmetric scattering as a result of specific nuclear forces and of the charge on the nucleus. The possibility of observing electrostatic scattering of neutrons arises from interference between $M.E._n$ and the matrix element, $M.E._{es}$,

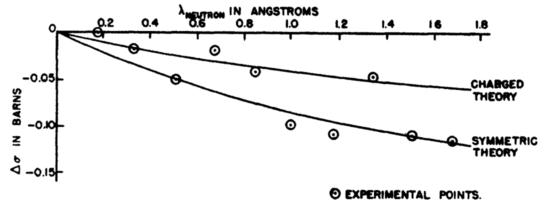

FIG. 2. The curves give the calculated change in the neutron elastic scattering cross section as a function of incident neutron wave-length according to a charged and a symmetric scalar meson theory for $g^2/4\pi=0.30$. The experimental points were obtained by Havens, Rabi, and Rainwater for scattering from lead.

for electric scattering by the atomic electrons, which introduces an asymmetry into the scattering. We can write a cross section by summing over final neutron states and dividing by the incident flux. For a scattering source containing various isotopes, labelled with subscripts i, present in proportions p_i,

$$\sigma d\Omega=\frac{L^6M^2}{4\pi^2}\sum_i p_i|(M.E._n)_i+M.E._{es}|^2d\Omega.$$

Writing $(M.E._n)_i$ in terms of a scattering length[6] $a_i=(ML^3/2\pi)(M.E._n)_i$, and keeping only the linear term, we get for the change in cross section because of electric scattering,

$$\begin{aligned}\Delta\sigma d\Omega&=(\sigma-\sigma_n)d\Omega\\&=(1/25.3)(g^2/4\pi)(e\phi_\Delta\Delta^2/\pi^2)(aM/\mu^2)d\Omega,\\&\qquad\text{(charged meson theory)}\\&=(2/25.3)(g'^2/4\pi)(e\phi_\Delta\Delta^2/\pi^2)(aM/\mu^2)d\Omega,\\&\qquad\text{(symmetric meson theory)}\end{aligned}$$

where a is an average scattering length, $p_1a_1+p_2a_2+\cdots+p_na_n$, for all isotopes. The Δth fourier component of the scattering potential, ϕ_Δ, is conveniently written in terms of the form factor, f, for an atom with Z electrons of charge $-e$, distributed in space with a density $\rho(\mathbf{r}')$.

$$\begin{aligned}\phi_\Delta&=-Ze\int e^{-i\Delta\cdot\mathbf{r}}d\mathbf{r}\int d\mathbf{r}'\rho(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|\\&=-(4\pi Ze/\Delta^2)f.\end{aligned}$$

We have then for the change in cross section

$$\begin{aligned}\Delta\sigma d\Omega&=-0.632Z\alpha(g^2/4\pi)(aM/\mu^2)(f/4\pi)d\Omega\\&\qquad\text{(charged meson theory)}\\&=-1.26Z\alpha(g'^2/4\pi)(aM/\mu^2)(f/4\pi)d\Omega\\&\qquad\text{(symmetric meson theory).}\end{aligned}\tag{15}$$

The functional dependence exhibited in (15) follows from (14). The numerical coefficient depends weakly on the mass ratio (M/μ). The above value is for $M/\mu=6.12$.

The experiments performed at Columbia and at Chicago provide us with two quantities to compare with theoretical results. Havens, Rabi, and Rainwater give points[7] for the change in the total cross section as a function of the wave-length of the incident neutron for scattering by lead. Fermi and Marshall give the asymmetry in the differential cross section for scattering by xenon atoms as

$$\frac{\sigma(135°)-\sigma(45°)}{\sigma(135°)}=0.0005\pm0.0085.\tag{16}$$

To compare with the Columbia result we introduce a scattering length $a=9.06\times10^{-13}$ cm for lead.[8] An expression for the integrated form factor as a function of incident neutron wave-length is obtained from the graph on page 148 of Compton and Allison.[9] The theoretical curves, for $g^2/4\pi=0.30$, and $g'^2/4\pi=0.30$, appear in Fig. 2 together with the experimental points. The calculated formula is consistent with experiment in indicating an *attraction* between neutrons and electrons and in displaying a wave-length variation characteristic of the atomic form factor.

No estimate of error is supplied in the report of this experiment; the experimental points were obtained only after application of large corrections which were probably valid but which could not be independently checked. In view of this, and of the theoretical uncertainties associated with this calculation, the order of agreement indicated in Fig. 2 must be considered satisfactory.

