

Now Eq. (4) is not essential in the formulation of variational principle for the *free state*, because it can be satisfied even if ψ is not an exact solution of the wave equation. Moreover in deriving Eq. (3) the condition $\xi=0$ is nowhere used. In fact, instead of using Eq. (4), we might with equal justification have used any integral involving $(H-k^2)\psi$ as the equation for $\tan\eta$. The use of $\xi=0$ would therefore seem to be arbitrary and not inherent for the variational principle derived from Eq. (3). The fact that Hulthén finds by his method two solutions for $\tan\eta$ is a consequence of this ambiguity resulting from the use of the condition $\xi=0$; he actually rejects one of the two solutions as not "good" though there is no way of deciding as to which of the two solutions is "good," except by comparison with results obtained by other methods. A rigorous method for computing the wave function for the continuous spectrum should, therefore, be based solely on Eq. (3) or its equivalent:

$$\delta\xi = k\delta\eta. \quad (7)$$

We now assume a trial wave function of the form

$$\psi(r) = f(r) \sin kr + g(r) \cos kr, \quad (8)$$

subject to the boundary conditions

$$\left. \begin{aligned} f(0) &= \text{finite}, & \lim_{r \rightarrow \infty} f(r) &= 1; \\ g(0) &= 0, & \lim_{r \rightarrow \infty} g(r) &= \lambda = \tan\eta. \end{aligned} \right\} \quad (9)$$

For ψ given by (8)–(9), the variational integral becomes

$$\xi = \int_0^\infty [f_1^2 \sin^2 kr + g_1^2 \cos^2 kr + k(fg_1 - gf_1) + V(f^2 \sin^2 kr + g^2 \cos^2 kr) + (g_1 f_1 + Vgf) \sin 2kr] dr, \quad (10)$$

where f_1 and g_1 denote the derivatives of f and g .

Equation (7) then becomes

$$\delta\xi = k\delta\lambda \quad (11)$$

when ξ is computed with the wave function (9). Now we determine λ and all the parameters by the equations

$$\partial\xi/\partial c_i = 0, \quad i=1, 2, \dots, n \quad (12)$$

and

$$\partial\xi/\partial\lambda = k. \quad (13)$$

Equations (12) express the stationary property of the phase shift, while Eq. (13) follows from Eq. (11) directly. Now if we assume for $f(r)$ and $g(r)$ the forms

$$\begin{aligned} f(r) &= 1 + \sum_{n=1} P_n(r) e^{-nr}, \\ g(r) &= (1 - e^{-r}) \left[\lambda + \sum_{n=1} Q_n(r) e^{-nr} \right], \end{aligned} \quad (14)$$

where $P_n(r)$ and $Q_n(r)$ may, for example, be polynomials in r , involving a set of coefficients c_1, c_2, \dots, c_n which we subject to the variation, Eqs. (12)–(14) will then give a set of $(n+1)$ linear equations for λ, c_1, \dots, c_n . The coefficients of this set of equations form a symmetrical determinant, which can be readily solved.

The method derived in the foregoing paragraph differs from Hulthén's in that by using Eq. (13) instead of Eq. (4) we have made the method more rigorous by conforming to the variational principle, Eq. (11); also by introducing the λ -formulation (as we may call it) we have essentially facilitated the actual calculations.

As an illustration of the foregoing method we shall consider the scattering by the potential

$$V(r) = l e^{-r}/r, \quad (15)$$

where l is a constant. In conformity with Eqs. (14) we write

$$\begin{aligned} f(r) &= 1 + (c_1 + c_2 r) e^{-r}, \\ g(r) &= (1 - e^{-r}) [\lambda + (c_3 + c_4 r) e^{-r}]. \end{aligned} \quad (16)$$

After some elementary calculations, a set of linear equations in λ, c_1, \dots, c_4 are finally obtained. The coefficients in these equations can best be expressed in terms of certain standard functions of k .⁴

In order to compare our results with Hulthén's, we have computed the phase shift for two values of l and k , namely $l = -1.5$,

TABLE I. The phase shifts and the constants for the expansion of the wave function at $k=0.8$.

