On the Interpretation of Neutron-Proton Scattering Data by the Schwinger Variational Method*

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Variational methods developed by Schwinger are applied to neutron-proton scattering at energies below 10 Mev. S-wave scattering alone is considered, and the tensor force is not taken into account. An expansion is obtained for the phase shift in powers of the energy. The coefficients can be evaluated explicitly from the wave function. The first term of the series is related to Fermi's scattering length, the second term involves an "effective range." The third and higher terms turn out to be negligible.

The results are used to define an "intrinsic range" for a potential well of arbitrary shape. Thus a reasonable comparison of potential wells of different shapes is made possible. The relation between intrinsic range and effective range is discussed.

The experimental data on coherent and incoherent neutron-proton scattering are discussed in terms of a "shape-independent" approximation. The best value for the effective range in the triplet state is $r_t = (1.56 \pm 0.13) \times 10^{-13}$ cm. The effective range in the singlet state is not well determined by the present data.

The effect of higher, shape-dependent terms in the expansion of the phase shift is considered. These terms become more important as the well shape becomes more long tailed, but they are found to be negligible within experimental uncertainties for the four well-shapes considered here (square, Gaussian, exponential, and Yukawa).

The results for the scattering phase shifts can be extrapolated to negative energy to give an approximate algebraic equation for the energy of the bound state of the deuteron.

All the numerical results are shown in graphical form; interpolation formulas are provided where higher accuracy may be needed.

I. INTRODUCTION

THE interpretation of nuclear scattering data commonly proceeds according to the scheme

(Experimental Cross Sections)→(Phase Shifts)
←(Theoretical Nuclear Potentials).   (1.1)

That is, the phase shifts[1] constitute the common meeting ground between theory and experiment. The classic example of this type of analysis is the pioneer work of Breit and associates[2] on proton-proton scattering.

The scheme (1.1) involves a large amount of computation. There has long been a feeling that the low energy data really do not yield enough information to merit such a detailed approach. This feeling was expressed by Landau and Smorodinsky[3,4] among others. They developed a semi-empirical formula for the phase shifts. Unfortunately, they did not succeed in giving a rigorous mathematical justification for their method of approach. Consequently they were unable to relate the parameters which they introduce to the correct wave mechanical description of the two-body system in a precise way.

During the last few years, however, Schwinger[5] has developed a powerful variational method which provides a rigorous basis for the Landau-Smorodinsky approach to the analysis of scattering data. This method shortens the labor of computation materially for any one assumed form of the nuclear potential. In addition, this method provides a simple way to determine just what properties of the nuclear potential can and cannot be inferred from the experimental data.

The essence of the approach of Schwinger and Landau and Smorodinsky is its prediction of a simple functional form for the variation of the phase shifts with energy under very general assumptions about the nuclear potentials. This functional form involves some undetermined parameters, of course. The analysis then proceeds according to the scheme:

(Experimental Cross Sections)→(Phase Shifts)
→(Variational Parameters)
←(Theoretical Potentials).   (1.2)

In contrast to (1.1), the meeting place between theory and experiment is not the phase shifts themselves but the values of the variational parameters implied by the phase shifts. This simplifies the numerical work since only a few parameters (actually 3 are sufficient for energies below 10 Mev) have to be computed for each choice of the theoretical potential energy, rather than a larger number of phase shifts. One can get a qualitative understanding of these parameters so that one can easily

* Assisted by the joint program of the ONR and the AEC.
[1] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, London, 1933), p. 24.
[2] G. Breit, E. U. Condon, and R. D. Present, Phys. Rev. 50, 825 (1936); G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. 55, 1018 (1939); L. E. Hoisington, S. S. Share, and G. Breit, Phys. Rev. 56, 884 (1939); G. Breit, A. A. Broyles, and M. H. Hull, Phys. Rev. 73, 869 (1948).
[3] L. Landau and J. Smorodinsky, J. Phys. U.S.S.R. 8, 154 (1944).
[4] J. Smorodinsky, J. Phys. U.S.S.R. 8, 219 (1944) and 11, 195 (1947).

[5] J. Schwinger, Phys. Rev. 72, 742A (1947); hectographed notes on nuclear physics, Harvard, 1947.

predict the difference in the scattering properties of, say, a "short-tailed" and a "long-tailed" well. The scheme (1.2) also makes it possible to give a clear-cut criterion of "best fit" to the data: one fits a definite functional form to the phase shifts, so that the parameters and the errors in them can be determined by least square methods.

The present paper deals with this scheme of analysis for neutron-proton scattering. (A companion paper on proton-proton scattering is in preparation.) For low energies (below about 10 Mev), S-wave scattering alone is of importance. For this case the quantity $k \cot\delta$ [$k =$ wave number of the neutron in the center-of-gravity system, $\delta =$ phase shift] can be expanded as a power series in k^2 (i.e., in the energy, since the two are proportional), as follows:

$$k \cot\delta = -1/a + \tfrac{1}{2} r_0 k^2 - T k^4 + \cdots. \qquad (1.3)$$

The existence of the series (1.3), and of its equivalent for proton-proton scattering, was realized before the work of Landau and Smorodinsky. Indeed, Breit, Condon, and Present[2] showed that the zero-range assumption does not give a good fit to the proton-proton scattering data. This assumption is equivalent to using only the first term of the proton-proton series corresponding to (1.3). The Russian papers served to focus attention on the use of (1.3) as the starting point for a complete scheme for analyzing the data. Schwinger's work in turn made it possible to relate the coefficients to the quantum-mechanical properties of the system. Since that time Bethe,[6] Peierls,[7] Hatcher, Arfken, and Breit,[8] and Chew and Goldberger[9] have all, independently, found ways of deriving the Schwinger expressions for the coefficients in (1.3) directly from the basic differential equation, without the use of variational methods. Ekstein[10] has independently succeeded in deriving the variation principle underlying the present work. Hulthen[11] has applied slightly different variational methods to similar problems.

The first two coefficients of the power series (1.3) have been written in a special form so as to give them simple physical meaning: the parameter a turns out to be Fermi's[12] "scattering length" evaluated at zero energy. r_0 is dimensionally a length and the factor $\tfrac{1}{2}$ in (1.3) makes its value fall somewhere near the "edge" of the potential well.

Hence r_0 is called the "effective range." The term "effective range" must be used with some caution since r_0 depends not only upon the range but also upon the depth of the potential well.

The result (1.3) does not look too promising at first sight since a power series has infinitely many terms. Hence we must, in principle, determine infinitely many parameters, i.e., the coefficient of every power of k^2 in (1.3). For low enough energies, however, the first few terms of the power series will clearly suffice. Suppose the first n terms of the series (1.3) are sufficient to give an adequate fit to the data over the energy range in question. Then the data determine no more than these n coefficients of the series (1.3). Two wells which lead to the same values of these coefficients are equivalent fits (equally good or equally bad as the case may be) to the experiments, even though their higher order parameters (coefficients of k^{2n}, k^{2n+2}, etc.) may differ considerably.

It will turn out that the third term of (1.3) and all the ones beyond it are so small that they can be neglected within the experimental error over the energy range in question. We then get the *shape-independent approximation formula:*

$$k \cot\delta \cong -1/a + \tfrac{1}{2} r_0 k^2. \qquad (1.4)$$

The name comes from the fact that (1.4) involves only *two* variational parameters, namely, the scattering length a and the effective range r_0. For any well shape we always have *two* adjustable well parameters at our disposal: the well depth and the range of the well. Having two well parameters with which to fit two variational parameters, we can obviously make a fit no matter what the shape of the well, provided only the well shape and energy range are such that the higher terms in (1.3) are indeed small.

In neutron-proton scattering, the data below 10 Mev are sufficiently inaccurate so that one cannot determine the coefficient T of k^4 except within very wide limits. Moreover, around 15–25 Mev the S-wave phase shift is near to 90° so that the S-wave cross section is close to $4\pi\lambda^2$, and the transmission experiments just measure the effective wave-length of the neutrons. At even higher energies, the data are extremely hard to interpret; higher orbital angular momenta enter significantly, and their contributions (phase shifts) are hard to separate. In addition, the spin-orbit coupling due to the tensor force also becomes significant, and this doubles the number of independent phase shifts which must be determined from the data. The present data[13] around 90 Mev do not have the accuracy necessary

[6] H. Bethe, Phys. Rev. **76**, 38 (1949).

[7] F. C. Barker and R. E. Peierls, Phys. Rev. **75**, 312L (1949).

[8] R. D. Hatcher, G. B. Arfken, and G. Breit, Phys. Rev. **75**, 1389 (1949).

[9] G. F. Chew and M. L. Goldberger, Phys. Rev. **75**, 1466A (1949); also Phys. Rev. **75**, 1637 (1949).

[10] H. Ekstein, Phys. Rev. **75**, 1322A (1949).

[11] L. Hulthen, Arkiv f. mat., astr. och fysik **35A**, No. 25 (1948).

[12] E. Fermi and L. Marshall, Phys. Rev. **71**, 66 (1947).

[13] L. J. Cook, E. M. McMillan, J. M. Peterson, and D. C. Sewell, Phys. Rev. **72**, 1264L (1947); J. Hadley, E. Kelly, C. Leith, E. Segrè, C. Wiegand, and H. York, Phys. Rev. **75**, 351 (1949).

for such an analysis. We are therefore restricted to the data below 10 Mev, and we can learn nothing at all about the shape of the nuclear potential well.