To compare with the Chicago experiment, we obtain an asymmetry

$$\begin{aligned}\frac{\sigma(135°)-\sigma(45°)}{\sigma(135°)}&=0.00206,\ \text{(charged meson theory)}\\&=0.00412,\ \text{(symmetric meson theory),}\end{aligned}\tag{17}$$

where we have taken from the paper of Fermi and Marshall[2] $f(45°)=0.776$ and $f(135°)=0.515$ as form factors for Xe, and 4.4 barns as its cross section. These results are larger than the observed asymmetry, but well within the limits of experimental uncertainty.

6. SPIN-FLIP CROSS SECTION

We calculate the value of matrix element (11) for the case in which the scattered neutron has its spin flipped. Again, as in (12), the leading order terms are proportional to the square of the neutron momentum. Contributions from schemes I_a, IV_a, II_b, IV_b, I_c, and IV_c, according to which the static field creates, scatters, or annihilates mesons, are neglected as higher order in (K_0/M). This follows from (6) where the β-matrix,

[6] We use the definition and sign convention of E. Fermi and L. Marshall, Phys. Rev. **71**, 666 (1947).

[7] See reference 1, p. 635.

[8] This is the value obtained in the most recent determination by Shull and Wollan (unpublished). It corresponds to a coherent scattering cross section of 10.3*b* for lead.

[9] A. H. Compton and S. K. Allison, *X-rays in Theory and Experiment* (D. Van Nostrand and Company, Inc., New York, 1938).

through which the neutrons and protons are coupled, does not contribute in leading order to a spin-flip process. The electric scattering cross section with spin flip for a charged meson theory is evaluated numerically for $M/\mu=6.12$:

$$\sigma_{\text{flip}}d\Omega=5.06(g^2/4\pi)^2\phi_\Delta K_0{}^4|(\mathbf{n}_0\times\mathbf{n}_f)\times\mathbf{n}_{\text{sp}}|^2 d\Omega \quad 10^{-33}\ \text{cm}^2.$$

$\mathbf{n}_0$ and $\mathbf{n}_f$ are unit vectors in the direction of initial and scattered neutron momentum, respectively, and $\mathbf{n}_{\text{sp}}$ is a unit vector along the spin axis. We have $\phi_\Delta=4\pi e/\Delta^2$ for scattering by a point charge e. If we put $g^2/4\pi=0.30$, we obtain a total cross section of less than 2×10^{-8} barn for transversely polarized neutrons which are detected by an experimental arrangement in which the neutron counter has an aperture of one minute of arc. (The answer is four times as large according to a symmetric meson theory since transitions schemes involving neutral mesons do not contribute.) This effect is much too small to be detected with present techniques.

The spin flip cross section vanishes for approximate calculations which treat nucleons as Schrödinger-Pauli particles or as infinitely heavy sources (no recoil).

7. DISCUSSION

The interaction of neutrons with electrons has been calculated. The neutron is regarded as one state of a Dirac nucleon which is coupled to a scalar meson field. The mechanism of the interaction is assumed to be the electrostatic coupling between the Coulomb field of the electrons and the virtual protons, pairs, and mesons which, according to field theory, exist in virtual intermediate states of the neutron. Scalar meson fields with charged particles only, and with a symmetric mixture of charged and neutral mesons have been considered. For both cases, qualitative agreement with experiment is established.

The greatest uncertainty in the assignment of a number to our formula for comparison with experiment arises from the choice of 0.30 as the value for the meson-nucleon coupling constant, $g^2/4\pi$. This is the approximate value indicated for a static interaction between two nucleons of the form of the Yukawa potential as derived from the scalar meson theory. We know, however, that the scalar theory used here does not correctly describe the deuteron, particularly as regards the spin dependence of the interaction. In fact there is no satisfactory theory of the deuteron as yet based on any assumed coupling. This is probably because the nucleon recoils have not been taken adequately into account in such calculations and important modifications, even of a qualitative nature, may be expected when account of such recoil is taken.

The static interactions derived from a pseudoscalar meson theory have spin dependences which seem suggestive of the types that may operate in the deuteron. It is therefore worth while to repeat the calculation of this paper for pseudoscalar mesons, although the difficulty of adjusting the coupling constant will be just as serious as for the scalar case.