$-l$	Approx.	λ	c_1	c_2	c_3	c_4	η
1.5	(c_1, c_2, c_3, c_4)	1.11469	0.02278	0.01492	0.92077	-0.06414	0.83958
	(c_1, c_3, c_4)	1.11524	0.05205	0.89689	-0.06512	0.83982
	(c_1, c_2, c_3)	1.11188	0.59506	-0.28842	0.44184	0.83832
	(c_1, c_2)	1.10430	0.03493	0.90190	0.83492
	Hulthén						0.83708
2.1	(c_1, c_2, c_3, c_4)	3.31023	0.61034	0.02359	3.48932	-0.49419	1.27742
	(c_1, c_3, c_4)	3.31111	0.65660	3.45157	-0.49574	1.27749
	(c_1, c_2, c_3)	3.28028	4.86017	-2.22788	-0.08688	1.27489
	(c_1, c_2)	3.22313	0.52837	3.47190	1.26996
	Hulthén						1.27515

$k=0.8$ and $l=-2.1$, $k=0.8$ respectively. The results are tabulated below together with Hulthén's best values of the phase shift. In Table I in addition to the results for the 4-parameter (c_1, c_2, c_3, c_4) trial wave functions (Eqs. (16)) we have also included the results derived for 3-parameter (c_1, c_3, c_4) and (c_1, c_2, c_3) and 2-parameter (c_1, c_2) trial functions.

The present method can be extended to electron scattering by the hydrogen atom, and to allow also for exchange effects. The calculations relating to these extensions are now in progress.

Finally, I should like to express my sincere thanks to Professor S. Chandrasekhar for his interest in this problem and also for his valuable discussions.

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Photo-Disintegration of the Deuteron at Intermediate Energies

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IN the photo-disintegration process, all but a small fraction of the γ -ray energy appears as relative energy of neutron and proton, while in N - P scattering half of the incident neutron energy appears as kinetic energy of the center of mass of the N - P system. Consequently, for energies well above the threshold, photo-disintegration experiments should yield as much information about the N - P interaction as N - P scattering experiments performed at twice the energy. Therefore, energies up to 20 Mev, which are of particular interest because of the large number of electrostatic generators and betatrons operating in this range, are of considerable theoretical importance. (A copy of a letter by Fuller¹ describing preliminary photo-disintegration experiments in this energy region arrived when this report was in preparation.)

In this energy range computations are relatively simple. Only a small number of multipoles are involved (electric dipole, magnetic dipole, and electric quadrupole). For well radii below 2.8×10^{-13} cm, range-corrected Bethe-Peierls formulas are essentially correct, and the total electric cross section is given by:

$$\sigma_e = \frac{8\pi e^2 \hbar^2}{3 \hbar c M} \frac{W_1^3 (\hbar\omega - W_1)^3 \sin^2(kr_1) \exp(2\alpha r_1)}{(\hbar\omega)^3 (1 + \alpha r_1)}, \quad (1)$$

where W_1 is the binding energy of the deuteron. (The photo-magnetic contribution can be neglected except for very low energies.) For a 50-50 mixture for which the outgoing nucleons may be treated as free, expression (1) is only slightly modified by the inclusion of tensor forces. The total cross section including tensor forces is given by (we use the approximation² in which contributions from inside the well are neglected. This approximation gives the right order of magnitude for our case, the

results being in agreement with the exact computations of Rarita and Schwinger³):

$$\frac{\sigma_{e, \text{ tensor}}}{\sigma_e} = \frac{1}{1+bq^2} \left\{ 1 + \frac{5}{18} q^2 \left(1 + \frac{3}{5} \frac{\hbar\omega - W_1}{W_1} \right)^2 \right\}, \quad (2)$$

where q is the ratio of the D component of the ground state wave function to the S component at $r=r_t$, and b is given by:

$$b = \frac{1}{9} + \frac{2}{3} \frac{(1+\alpha r_t)^2}{(\alpha r_t)^3}. \quad (3)$$

In the energy region considered, the total cross section is decreased slightly by the inclusion of tensor forces. For $r_t = 2.8 \times 10^{-13}$ cm, the correction is less than two percent, and for $r_t = 1.8 \times 10^{-13}$ cm, the correction is less than five percent.

The angular distribution for unpolarized γ -rays, in the center-of-mass system is given by:[†]

$$f(\theta) = a + \sin^2\theta(1 + 2\beta \cos\theta), \quad \beta = \frac{v}{c} = \left\{ \frac{\hbar\omega - W_1}{Mc^2} \right\}^{\frac{1}{2}}. \quad (4)$$

The term $2\beta \cos\theta$, which is a correction of the first order in β , arises from interference between electric dipole and electric quadrupole terms and leads to a distinct asymmetry.[‡] In the laboratory system the finite momentum of the incident photon will introduce a further asymmetry. The distribution corresponding to (4) can be obtained by replacing $\sin^2\theta$ by:

$$\sin^2\theta \left[1 + 2\beta \left(1 + \frac{W_1}{\hbar\omega - W_1} \right) \cos\theta \right]. \quad (5)$$

The photoelectric part of the angular distribution (4) in the laboratory system is identical with that obtained by Sommerfeld for atomic hydrogen, because of the kinematic nature of the retardation effect which is independent of the particular atomic or nuclear system involved. It is of interest to point out that this asymmetry sets in at energies as low as 10 Mev, while qualitatively one would expect the quadrupole term to play a role around 70 Mev where $\lambda = r_t$.