There is a slight correction to the conclusion of the last paragraph; one can of course conceive of wells with values of T so abnormally large that they would make the third term of (1.3) predominate even at moderately low energies. Even the rather inaccurate and low energy data will then give a poor fit to (1.3). Peculiar wells with such abnormally large values of T *are* excluded by the experiments. But we shall see that all the commonly assumed well shapes give quite small values of Tk^4 below 10 Mev. Well shapes which lead to large values of T must have a very long tail (longer than the Yukawa well). Extremely long-tailed wells are probably excluded by Wigner's[14] original argument for short-range forces (based upon a comparison of the binding energies of H^2, H^3 and He^4). Hence *we conclude that, except for unusual well shapes likely to be excluded by other considerations, the experimental data about the neutron-proton system at low energies tell us nothing at all about the shape of the nuclear potential well.*

An analysis given by one of us,[15] based upon the shape-independent approximation, shows that it is impossible to determine the parameters uniquely even with this simplifying assumption. The effective range r_s in the singlet spin state is known at present only within limits so wide as to be almost meaningless. However, more accurate data would remedy that situation, and the accuracy involved appears to be within the limits of present experimental techniques. On the other hand, we feel that it is useless at this time to analyze the neutron-proton data by any more detailed method than the shape-independent approximation.

Section 2 of this paper is devoted to an exposition of the Schwinger variational method for scattering problems. In Section 3 we use this method to derive the shape-independent approximation (1.4). An alternative, simpler derivation of (1.4) without the use of variational techniques is given in a companion paper by Bethe.[6] Section 4 is devoted to the elaboration of these results. We define the "intrinsic range" and "well-depth parameter" of a well of arbitrary shape and we discuss the relation between the effective range and intrinsic range. Section 5 gives an approximate formula for the energy of the ground state of the deuteron. Section 6 is devoted to the analysis of the experiments in terms of the shape-independent approximation, bringing the results of reference 15 up to date. The formula for T in (1.3) is derived in Section 7, and some computed values are given there to show that T can indeed be neglected for the usual well shapes.

[14] E. Wigner, Phys. Rev. **43**, 252 (1933).
[15] J. M. Blatt, Phys. Rev. **74**, 92 (1948).

Finally in Section 8 we give a more accurate expression for the ground state energy of the deuteron. We also give interpolation formulas for the necessary well depth as a function of the intrinsic range b of the well for the four conventional well shapes.

The tensor force has been neglected throughout this work. An investigation of this aspect of the problem is being carried out by L. C. Biedenharn, and will be presented in a later publication.

II. THE ESSENTIALS OF THE SCHWINGER VARIATIONAL METHOD

The Schwinger analysis,[5] upon which this whole work is based, can be summarized for our purposes as follows. We shall assume S-scattering only and no long-range (Coulomb) forces for the sake of simplicity of presentation. Let $\varphi(r)$ be the radial part of the wave function in the center-of-gravity system, and let

$$u(r) = r\varphi(r). \tag{2.1}$$

Then $u(r)$ satisfies the Schrödinger equation:

$$[-d^2/dr^2 - k^2 + 2mV(r)/\hbar^2]u(r) = 0. \tag{2.2}$$

Here $k^2 = 2mE/\hbar^2$ is the square of the wave number associated with the relative motion, E is the energy associated with relative motion, $m = m_1m_2/(m_1+m_2)$ is the reduced mass, and $V(r)$ is the nuclear potential. This differential equation is to be solved subject to the boundary condition

$$u(0) = 0. \tag{2.3}$$

Since $V(r)$ approaches zero rapidly as soon as r exceeds the range b of the nuclear force, the solution of (2.2) and (2.3) will behave for $r \gg b$ like

$$u(r) \sim \sin(kr) + \tan\delta \, \cos(kr). \tag{2.4}$$

(2.4) defines the phase shift δ. The cross section is given by[1]

$$\sigma = 4\pi \sin^2\delta/k^2. \tag{2.5}$$

Schwinger now proceeds to replace the differential equation (2.2) by an integral equation. He writes (2.2) as

$$(-d^2/dr^2 - k^2)u(r) = -[2mV(r)/\hbar^2]u(r)$$

and introduces a Green's function for the operator on the left-hand side. The Green's function is the one appropriate to standing waves. It satisfies the conditions:

$$(-\partial^2/\partial r^2 - k^2)G(r, r') = \delta(r-r'),$$
$$G(0, r') = 0, \quad G(r, r') \sim \cos(kr) \text{ for } r > r'. \tag{2.6}$$

This Green's function is given by

$$G(r, r') = k^{-1} \sin(kr_<) \, \cos(kr_>), \tag{2.7}$$

where $r_<$ stands for the smaller one of r, r', and $r_>$ stands for the larger one of r, r'.

We also introduce the notation

$$W(r) \equiv -2mV(r)/\hbar^2. \qquad (2.8)$$

The differential Eq. (2.2) can now be written as an integral equation:

$$u(r) = \sin(kr) + \int_0^\infty dr' G(r, r') W(r') u(r'). \qquad (2.9)$$

We observe that if we add any multiple of $\sin(kr)$ or $\cos(kr)$ to the right-hand side of (2.9), $u(r)$ will still satisfy the differential Eq. (2.2). The particular choice (2.9) was made so as to satisfy the boundary conditions (2.3) and (2.4). (2.3) is obviously satisfied. To see whether (2.4) is satisfied, we let r be much larger than the range b of the forces. The integral extends practically only over values of r' of order b, which is much smaller than r by assumption. We therefore substitute $r_< = r', r_> = r$ in $G(r, r')$ (2.7). This gives

$$u(r) \simeq \sin(kr)$$

$$+ \left[k^{-1} \int_0^\infty dr' \sin(kr') W(r') u(r') \right] \cos(kr).$$

Comparison with (2.4) shows that the phase shift

δ is given by

$$\tan\delta = k^{-1} \int_0^\infty dr' \sin(kr') W(r') u(r'). \qquad (2.10)$$

Schwinger now rewrites the integral Eq. (2.9) in the form of a variation principle. To do this, he multiplies both sides of (2.9) by $W(r)u(r)$ and integrates over r. The result is

$$\int_0^\infty W(r) u^2(r) dr = \int_0^\infty W(r) u(r) \sin(kr) dr$$

$$+ \int_0^\infty dr \int_0^\infty dr' W(r) u(r) G(r, r') W(r') u(r').$$

He divides both sides of this equation by the square of the expression (2.10), i.e., by $\tan^2\delta$. This gives, after rearrangement of terms,

$$\frac{\displaystyle \int_0^\infty W u^2 dr - \int_0^\infty dr \int_0^\infty dr' W(r) u(r) G(r, r') W(r') u(r')}{\left[k^{-1} \displaystyle\int_0^\infty W(r) u(r) \sin(kr) dr \right]^2}$$

$$= k^2 \left[\int_0^\infty W(r) u(r) \sin(kr) dr \right]^{-1}.$$

The right-hand side of this equation is equal to $k \cot\delta$ by formula (2.10). We therefore get the *fundamental equation of the Schwinger theory*:

$$k \cot\delta = \frac{\displaystyle \int_0^\infty W(r) u^2(r) dr - \int_0^\infty dr \int_0^\infty dr' W(r) u(r) G(r, r') W(r') u(r')}{\left[k^{-1} \displaystyle\int_0^\infty W(r) u(r) \sin(kr) dr \right]^2}. \qquad (2.11)$$

Schwinger now observes that (2.11) can be considered as a variation principle for $k \cot\delta$. In other words, the function $u(r)$ which satisfies the integral Eq. (2.9), is precisely the function which makes the expression (2.11) stationary. Conversely the value of $k \cot\delta$ computed from that function $u(r)$ which makes (2.11) stationary will give the correct phase shift which satisfies (2.10). These statements can be verified directly. One merely has to replace the correct wave function $u(r)$ by $u(r) + \delta u(r)$, expand to terms linear in $\delta u(r)$, and observe that these terms vanish no matter what form is assumed for $\delta u(r)$, merely as a result of the fact that $u(r)$ satisfies (2.9) and (2.4).

A useful corollary of the stationary property of the expression (2.11) is the fact that the error in $k \cot\delta$ computed from (2.11) will be of the order of magnitude of the *square* of the error in the wave function. A qualification is necessary here. $u(r)$

enters (2.11) only in the combination $W(r)u(r)$. Hence an error in $u(r)$ is important only if it occurs within the range of nuclear forces.

Another advantage of (2.11) is its homogeneity in $u(r)$. Multiplication of $u(r)$ by a constant factor does not change the variational expression for $k \cot\delta$. Hence we do not need to normalize the trial wave function.

In addition, Schwinger is able to give an explicit method for improving the trial wave function systematically. Suppose we start with a trial function $u_0(r)$ not equal to the correct $u(r)$. We substitute $u_0(r)$ in place of $u(r)$ into (2.11) to get an approximation $k \cot\delta_0$. We can then use the integral equation to iterate the wave function, getting a better approximation $u_1(r)$ from $u_0(r)$. This gives an improved value of $k \cot\delta_1$, which can be used to iterate again on the wave function, and so on. This method converges very rapidly if $u_0(r)$ is chosen

properly. An example of this procedure is given in Section 7 of this paper.