To transform our calculations to a corresponding one in which pseudoscalar mesons are coupled to the nucleon field with pseudoscalar coupling, we simply replace g by $-iG\gamma_5$, where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ is a Dirac matrix appearing between the spinor functions of the neutron and proton fields, in Eqs. 1 and 2. The quantization procedure is unaltered. The result corresponding to (13) is

$$M.E.=(1/287)(G^2/4\pi)(e\phi_\Delta/\pi L^3)(\Delta^2/\mu^2). \quad (18)$$

Following Villars,[10] Dyson,[11] and Luttinger,[12] we take $G^2/4\pi=36$, and obtain a change in cross section, $\Delta\sigma$, 10.6 times as large as those given in Eq. (15) for the scalar theory.

We note from Eqs. (16) and (17) that the above result (18) for pseudoscalar mesons exceeds, by a factor of 2.5, the upper limit for this effect placed by Fermi and Marshall.[2] This would appear to argue against the pseudoscalar theory, but for the reasons noted above definite conclusions cannot be drawn until the situation regarding the two nucleon problem is clarified.

Note added in proof: The value $G^2/4\pi\approx4$ suggested by Bethe at the Spring 1949 meetings in Washington, D. C. (B.A.P.S., vol. 24, no. 4) resolves the discrepancy between (18) and (16) and indicates approximately the same change in cross section on the basis of a pseudoscalar calculation as that given by the scalar theory.

APPENDIX

We indicate briefly the covariance of our formalism for describing the interaction of quantized meson and Dirac nucleon fields. A charge-current conservation equation is exhibited. We consider here a charged scalar meson field.

In the absence of external electromagnetic fields, the Lagrangian density, canonically conjugate variables, and Hamiltonian density are those given in Section 2 of the text (Eqs. (1) and (2)) with $A_\mu=0$. The canonical equations of motion take the form

$$\begin{aligned}&\dot\psi=\pi^*;\quad \dot\psi^*=\pi;\\&\dot\pi=(\Delta-\mu^2)\psi^*-g(\Psi_P{}^*\beta\Psi_N);\\&\dot\pi^*=(\Delta-\mu^2)\psi-g(\Psi_N{}^*\beta\Psi_P);\\&\dot\Psi_P=-(\boldsymbol{\alpha}\cdot\mathbf{grad}+iM\beta)\Psi_P-ig\beta\Psi_N\psi;\\&\dot\Pi_P=-(\mathbf{grad}\Pi_P\cdot\boldsymbol{\alpha}-iM\Pi_P\beta)-g\Psi_N{}^*\beta\psi^*;\end{aligned} \quad (a)$$

with equations analogous to the last two for $\dot\Psi_N$ and $\dot\Pi_N$. Commutation relations (4) and (5) are introduced in order that the meson field satisfy Bose statistics and the nucleon fields satisfy Fermi statistics. We wish to demonstrate that these commutator relations are of covariant form. That is, the values of the various commutators and anticommutators corresponding to simultaneous events in one Lorentz frame should imply the same values for simultaneous events in an other. To study this question, we use the method of Heisenberg and Pauli.[3] We consider, for example, the quantity

$$[\Pi_{N\sigma}(\mathbf{r},t),\Psi_{P\nu}(\mathbf{r}',t')]_+$$

which has been assumed zero for $t=t'$. If we now assume that t and t' are nearly equal, we can calculate the value of this anticommutator bracket to first order in $(t-t')$ by the use of the equations of motion (a) plus the various commutation relations

[10] F. Villars, Helv. Phys. Acta **XX**, 476 (1947).
[11] F. J. Dyson, Phys. Rev. **73**, 929 (1948).
[12] J. M. Luttinger, Helv. Phys. Acta **XXI**, 483, (1948).

assumed above to hold for simultaneous events. We obtain neglecting terms in $(t-t')^2$,

$$[\Pi_{N\sigma}(\mathbf{r},t),\Psi_{P\nu}(\mathbf{r}',t')]_+=g\psi(\mathbf{r})\beta_{\sigma\nu}\{(t-t')\delta(\mathbf{r}-\mathbf{r}')\}. \tag{b}$$

The quantity in the brace is, to first order, an invariant,† having the value zero for $|t-t'|<|\mathbf{r}-\mathbf{r}'|$. Consider the system in which the events $(\mathbf{r},t)$ and $(\mathbf{r}',t')$ transform into simultaneous events $(\bar{\mathbf{r}},\bar{t})$ and $(\bar{\mathbf{r}}',\bar{t})$. This will represent an infinitesimal Lorentz transformation, since it was assumed that t and t' were nearly equal. The expression on the left of (b) therefore takes on the transformed value

$$[\Pi_{N\sigma}(\bar{\mathbf{r}},\bar{t}),\Psi_{P\nu}(\bar{\mathbf{r}}',\bar{t})]_+=0.$$