The data obtained by Fuller¹ indicate that the total cross section decreases less rapidly with energy than that given by Eq. (1), but by fitting at 13 Mev rather than at 7 Mev as he has done, Eq. (1) is found to yield agreement within the experimental error. However, the large statistical errors in these data do not permit any definite conclusions in this respect. (The approximate expression (2) and the exact calculations of Hu and Massey⁴ indicate that at energies greater than 20 Mev the cross section should fall off less rapidly than indicated by (1), because of the influence of tensor forces. In the energy range of Fuller's experiments, however, this effect should not be important.) In Fig. 1 the angular distribution (4) ($f(\theta)$), (computed for $\hbar\omega = 17$ Mev) in the laboratory system, is compared with Fuller's data ($F(\theta)$) for the energy range 14.0 to 20.3 Mev. It is seen that the agreement is quite good but the experimental error is too large to justify any definite conclusions.

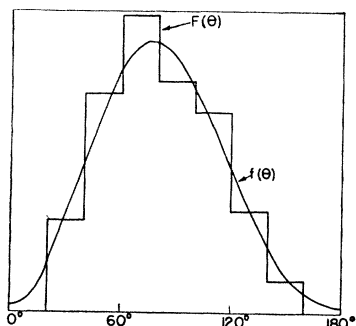


FIG. 1. Comparison of the angular distribution $f(\theta)$ with Fuller's data $F(\theta)$ in the energy range 14.0 to 20.3 Mev.

Further experiments in this energy region would certainly have considerable significance, since they would yield information on the following aspects of the N - P interaction.

(a) Percentage mixture. Rarita and Schwinger and Hu and Massey have shown that the magnitude of the isotropic term in Eq. (2) is very sensitive to the choice of mixture. (For zero percent charge exchange $a=0.4$ at 17.5 Mev, while for mixtures near 50-50, $a=0.02$ including magnetic dipole contributions.) Furthermore the asymmetry in the angular distribution will be greater for pure ordinary forces than that given by (4).

(b) Range of the N - P interaction. Since the magnitude of the total cross section (for intermediate energies) depends only on the triplet well radius and on the deuteron binding energy, measurements of total cross sections should give a good indication of the range of the N - P interaction.

A more complete treatment including the influence of the shape of the interaction potential and a more detailed analysis of the effect of tensor forces is in preparation.

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† This expression is rigorously correct for zero-range N - P interaction, but for energies up to 20 Mev it is an excellent approximation for both square-well and Hulthén potentials.

‡ There is some confusion about this asymmetry in the literature. A. Pais [Kgl. Danske Vid. Sels. Math.-fys. Medd. Bind 20, No. 17 (1943)] and J. M. Jauch [Phys. Rev. **69**, 276 (1946)] conclude that there can be no interference between terms corresponding to final states of different parity because such states correspond to different isotopic spin functions. However, L. Rosenfeld [Nuclear Forces (Interscience Publishers, Inc., New York, 1949), Vol. I, p. 47] and others have shown that the inclusion of isotopic spin and the assumption that proton and neutron are simply different states of the same nucleon leads to the same results in any physical processes as the assumption that proton and neutron are distinct particles. We wish to thank Dr. L. L. Foldy for an illuminating discussion of this isotopic spin formalism.

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High Energy Photo-Disintegration of the Deuteron

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COMPUTATIONS of total cross sections and angular distributions for the photo-disintegration process have been carried out for square-well radii of 1.8 and 2.8×10^{-13} cm in the energy range 20-100 Mev, and for zero radius in the range 20-300 Mev on the following assumptions.

(a) The electromagnetic interaction is represented by $\mathbf{E} \cdot \mathbf{r}$ (rather than $\mathbf{A} \cdot \mathbf{v}$) and $\mathbf{p} \cdot \mathbf{H}$ for electric and magnetic dipole

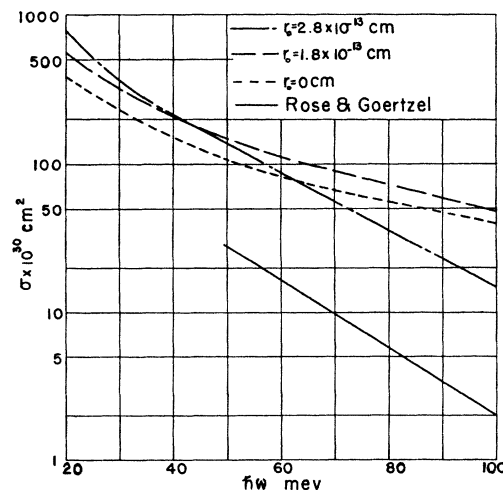


FIG. 1. Total cross section.