Unlike the usual variation principle for the energy which gives an upper bound for the energy of the ground state of a system,[16] *the variation principle (2.11) gives neither an upper nor a lower bound for $k \cot\delta$ but rather a value close to $k \cot\delta$ with an error whose direction we do not known, even though we can estimate its magnitude.*

The generalization of (2.11) for arbitrary orbital angular momenta, $l \neq 0$, and to include Coulomb forces is trivial. However, Schwinger has generalized the variational approach so that it is useful for a much wider range of problems. Examples are (1) the inclusion of spin-orbit coupling (tensor force),[5] (2) improvement of the Born approximation at high energies,[5,17] (3) a general discussion of the scattering matrix,[18] of reciprocity theorems and conservation laws; the Heitler theory of radiation damping[19] emerges as a particular case, in which certain terms are neglected, (4) the analysis of the effect of obstacles in wave guides[20] and of diffraction problems in optics,[21] and (5) a discussion of boundary conditions in neutron diffusion problems.

The analysis in the form given here is *not* applicable to nuclear reactions[22] where the "particles" which collide are highly complex, and where rearrangement of components to form reaction products of different kinds is possible.

The preceding remarks are intended merely to sketch in some of the background of this particular application of the Schwinger formalism. It is hoped that Professor Schwinger will soon find the time to publish the general formalism in detail.

III. THE SHAPE-INDEPENDENT APPROXIMATION FORMULA**

It is the objective of the Schwinger method to obtain an approximation to the energy dependence of the phase shift in the simple form (1.3). In order to do this, Schwinger replaces the wave function $u(r)$ in the variation principle (2.11) by a trial wave function $u_0(r)$ which is the correct expression for $u(r)$ at some particular value of the energy. It turns out that the wave function $u_0(r)$ appropriate to *zero* energy is best adapted for our purposes. In other words, we replace $u(r)$ which satisfies (2.2)

by $u_0(r)$ which satisfies

$$-d^2u_0/dr^2 = W(r)u_0. \qquad (3.1)$$

One reason for this choice is the fact that (3.1) does not lead to an eigenvalue problem for the determination of $u_0(r)$. Indeed, once a form for $W(r)$ is assumed, (3.1) can be integrated numerically without any difficulty.

Another reason is the simple behavior of $u_0(r)$ outside the range of the forces. Equation (3.1) shows that outside the nuclear range, $u_0(r)$ behaves like a straight line. We shall write

$$u_0(r) \simeq 1 - \alpha r \qquad (3.2)$$

for $r \gg b$, b being the range of the nuclear force (we shall see later on how one can give an unambiguous definition of b). We observe that (3.2) is normalized in a certain way. This does not lead to any difficulty in practice. One merely integrates (3.1) numerically until $r \gg b$, and normalizes the straight line obtained there to the form (3.2).

The constant α has a simple interpretation. We observe that $u_0(r) = 0$ when $r \simeq \alpha^{-1}$, provided α^{-1} is large enough so that the asymptotic form (3.2) is valid there. $a = \alpha^{-1}$ *is therefore the Fermi*[12] *scattering length evaluated at zero energy.*

One might think at first sight that $u_0(r)$ is a very poor choice of a trial function for $E \neq 0$ since its general behavior differs radically from the oscillating form of the correct $u(r)$ outside the nuclear well. We recall, however, that $u_0(r)$ needs to approximate $u(r)$ only for $r \lesssim b$ since $u(r)$ enters (2.11) only in the combination $W(r)u(r)$. *Inside* the nuclear range, however, $u_0(r)$ is a good approximation to $u(r)$, the difference being of order k^2. Since the error in $k \cot\delta$ is of the order of the square of the error in the trial wave function, we conclude that *our result for $k \cot\delta$ will be accurate up to, but excluding, terms of order k^4 in an expansion in powers of k.* In other words, we will get the first two terms of the series (1.3) correctly.

The various reductions which will be made in this section to derive formula (1.4) amount to a replacement of $V(r)u_0(r)$, wherever it occurs, by $(E_0 - T)u_0(r)$, T being the kinetic energy operator ($E_0 = 0$ with our choice of the trial wave function). One then uses the fact that $(E_0 - T)u_0(r) = 0$ outside the range of the nuclear forces to split off the asymptotic behavior of the trial wave function. As a result of this procedure, *the final formula (1.3) for $k \cot\delta$ does not contain the potential $V(r)$ explicitly, only implicity in its effect upon the trial wave function $u_0(r)$.* In particular,[23] the final result does not even depend upon the assumption of a potential of the usual type:

$$Vu = V(r)u(r) \qquad (3.3)$$

[16] See, for example, L. C. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1935), Chapter VII.

[17] J. Eisenstein and F. Rohrlich, unpublished theses, Harvard University.

[18] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937). W. Heisenberg, Zeits. f. Physik **120**, 513, 673 (1943). C. Møller, Kgl. Danske Vid. Sels. Math.-Fys. Medd. **23**, No. 1 (1945).

[19] W. Heitler and H. W. Peng, Proc. Camb. Phil. Soc. **38**, 296 (1942). J. M. Blatt, Phys. Rev. **72**, 466 (1947).

[20] *Waveguide Handbook*, M.I.T. Radiation Laboratory Series (McGraw-Hill Book Company, Inc., New York, in press).

[21] H. Levine and J. Schwinger, Phys. Rev. **74**, 958 (1948); **74**, 1212A (1948).

[22] See, however, the recent paper by W. Kohn, Phys. Rev. **74**, 1763 (1948).

** The following derivation, except for a few trivial changes is reproduced from lecture notes on a course in nuclear physics given by Professor Schwinger at Harvard, Spring 1947.

[23] This was observed by Breit for proton-proton scattering. G. Breit and W. G. Bouricius, Phys. Rev. **74**, 1546L (1948).

but follows just as well from a more general potential operator

$$Vu = \int_0^\infty V(r,r')u(r')dr' \qquad (3.4)$$

(the "velocity-dependent force" of Wheeler[24]). Thus the form (1.4) is quite general and depends for its validity only upon these conditions:

(1) The state of the nuclear two body system can be described by a wave function ψ.

(2) The usual symmetries hold (i.e., the center-of-mass motion and the angular motion can be factored out of ψ).

(3) The wave function satisfies a Schrödinger-type equation

$$(T+V)\psi = E\psi$$

with V a very general operator, restricted only by the assumption of "short-range." In this connection "short-range" merely means that the wave function approaches its asymptotic form rapidly enough so that the effective range (3.9) turns out to be finite.

Since the final result is so general and does not involve the potential $V(r)$ explicitly, one suspects that it can be derived in a simpler way, by eliminating $V(r)$ at a much earlier point in the derivation. This is indeed the case. Bethe[6] has succeeded in giving an alternative derivation which uses the three postulates above directly. The Bethe derivation shows that the expansion for $k\cot\delta$ is closely related to the orthogonality condition on the wave functions belonging to the same Hamiltonian operator but to different values of the energy.

It is rather gratifying to see that a considerable amount of useful information can be derived on the basis of such general assumptions. On the other hand, some people may consider it disconcerting to find just how little the experiments really tell us unequivocally about the nature of nuclear forces.

Since Bethe has given a simpler derivation of (1.4), we shall only give the bare outline of the reduction from (2.11) to (1.4). We define the function $g(r)$ by

$$u_0(r) \equiv 1 - \alpha r - g(r), \text{ all } r. \qquad (3.5)$$

(3.2) shows that $g(r)$ is zero for $r \gg b$, while the boundary condition (2.3) on $u_0(r)$ implies that $g(0) = 1$. We also observe that $-d^2u_0/dr^2 = +d^2g/dr^2 = W(r)u_0(r)$ from formula (2.1). We first treat the numerator of (2.11). We need the integral

$$J(r) = \int_0^\infty G(r,r')W(r')u_0(r')dr'$$

$$= \int_0^\infty G(r,r')d^2g/dr'^2dr'.$$

[24] J. A. Wheeler, Phys. Rev. 50, 643 (1936).

A double integration by parts gives

$$J(r) = \cos(kr) + \int_0^\infty \partial^2G/\partial r'^2 g(r')dr'.$$

In the second term, we use (2.6) to eliminate $(\partial^2G/\partial r'^2)$. $J(r)$ then assumes the form

$$J(r) = \cos(kr) - g(r) - k^2\int_0^\infty G(r,r')g(r')dr'.$$

The numerator N of (2.11) is

$$N = \int_0^\infty W(r)u_0^2(r)dr - \int_0^\infty W(r)u_0(r)J(r)dr,$$

$$N = \int_0^\infty d^2g/dr^2[u_0(r) - J(r)]dr.$$

Using integrations by parts similar to the ones above, we get the result

$$N = -\alpha + k^2\int_0^\infty [2g(r)\cos(kr) - g^2(r)]dr$$

$$-k^4\int_0^\infty\int_0^\infty g(r)G(r,r')g(r')drdr'. \qquad (3.6)$$

We can estimate the magnitude of these integrals as follows: $g(r)$ is of order unity inside the nuclear range b, while it drops to zero rapidly for $r \gg b$. Hence the integral $\int_0^\infty g^2(r)dr$ is of order b. In the other term of the single integral we expand $\cos(kr)$ in a power series. We can do this consistently since we are trying to get an expansion in powers of the energy k^2. Every term in the power series leads to a term of form k^{2n} multiplied by an integral which is of order of magnitude b^{2n-1}.

We recall that because of our choice of trial function our final expression for $k\cot\delta$ will be correct only up to, but not including, terms of order k^4. We are therefore justified in replacing $\cos(kr)$ by unity in (3.6) and in neglecting the double integral (or order k^4b^3) altogether.