The linearity property of the Lorentz transformation then permits us to write in the transformed system

$$[\bar{\Pi}_{N\sigma}(\bar{\mathbf{r}},\bar{t}),\bar{\Psi}_{P\nu}(\bar{\mathbf{r}}',\bar{t})]_+=0,$$

verifying covariance of this relation for infinitesimal Lorentz transformations. Since a finite transformation can be represented as a sequence of infinitesimal ones, the general covariance follows.

† It is the small argument expansion of the invariant D function of Jordan and Pauli.

The proofs of the covariance of the other relations follow similar patterns and will not be given. We remark only that it does not seem possible to quantize using other commutation rules than those assumed in the text to operate between two field quantities, each belonging to a *different* type of field.

Charge and Current

In the absence of external electromagnetic fields we define charge and current densities as follows:

$$\rho^\mu=-ie(\psi\pi-\psi^*\pi^*);$$
$$\mathbf{s}^\mu=ie(\psi\,\mathbf{grad}\psi^*-\psi^*\,\mathbf{grad}\psi);$$
$$\rho^M=-ie(\Pi_P\Psi_P);$$
$$\mathbf{s}^M=-ie(\Pi_P\boldsymbol{\alpha}\Psi_P).$$

Then, by virtue of the Hamiltonian density and commutation rules that have been assumed, we obtain the differential conservation law:

$$\partial(\rho^\mu+\rho^M)/\partial t+\mathbf{div}(\mathbf{s}^\mu+\mathbf{s}^M)=0.$$

Here, as always,

$$\partial\rho/\partial t=i[\textstyle\int\mathcal{H}d^3x,\rho].$$

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Correlated Probabilities in Multiple Scattering*

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The correlated probabilities of lateral and angular displacements of cloud-chamber tracks, resulting from multiple small-angle scattering, have been calculated for several cases of interest. The results are applicable to curvatures and other measurements taken in the presence of a magnetic field. The usual Gaussian-type scattering law has been used in the form of the fundamental correlated distribution function derived by Fermi. One direct application of this function is to the effect of scattering on angle measurements in nuclear "stars."

A "three-point formula" is derived, involving a correlated distribution of two successive lateral displacements with the resultant angular displacement. The distribution of scattering-produced curvatures, originally derived by Bethe, is calculated. A "four-point formula" allows a quantitative discussion of the tendency of scattered tracks to appear circular rather than skewed or S-shaped.

Finally, a formula is derived for the distribution of the successive chord angles for a track observed at several points, and used to discuss the best method of averaging the observations to reduce scattering-produced curvature errors. The error produced by scattering is not appreciably diminished by taking the best mean for an observation of the track at a large number of points, instead of a single observation of chord and sagitta (three points).

INTRODUCTION

THE multiple scattering of charged particles is of considerable importance for several types of cloud-chamber experiments, and has been treated by various authors.[1] Several problems of interest involving correlated probabilities of angular and lateral displacements may, in fact, be discussed using a fundamental distribution function due to Fermi.[2] It is the purpose of this paper to derive and discuss some of these results, in particular, those dealing with the measurement of track curvatures in magnetic fields.

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[1] H. A. Bethe, Phys. Rev. **70**, 821 (1946); R. Richard-Foy, J. de phys. et rad. **7**, 370 (1946). Other references are quoted in these papers.

[2] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).

I. THE FUNDAMENTAL SOLUTION

We proceed to derive the fundamental distribution function[2,3]

$$W(y,\eta|x)dyd\eta=\frac{\lambda\sqrt{3}}{2\pi x^2}\exp\left\{-\frac{\lambda}{x}\left[\eta^2-\frac{3\eta y}{x}+\frac{3y^2}{x^2}\right]\right\}, \tag{1}$$

which gives the probability that a particle in traversing a distance x in a scattering material suffers a lateral displacement between y and $y+dy$ projected on a plane of observation containing x, and a net change of direction between η and $\eta+d\eta$ projected on the same plane.

The equation satisfied by this function may be derived

[3] We use the vertical bar | to separate given entities on the right from entities whose distribution is under consideration on the left, and shall in this way use the same function symbol W to denote several different distribution functions.