Now consider the denominator of the variational expression (2.11). It is the square of the integral

$$D^{\frac{1}{2}} = \int_0^\infty k^{-1}\sin(kr)W(r)u_0(r)dr$$

$$= k^{-1}\int_0^\infty \sin(kr)d^2g/dr^2dr.$$

Integration by parts yields

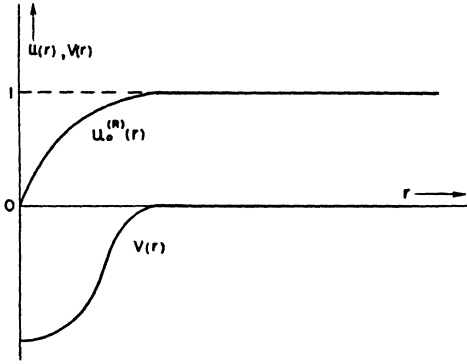$$D^{\frac{1}{2}} = 1 - k\int_0^\infty \sin(kr)g(r)dr \cong 1 - k^2\int_0^\infty rg(r)dr.$$

FIG. 1. Resonance wave function $u_0^{(R)}(r)$. The potential $V(r)$ is chosen to give a scattering resonance at zero energy (the scattering length a is infinite).

We want D^{-1} which, to order k^2 inclusive, is given by

$$D^{-1} \simeq 1 + 2k^2 \int_0^\infty rg(r)dr. \qquad (3.7)$$

We now combine (3.6) and (3.7) to get the shape-independent approximation

$$k \cot\delta = ND^{-1} = -\alpha + \tfrac{1}{2}r_0 k^2$$
$$+ \text{(terms of order } k^4 r_0^3) \quad (3.8)$$

where α is given by (3.2) and the "effective range" r_0 is defined by

$$r_0 = 2 \int_0^\infty [2g(r) - g^2(r) - 2\alpha rg(r)]dr,$$
$$\qquad (3.9)$$
$$r_0 = 2 \int_0^\infty [(1-\alpha r)^2 - u_0^2(r)]dr.$$

IV. THE SPECIFICATION OF A NUCLEAR POTENTIAL BY AN INTRINSIC RANGE b AND A WELL-DEPTH PARAMETER s. THE RELATION BETWEEN THE INTRINSIC RANGE AND THE EFFECTIVE RANGE

Formula (3.9) for the effective range enables us to give an unambiguous definition for the *intrinsic range of a nuclear potential*. Let $V(r)$ be the potential in question. In general, the scattering length α^{-1} will have some finite value. However, we can change the well depth (i.e., multiply $V(r)$ by a constant) until the first resonance occurs at zero energy. The wave function for that case is illustrated in Fig. 1. We see that $\alpha = 0$, i.e., the scattering length a is infinite. The scattering cross section at zero energy is in general given by (see (2.8) and (2.5))

$$\sigma_0 = 4\pi a^2. \qquad (4.1)$$

When a becomes infinite, so does the scattering cross section σ_0. (This does not contradict the usual rule that σ must not exceed $4\pi\lambdabar^2$ since $\lambdabar = k^{-1} = \infty$ at zero energy.) This explains the use of the term "resonance."

Having adjusted the well depth in the way described, we shall call the potential $V(r)$ for this resonance case $V^{(R)}(r)$, and the wave function $u_0^{(R)}(r)$. It is understood that the well depth chosen is the *smallest* one which will give a resonance at zero energy. Formula (3.9) then gives

$$r_0^{(R)} \equiv b = 2 \int_0^\infty [1 - u_0^{(R)}(r)^2]dr. \qquad (4.2)$$

No extraneous length (such as α^{-1}) enters into formula (4.2). Furthermore, the definition of $u_0^{(R)}$ depends only upon the shape and range of the potential $V(r)$ since its depth has been adjusted. It is therefore reasonable *to define $b(4.2)$ to be the intrinsic range of the nuclear potential $V(r)$*. We remark that for a square well potential, the definition (4.2) gives just the ordinary range of the square well.

Having defined the intrinsic range of the well, we shall now introduce a well-depth parameter. We obtained $V^{(R)}(r)$ from $V(r)$ by multiplication by a constant, i.e.,

$$V(r) = s V^{(R)}(r). \qquad (4.3)$$

(4.3) defines the well-depth parameter s. $s = 1$ if there is a resonance at zero energy; $s < 1$ implies a virtual level, $s > 1$ a real level of the two body system.

We give the forms of various commonly assumed potentials in our notation. We will *express $V(r)$ in Mev and the intrinsic range b in units of 10^{-13} cm.* Then the square well, exponential well, Yukawa well, and Gaussian well are given by

Square well (S)

$$-V(r) = s(102.276)b^{-2}, \quad (r < b),$$
$$= 0, \qquad (r > b), \qquad (4.4S)$$

Gaussian well (G)

$$-V(r) = s(229.208)b^{-2}$$
$$\times \exp[-2.0604(r/b)^2], \quad (4.4G)$$

Exponential well (E)

$$-V(r) = s(751.541)b^{-2} \exp[-3.5412r/b], \quad (4.4E)$$

Yukawa well (Y)

$$-V(r) = s(147.585)b^{-2}(b/r)$$
$$\times \exp[-2.1196r/b], \quad (4.4Y)$$

where the bracketed numbers in each coefficient are in $\text{Mev} \times 10^{-26}$ cm^2. For the Yukawa well, the corresponding meson mass μ is given by

$$(\mu/m_e) = 818.57/b,$$

where m_e is the electron mass, and b is in 10^{-13} cm.

The corresponding expressions for $W(r)$ are (in cm^{-2} if b is measured in cm)

Square well

$$W(r) = s(\pi^2/4)b^{-2}, \quad (r<b)$$
$$= 0, \qquad\qquad (r>b), \qquad\qquad (4.5S)$$

Gaussian well

$$W(r) = s(5.5296)b^{-2} \exp[-2.0604(r/b)^2], \quad (4.5G)$$

Exponential well

$$W(r) = s(18.1308)b^{-2} \exp[-3.5412r/b], \quad (4.5E)$$

Yukawa well

$$W(r) = s(3.5605)b^{-2}(b/r)$$
$$\times \exp[-2.1196r/b]. \quad (4.5Y)$$

In Fig. 2 we have plotted $b^2W(r)$ vs. (r/b) for these four wells with $s=1$. The wells in Fig. 2 give equivalent results in the shape-independent approximation.

We now discuss the dependence of the Schwinger parameters, α and r_0, upon the well depths, s. Let us start with the well depth adjusted to give resonance, $s=1$ (see Fig. 1). Now increase s somewhat. The curvature of the wave function will look as indicated in Fig. 3. We see that the scattering length is now positive.

To a first approximation, $g(r)$ is unchanged from its behavior at resonance, $g^{(R)}(r)$. Formula (3.9) (in its first form, involving $g(r)$) then tells us that r_0 will be somewhat *less* than b. Indeed,

$$r_0 \cong b - 4\alpha \int_0^\infty rg^{(R)}(r)dr. \quad (4.6)$$

The effective range decreases with increasing well depth. The order of magnitude of the decrease can be estimated easily. The integral (4.6) is estimated by putting $g^{(R)}(r) = 1 - \sin(\pi r/2b)$ for $r<b$, $g^{(R)}(r) = 0$ for $r>b$. This is, of course, the correct function for a square well. Then (4.6) gives

$$r_0 \cong b[1 - 0.38(\alpha b)]. \quad (4.7)$$

The numerical coefficient of (αb) will of course differ from well to well. For a long-tailed well, $g^{(R)}(r)$ will extend farther out than $r=b$. Since $g^{(R)}(r)$ is multiplied by r in the integral (4.6), we see that bigger values of r are weighted more heavily. Hence we conclude that the numerical coefficient in (4.7) will increase with the length of the well-tail. *For a given value of* (αb), *the effective and intrinsic ranges will differ more the more "long-tailed" the well shape.* We have remarked before that all well shapes give identical results for the scattering as far as the shape-independent formula (3.8) is concerned. This does not contradict the fact that the relation between the effective range r_0 and the intrinsic range b is different for different well shapes. The intrinsic range b is never observed
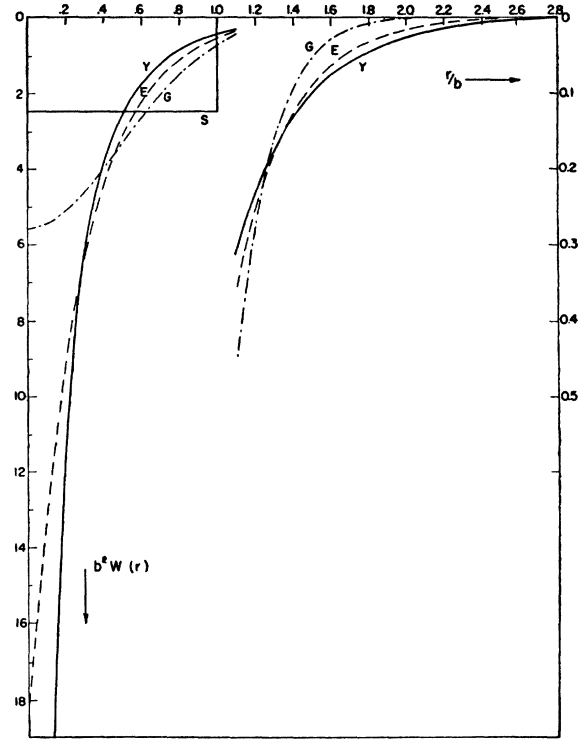

FIG. 2. Ordinate: $b^2W(r)$; Abscissa: r/b. The dimensionless representation of the nuclear potential $b^2W(r)$ $(=(-2mb^2/\hbar^2)V(r))$ is shown as a function of r/b for the square (S), Gaussian (G), exponential (E), and Yukawa (Y) well shapes for $s=1$ (scattering resonance at zero energy). These potential wells are essentially equivalent in the shape-independent approximation. The left-hand ordinate scale is to be used when $r/b<1.1$, the right-hand one when $r/b>1.1$.

directly but must be inferred from the experimental values of r_0 and α. Hence a given experimental r_0 leads to different intrinsic ranges b for various assumed well shapes, but we cannot infer anything about the well shape from an experimental knowledge of r_0.

We conclude this section by giving the results of some calculations with various well shapes. The dependence of α and r_0 upon the well depth is illustrated in Fig. 4 and Fig. 5 where the dimensionless quantities (αb) and (r_0/b) have been plotted against the well-depth parameter s. As a supplement
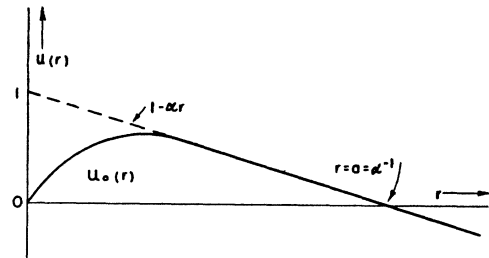

FIG. 3. Wave function $u_0(r)$ for zero energy with the well depth adjusted to give a bound state $(s>1)$. The scattering length a is finite and positive.
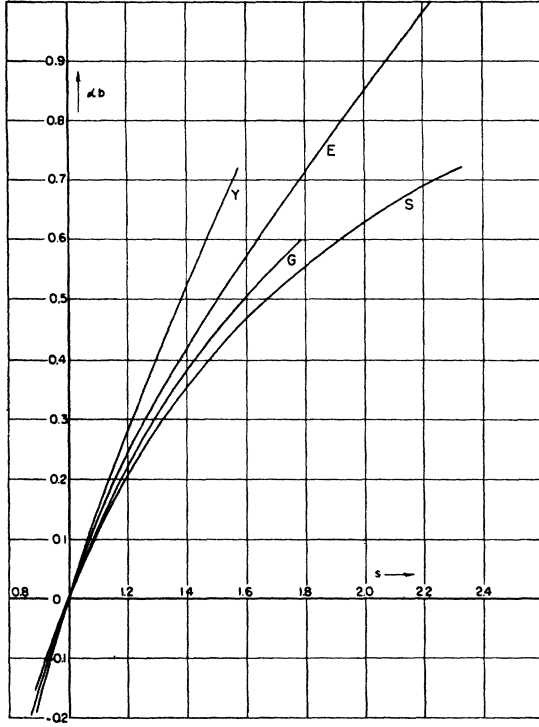
FIG. 4. Ordinate: αb; Abscissa: s. The ratio of the intrinsic range b of the potential well to the scattering length a ($=\alpha^{-1}$) is shown for the square (S), Gaussian (G), exponential (E), and Yukawa (Y) well shapes as a function of the well-depth parameter s. $s>1$ means a well deep enough to allow a bound state; $s<1$ has only a virtual state.

to Fig. 5, Table I contains interpolation formulas for (r_0/b) as a function of s for the various well shapes. The range of validity of these interpolation formulae can be inferred from the curves in Fig. 5.

One can eliminate the well depth s between αb and r_0/b once a well shape (e.g., exponential) is assumed. This gives a plot of r_0/b vs. αb. In practice it is more useful to have b/r_0 plotted against (αr_0) since one determines α and r_0 from the experiments and then wants to know the intrinsic range b which will give the best fit. Plots of (b/r_0) vs. (αr_0) are given in Fig. 6 for the various well shapes.

V. THE SCATTERING MATRIX AND THE GROUND STATE OF THE DEUTERON

We can use the shape-independent formula (3.8) to get an approximate expression for the energy of the bound state of the deuteron. The asymptotic behavior (2.4) of $u(r)$ can be rewritten as

$$u(r) \sim e^{-ikr} - Se^{ikr}, \tag{5.1}$$

where $S \equiv \exp(2i\delta)$ is the matrix element S_{00} of the scattering matrix $S_{ll'}$. (The off-diagonal elements $S_{ll'}$, $l \neq l'$ vanish due to our neglect of tensor forces, while $S_{ll} \cong 1$ for $l \geq 1$ at low energies.) If we replace k in (5.1) by $-i\gamma$ (γ real and positive) the first

term gives an exponentially decreasing contribution while the second term goes like $\exp(+\gamma r)$. Hence we get an acceptable wave function for a bound state of our two-body system if and only if S vanishes when $k = -i\gamma$. We therefore obtain the condition for the energy $E = -\hbar^2\gamma^2/2m$ of a bound state:

$$S(-i\gamma) = 0, \quad \gamma > 0. \tag{5.2}$$

Since $S = \exp(2i\delta)$, (5.2) can be rewritten as

$$\cot\delta = -i \quad \text{for} \quad k = -i\gamma. \tag{5.2'}$$

We now substitute (3.8) to get the approximate equation

$$\gamma \cong \alpha + \tfrac{1}{2}r_0\gamma^2. \tag{5.3}$$

If we neglect the range correction (i.e., put $r_0 = 0$), this gives

$$\gamma \cong \alpha. \tag{5.4}$$

(5.4) shows that a bound state of the two-body system can be expected if α is positive. If α is negative, there exists a "virtual level" at the positive energy $E \cong \hbar^2\alpha^2/2m$, but there cannot be a bound state very near to zero energy.

For the triplet state, (5.4) tells us that the triplet scattering length a_t is approximately equal to the
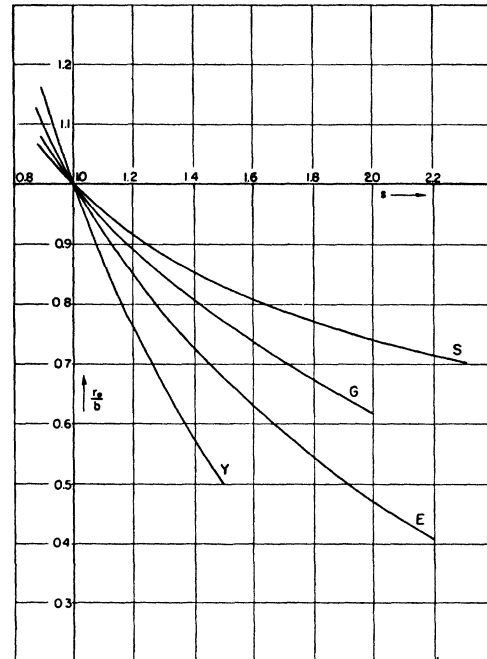

FIG. 5. Ordinate: r_0/b; Abscissa: s. The ratio of the effective range r_0 to the intrinsic range b is shown as a function of the well-depth parameter s for the four well shapes. For a virtual state ($s<1$) $r_0>b$, while for a bound state ($s>1$) $r_0<b$. The figure shows that the ratio of effective to intrinsic range for wells of the same depth s depends strongly upon the well shape, r_0 deviating from b more as the well gets more long-tailed.

TABLE I. Interpolation formulae for the ratio r_0/b of the effective range to the intrinsic range as a function of the well-depth parameter s for various well shapes.

Shape	r_0/b
Square well	$r_0/b = 1 - 0.500(s-1) + 0.486(s-1)^2 - 0.476(s-1)^3 + 0.378(s-1)^4 - 0.158(s-1)^5$
Gaussian well	$r_0/b = 1 - 0.641(s-1) + 0.568(s-1)^2 - 0.721(s-1)^3 + 1.172(s-1)^4 - 1.015(s-1)^5$
Exponential well	$r_0/b = 1 - 0.904(s-1) + 0.745(s-1)^2 - 0.543(s-1)^3 + 0.174(s-1)^4$
Yukawa well	$r_0/b = 1 - 1.369(s-1) + 1.093(s-1)^2 - 1.127(s-1)^3 + 1.005(s-1)^4 - 0.324(s-1)^5$

"radius of the deuteron" $\gamma^{-1} = 4.332 \times 10^{-13}$ cm.[25] In other words, the s scattering of neutrons by protons in the triplet state is approximately represented by scattering from a hard sphere of the radius of the deuteron.

There is no sense in solving (5.3) exactly since it is only an approximate expression for γ. Instead we use (5.3) to iterate on γ, starting with $\gamma = \alpha$ as a first approximation. This gives

$$\gamma = \alpha + \tfrac{1}{2}r_0\alpha^2 + \tfrac{1}{2}r_0^2\alpha^3 + \text{terms of order } (r_0^3\alpha^4). \quad (5.5)$$

The terms of order $r_0^3\alpha^4$ would not be given correctly anyhow since the expression (3.8) for $k \cot\delta$ is only correct up to, but not including, terms of order $r_0^3 k^4$.

Conversely, if we know the triplet scattering length a_t and the binding energy of the deuteron experimentally, we can determine the triplet effective range r_t from (5.3). This is actually the way r_t is found (see Section VI).

VI. COMPARISON OF THE SHAPE-INDEPENDENT THEORY WITH EXPERIMENT

We have already pointed out that formula (3.8) represents the entire effect of the nuclear potential upon the scattering cross section by two constants, the Fermi scattering length (evaluated at zero energy) $a = \alpha^{-1}$, and the effective range r. To the extent that the higher terms in the Schwinger expansion can be neglected, we get a simple closed formula for the cross section σ as a function of the neutron energy E.

$$\sigma = 4\pi k^{-2} \sin^2\delta = 4\pi[(-\alpha + \tfrac{1}{2}rk^2)^2 + k^2]^{-1}. \quad (6.1)$$

We first give a qualitative discussion of (6.1) for the triplet and singlet scattering separately. For the *triplet* scattering, we can use the relation (5.3) to write

$$\sigma_t = 4\pi(k^2 + \gamma^2)^{-1}[1 - \gamma r_t + \tfrac{1}{4}r_t^2(k^2 + \gamma^2)]^{-1}. \quad (6.2)$$

To the extent that the term in r_t^2 can be neglected, this gives Bethe's[26] first-order range correction

$$\sigma_t \cong 4\pi(1 - \gamma r_t)^{-1}(k^2 + \gamma^2)^{-1}. \quad (6.2')$$

(6.2) is correct to terms in r_t^2 *inclusive* since the coefficient T in (1.3) is of order r_t^3. Hence formula (6.2) is more accurate than (6.2'). On the other hand, (6.2') is somewhat simpler in form, showing that the zero-range cross section $4\pi(k^2 + \gamma^2)^{-1}$ is approximately multiplied by an energy-independent factor $(1 - \gamma r_t)^{-1}$. The calculations in this paper and in reference 15 were based on formula (6.2) for the triplet state.

In the *singlet state*, it is possible to introduce a "virtual level" with an energy $E' = \hbar^2\gamma'^2/2m$, γ' being defined by (5.3) with the singlet state scattering length a_s used in computing α. Since a_s, and hence $\alpha_s = a_s^{-1}$, is negative, γ' will be negative also. With this definition of γ', (6.2) can be used for the singlet state.

This is not very convenient, however since (unlike the triplet state) the quantity γ' itself depends upon the effective range one assumes. To see the nature of the result, it is better to use (6.1) directly. We observe that the singlet scattering has a (very broad) resonance rather near to zero energy. In terms of formula (6.1) this means that α_s is small compared to k for energies above the energy of the
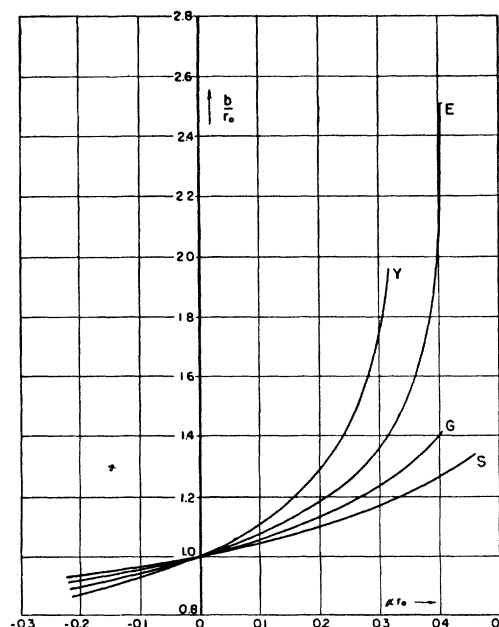

FIG. 6. Ordinate: b/r_0; Abscissa: αr_0. The ratio (intrinsic range/effective range) is shown as a function of (effective range/scattering length) for the four well shapes. The values of $\alpha(=a^{-1})$ and r_0 can in principle be inferred from experiment. Knowledge of α and r_0 then allows determination of the intrinsic range b for any assumed shape of the potential well.

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[25] We use $\epsilon = 2.208 \pm 0.007$ Mev as the binding energy of the deuteron. This value lies between the old value (2.185) and the value (2.237) reported recently by R. E. Bell and L. G. Elliot, Phys. Rev. **74**, 1552 (1948). Reasons for using the value quoted are given in reference 6.

[26] H. Bethe and R. Bacher, Rev. Mod. Phys. **8**, 119 (1936).

virtual level. Hence it is useful to write (6.1) in the form ($\lambda = k^{-1} =$ de Broglie wave-length$/2\pi$).

$$\sigma_s = 4\pi\lambda^2[1 + (\alpha\lambda)^2 - (\alpha r) + \tfrac{1}{4}r^2k^2]^{-1}. \quad (6.3)$$

To interpret this formula, we notice first that setting $\alpha = 0$ (resonance exactly at zero energy) gives

$$\sigma = 4\pi\lambda^2[1 + \tfrac{1}{4}r^2k^2]^{-1} \cong 4\pi\lambda^2 - \pi r^2. \quad (6.3')$$

In words: under the assumption of a resonance exactly at zero energy, the cross section is approximately given by the maximum possible one, $4\pi\lambda^2$, minus the area of a circle of radius r.

This, however, is *not* a good approximation for the physical situation. Between 2 and 4 Mev one gets a better (but still very rough) estimate by neglecting both $(\alpha\lambda)^2$ and $\tfrac{1}{4}r^2k^2$, i.e., by using

$$\sigma_s \cong 4\pi\lambda^2[1 - \alpha_s r_s]^{-1}. \quad (6.3'')$$

This shows that σ_s is depressed ($\alpha_s < 0$!) below its maximum possible value $4\pi\lambda^2$ by roughly a constant factor in this energy region. We would like to emphasize that (6.3'') is not nearly as good an approximation to (6.3) as the Bethe expression (6.2') is to (6.2).

Some representative values for the contribution of the singlet and triplet scattering to the total scattering cross section

$$\sigma = \tfrac{1}{4}\sigma_s + \tfrac{3}{4}\sigma_t \quad (6.4)$$

are given in Table II for the special assumption $r_t = 1.56 \times 10^{-13}$ cm, and the values of r_s ranging from 0 to 3×10^{-13} cm. α_s was adjusted to give 20.36 barns for the cross section at zero energy (see later on for the origin of these numbers). We see from Table II that $\tfrac{1}{4}\sigma_s$ predominates for low energies, becomes equal to $\tfrac{3}{4}\sigma_t$ around 1.5 Mev, and is smaller than $\tfrac{3}{4}\sigma_t$ from then on. The percentage change in $\tfrac{1}{4}\sigma_s$ between $r_s = 0$ and $r_s = 3 \times 10^{-13}$ cm is ~ 10 percent at $\tfrac{1}{2}$ Mev and ~ 20 percent at 5 Mev (it is of course 0 at epithermal energies). The resulting percentage change in the total cross section σ, (6.4), is however practically constant with energy, being ~ 6 percent all the way from $\tfrac{1}{2}$ Mev to 5 Mev.

Conversely, we see from Table II that a cross section measurement accurate to ± 1 percent will determine r_s to $\sim \pm 0.5 \times 10^{-13}$ cm if the effective beam energy is known. An error of ± 1 percent in the effective energy of the beam will lead to an error of about $\pm 0.3 \times 10^{-13}$ cm in r_s. Both these estimates are practically independent of energy between 0.5 and 5 Mev. Nevertheless it is advisable not to go to energies much beyond 3–4 Mev since the shape-independent approximation itself becomes worse at the higher energies. One percent accuracy in the cross section and in the effective beam energy is probably within the limits of present

technique. The present data available at this time,[27] however, are not that accurate.

We just saw that measurements of accuracy ± 1 percent anywhere between 0.5 and 4 Mev will determine the singlet effective range to within $\pm 0.8 \times 10^{-13}$ cm *provided the triplet effective range is known exactly*. The results of reference 15 (Fig. 2 there) show that an uncertainty of as little as $\pm 0.1 \times 10^{-13}$ cm in r_t implies an uncertainty of about $\pm 0.7 \times 10^{-13}$ cm in r_s.

Since the value of the triplet effective range r_t is so important for this analysis, we briefly recapitulate the factors entering into its determination. They are (1) the epithermal cross section

$$\sigma_0 = \tfrac{3}{4}(4\pi a_t^2) + \tfrac{1}{4}(4\pi a_s^2), \quad (6.5)$$

and (2) the coherent scattering amplitude

$$f = 2(\tfrac{3}{4}a_t + \tfrac{1}{4}a_s). \quad (6.6)$$

(6.5) and (6.6) determine a_t, and a_t together with the binding energy determines r_t through (5.3).

The best value of σ_0 comes from neutron velocity spectrometer measurements.[28] It is

$$\sigma_0 = (20.36 \pm 0.10) \text{ barns}. \quad (6.7)$$

This value differs from the value used in reference 15. This changes the resulting estimate of r_t in the upward direction.

The coherent scattering amplitude f can be determined in two independent ways: scattering of neutrons from parahydrogen, and scattering of neutrons from crystals containing hydrogen atoms. The present values are[29, 30]

$$\begin{matrix} f = -(3.95 \pm 0.12) \times 10^{-13} \text{ cm (parahydrogen)}, \\ f = -(3.96 \pm 0.20) \times 10^{-13} \text{ cm (crystals)}. \end{matrix} \quad (6.8)$$

The perfect agreement is rather gratifying, even though probably somewhat accidental. The statistical error is smaller on the parahydrogen measurement; on the other hand, the systematic error of this measurement (unknown admixture of ortho-hydrogen) could be quite large. Since this particular systematic error does not influence the crystal measurement, and since the two measurements of f agree, it is reasonable to assume that the para-hydrogen determination does not in fact have a large systematic error concealed in it.

In reference 15 these data were shown graphically on a plot of $(-f)$ *vs.* σ_0 (Fig. 3). This is done here in Fig. 7 with the presently accepted values (6.7)

[27] C. D. Bailey, W. E. Bennett, T. Bergstralh, R. C. Nucholls, H. T. Richards, and J. H. Williams, Phys. Rev. **70**, 583 (1946); D. Frisch, Phys. Rev. **70**, 589 (1946).

[28] W. B. Jones, Jr., Phys. Rev. **74**, 364 (1948). E. Melkonian, L. J. Rainwater, and W. W. Havens, Jr., Phys. Rev. **75**, 1295A (1949).

[29] R. B. Sutton *et al.*, Phys. Rev. **72**, 1147 (1947).

[30] C. G. Shull, E. O. Wollan, G. A. Morton, and W. L. Davidson, Phys. Rev. **73**, 842 (1948).

and (6.8) as well as the new value of the binding energy of the deuteron.[25] We see that the data give the following estimate for r_t

$$r_t = 1.56 \pm 0.13 \times 10^{-13} \text{ cm (preferred)}$$
$$1.2 < r_t < 1.9 \times 10^{-13} \text{ (outer limit).}$$
(6.9)

The "preferred" value corresponds to the quoted experimental errors. The "outer limits" were obtained by using three times the quoted errors.

The analysis of the incoherent scattering data by the use of two linear plots, the "singlet plot" and "triplet plot," was discussed in reference 15. The results were summarized in Fig. 2 of that reference. We do *not* reproduce that figure here for the following reason: it seems possible that the energy of the beam was systematically overestimated by an amount which is not known.[31] If that were true, the necessary correction would yield a larger value of the singlet effective range (for any given triplet effective range) than that predicted by Fig. 2 in reference 15. This can be seen most easily from an ordinary plot of cross section *vs.* energy. In Fig. 8 we have plotted the experimental cross sections with their errors. Also drawn there are theoretical curves taken from Table II (i.e., with $r_t = 1.56 \times 10^{-13}$ cm).

Figure 8 shows that the experimental points are fitted best by a value of r_s between 0 and 1×10^{-13} cm and that $r_s \sim 3 \times 10^{-13}$ cm is not a good fit, in contradiction to the assumption of charge independence of nuclear forces.[32] (This lack of agreement would have been a little more pronounced had we used the old value of the epithermal cross section which led to a shorter triplet range.) On the other hand, suppose that the effective energy of the beam had been overestimated systematically. The correction for that error would shift all the experimental points to the left on the figure, towards larger singlet ranges.

The present situation can therefore be summarized as follows:

(1) It may not be possible to obtain an excellent fit to all the present low energy data on the neutron-proton system by a singlet n-p range as long as the (singlet) range of the proton-proton force ($\sim 2.6 \times 10^{-13}$ cm).[2]

(2) It is however quite possible to use the known experimental errors to get agreement with such a long singlet range.

(3) In addition, the fast neutron cross-section measurements may contain a systematic error of unknown size but in a direction which would lead us to underestimate the true singlet range.

The fast neutron scattering experiments are now being repeated with improved technique.[33] It seems

[31] D. Frisch, private communication.
[32] G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. 55, 1018 (1939); G. Breit and J. R. Stehn, Phys. Rev. 52, 396 (1937).
[33] J. H. Williams, private communication.

TABLE II. The total neutron-proton scattering cross section is

$$\sigma = \tfrac{3}{4}\sigma_t + \tfrac{1}{4}\sigma_s,$$

where

$$\sigma_t = 4\pi[k^2 + (-1/a_t + \tfrac{1}{2}r_t k^2)^2]^{-1}$$
$$\sigma_s = 4\pi[k^2 + (-1/a_s + \tfrac{1}{2}r_s k^2)^2]^{-1}$$

in the shape-independent approximation. The values of a_t and a_s were determined from the experimental values of σ_0^* and f^{**} while the value of $r_t = 1.56 \times 10^{-13}$ cm was determined from the values of σ_0 and f, and the binding energy of the deuteron*** (see Fig. 7).

$$k^2 = 1.206 \times 10^{24} E(\text{Mev}) \text{ cm}^{-2}.$$

k^2 (in 10^{24} cm^{-2})	$\tfrac{3}{4}\sigma_t$ (in 10^{-24} cm^2)	$\tfrac{1}{4}\sigma_s$ (in 10^{-24} cm^2)				E (in Mev)
		$r_s=0$	$r_s=1$ ×10⁻¹³ cm	$r_s=2$ ×10⁻¹³ cm	$r_s=3$ ×10⁻¹³ cm	
0	2.63	17.73	17.73	17.73	17.73	0
0.5	2.39	4.64	4.50	4.35	4.21	0.415
0.75	2.29	3.39	3.27	3.16	3.04	0.622
1.0	2.20	2.67	2.57	2.47	2.37	0.829
1.5	2.03	1.87	1.80	1.72	1.64	1.244
2.0	1.88	1.44	1.38	1.32	1.25	1.658
3.0	1.64	0.99	0.95	0.89	0.84	2.488
4.0	1.45	0.75	0.72	0.67	0.62	3.317
5.0	1.30	0.61	0.58	0.54	0.49	4.146
6.0	1.17	0.51	0.48	0.45	0.41	4.975
7.0	1.07	0.44	0.41	0.38	0.34	5.804
8.0	0.98	0.38	0.36	0.33	0.30	6.634
10.0	0.84	0.31	0.29	0.26	0.23	8.292

* See reference 28.
** See references 29 and 30.
*** See reference 25.

reasonable to us, therefore, to wait until the new data are available for analysis.

In view of the fact that a small error in r_t implies a large error in the resulting estimate for r_s, we would like to recommend a careful redetermination of the coherent scattering length f by means of neutron scattering in parahydrogen. Figure 7 shows that the present uncertainty in r_t is mostly due to the uncertainty in f. If the statistical error in f were cut by a factor of 2 and if in addition proper precautions were taken to avoid systematic errors, we would know r_t to $\pm 0.07 \times 10^{-13}$ cm rather than the present $\pm 0.13 \times 10^{-13}$ cm. An uncertainty of $\pm 0.07 \times 10^{-13}$ cm in r_t would imply an uncertainty of only about $\pm 0.4 \times 10^{-13}$ cm in the effective singlet range r_s. Since the likely errors of the fast neutron cross-section measurements will give this kind of uncertainty also, with present techniques it does not seem useful to us to push the measurement of f much *beyond* twice its present accuracy. It appears very much worth while, however, to push it that far. The use of the magnetic resonance absorption method of Purcell *et al.*[34] to monitor the admixture of orthohydrogen while the experiment is in progress would eliminate the major source of systematic error.

[34] E. M. Purcell, R. V. Pound, and N. Bloembergen, Phys. Rev. 70, 986L (1946).

VII. THE THIRD TERM IN THE EXPANSION FOR $k \cot\delta$

We saw in Section 3 that the trial wave function u_0 is not accurate enough to get the terms of order k^4 or higher correctly in the expansion for $k \cot\delta$. The variational principle tells us that if we wish to get the expansion exact to terms in k^4 (and also k^6) we must have our trial wave function correct to terms in k^2 inclusive. There are alternative ways of proceeding towards this goal, but before indicating the actual methods used to find the improved trial wave function, it is better to examine the extension of the expansion for $k \cot\delta$ under the assumption that we know the trial function correctly to terms of order k^2 inclusive. We write the correct wave function $u(r)$ as an expansion in k^2,

$$u(r) = u_0(r) + k^2 v(r) + \text{higher terms}, \qquad (7.1)$$

where $u(r)$ satisfies (2.2), and $u_0(r)$ satisfies (3.1). We assume that by some means or other we have determined $v(r)$ to terms independent of k^2. To find $k \cot\delta$ correct to terms to k^6 inclusive, we substitute (7.1) without the higher terms into the variational principle (2.11). We expand the numerator and denominator in powers of k^2. We can keep terms up to k^6 inclusive, but we have restricted ourselves to terms up to k^4 inclusive since the numerical work involved in computing the coeffi-

FIG. 7. Ordinate: $(-f)$; Abscissa: σ_0. The experimental values (see references 29 and 30) of the coherent scattering amplitude $f = 2(\frac{3}{4}a_t + \frac{1}{4}a_s)$ are plotted against the experimental value (see reference 28) of the epithermal incoherent scattering cross section $\sigma_0 = 4\pi(\frac{3}{4}a_t{}^2 + \frac{1}{4}a_s{}^2)$. Curves of constant r_t are superposed. The experimental values of f and σ_0 imply 1.2×10^{-13} cm $< r_t < 1.9 \times 10^{-13}$ cm as outer limits, $r_t = (1.56 \pm 0.13) \times 10^{-13}$ cm as most probable. *Note in proof:* Ordinate label on figure is incorrect. Quantity plotted is $(-f)$.

cient of k^4 as a function of s (i.e., of the well depth) is already quite large. Furthermore, it turns out that the k^4 term exerts a small effect for energies less than 6 Mev, so that it is reasonable to assume that the k^6 term can safely be neglected for the same energy interval.

In writing down the result it helps to define the function

$$w(r) \equiv g(r) - W(r)v(r). \qquad (7.2)$$

We then get

$$k \cot\delta = -\alpha + \tfrac{1}{2}r_0 k^2 - Tk^4 + \cdots \qquad (7.3)$$

where r_0 is given by (3.9) (this must be so since (3.8) was already correct to that order) and where the coefficient T is:

$$T = \int_0^\infty \int_0^\infty w(r) r_< w(r') \, dr \, dr' - \int_0^\infty W(r) v^2(r) \, dr$$

$$- \int_0^\infty r(r_0 - r) w(r) \, dr - (1/3)\alpha \int_0^\infty r^3 w(r) \, dr$$

$$- \alpha \left[\int_0^\infty r w(r) \, dr \right]^2. \qquad (7.4)$$

The coefficient T has the dimensions of a volume. We can therefore write T in two alternative forms with non-dimensional coefficients P, P^*:

$$T = P r_0{}^3 = P^* b^3. \qquad (7.5)$$

We remark here that a similar expansion of the phase shift (actually $\sin^2\delta$) and cross section for the case of a square well has been made by Kittel and Breit.[35] Their method of expansion is somewhat different from ours since they solve the boundary value problem directly and then expand the resulting relation for $\sin^2\delta$. In the energy region where (7.3) is a good approximation to $k \cot\delta$ (for energies less than 10 Mev), their results and ours for the square well are equivalent.

We now turn to the problem of determining $v(r)$ in the expansion (7.1). The first method which suggests itself is that of modifying the integral Eq. (2.9) in order to get an iteration method for improving the wave function. To do this, we use (2.10) to write the coefficient (= unity) of $\sin(kr)$ in (2.9) as

$$1 = (k \cot\delta) k^{-2} \int_0^\infty dr' \sin(kr') W(r') u(r').$$

The integral Eq. (2.9) then assumes the form

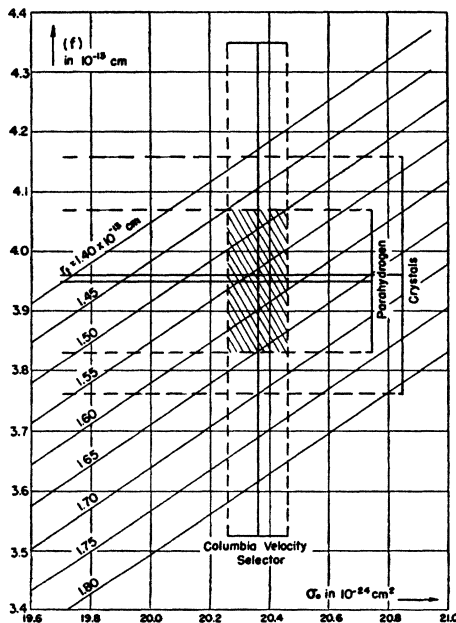$$u(r) = \int_0^\infty G^*(r, r') W(r') u(r') dr', \qquad (7.6)$$

[35] G. Breit and C. Kittel, Phys. Rev. **56**, 744 (1939).

where the kernel $G^*(r, r')$ is given by

$$G^*(r, r') = (k \cot\delta)(\sin(kr)/k)$$
$$\times (\sin(kr')/k) + G(r, r'). \quad (7.7)$$

We observe that (7.6) and (7.7) insure that $u(r)$ has the correct asymptotic form for large values of r, as well as that $u(0) = 0$.

We use (7.6) as the basis of an iteration method in the following way. We start with a trial function u_0 and compute $k \cot\delta = k \cot\delta_0$ from the variational expression (2.11). We then insert this value of $k \cot\delta_0$ into the Green's function (7.7) and compute a better trial function $u_1(r)$ from (7.6). From then on the process is repeated until sufficient accuracy is obtained for the purpose one has in mind. The improvement of the wave function afforded by each step of this method is considerable. However, the wave function cannot be said to be correct to one order higher in k^2 after each step of the iteration. The results obtained by means of this method are such that the first two coefficients of the power series (1.3) for $k \cot\delta$ will be exact (since they depend only on u_0), while succeeding coefficients will be only approximately correct (their correctness depending on the goodness of the iterated wave function). The numbers obtained for the coefficient of k^4 in what follows were obtained by the above iteration method. An estimate of the resultant error in T will be given later.

An alternative method of finding an improved wave function, correct to terms in k^2 inclusive, is to substitute (7.1) directly into the differential Eq. (2.2). This leads to the following differential equation for $v(r)$:

$$-d^2v/dr^2 - W(r)v(r) = u_0 = 1 - \alpha r - g(r). \quad (7.8)$$

$v(r)$ must vanish at $r = 0$, by (2.3). It is not necessary to impose another boundary condition on the solution of (7.8) since a constant multiple of $u_0(r)$ added to $v(r)$ in (7.1) does not influence the final result for $k \cot\delta$. This can be seen from the fact that, to order k^2 inclusive ($\beta = $ constant):

$$u_0 + k^2(v + \beta u_0) + \cdots = (1 + k^2\beta)(u_0 + k^2v + \cdots) \quad (7.9)$$

so that the addition of βu_0 to v in (7.1) merely amounts to a change of normalization of the trial wave function in the variation principle. The variation principle (2.11) is invariant under such a change of normalization.

It turns out that (7.8), in addition to yielding the exact iterate $v(r)$, also makes it possible to simplify the expression (7.4) for T. In particular it makes it possible to get rid of the double integral which is rather tedious to compute. On the other hand, the iteration method based upon the integral Eq. (7.6) proceeds entirely by quadratures, whereas the iteration method (7.8) involves the solution of a second-order differential equation. From a compu-

tational point of view the work involved in the two methods of iteration is substantially the same. We might add that the Bethe derivation of the Schwinger formula leads directly to the result obtained with the (exact) differential equation iteration method (7.8).

For completeness's sake we write down the expression for $v(r)$ obtained by the integral equation iteration method (7.6). It is

$$v(r) \cong \tfrac{1}{2}r(r_0 - r) + (1/6)\alpha r^3 + (\alpha r)\int_0^\infty r'g(r')dr'$$
$$-\int_0^\infty r_< g(r')dr'. \quad (7.10)$$

This approximate expression for $v(r)$ is compared with the exact function in Fig. 9 for the case of a square well. In accordance with (7.9), a multiple of $u_0(r)$ was added to (7.10) to make the comparison meaningful. The dot-dashed curve was adjusted to have the same asymptotic behavior (for $r > b$) as the exact $v(r)$, while the dashed curve was adjusted to give the best fit in the region of interest (i.e., within the range of the nuclear forces). One sees that the general form of the curves is the same, and that the numerical agreement is fairly good. More important
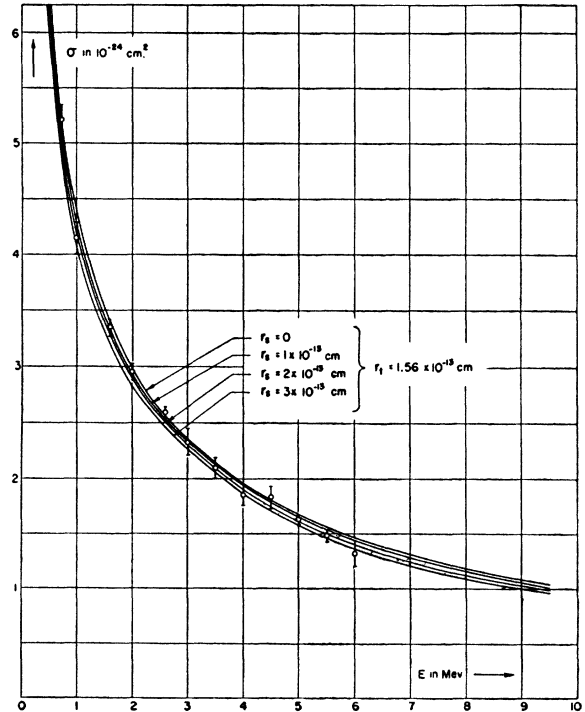

FIG. 8. Ordinate: σ; Abscissa: E. The total incoherent scattering cross section σ is shown as a function of E(Mev) for $r_t = 1.56 \times 10^{-13}$ cm, and $r_s = 0$, 1, 2, 3×10^{-13} cm. The experimental measurements (see reference 27) with their uncertainties are also indicated. The experimental points are compatible with $r_t = 1.56 \times 10^{-13}$ cm, $r_s \gtrless 0$ or 1×10^{-13} cm.
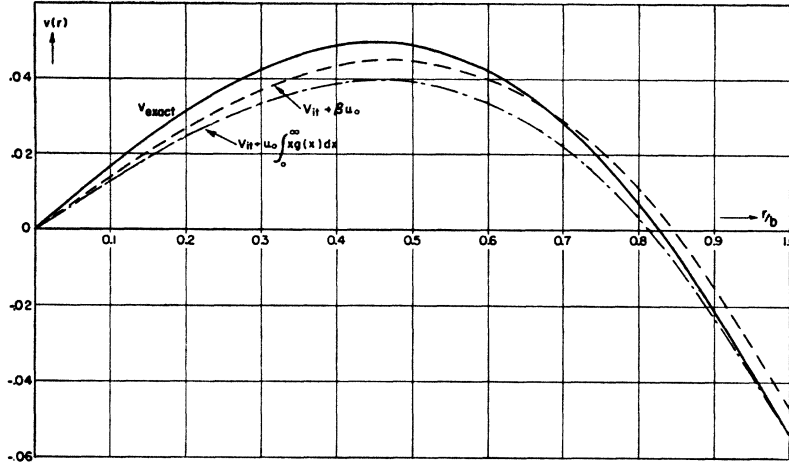
FIG. 9. Ordinate: $v(r)$; Abscissa: r/b. The approximate and exact expressions for $v(r)$ are compared for a typical case with a square well potential. $v(r)$ is the coefficient of k^2 in an expansion of the wave function $u(r)$ in powers of k^2 $[u(r) = u_0(r) + k^2 v(r) + \cdots]$. $v(r)$ occurs in the evaluation of the coefficient T in $k \cot\delta (= -1/a + \frac{1}{2}r_0 k^2 - Tk^4 + \cdots)$. $v(r)$ is small in magnitude compared to unity, and in addition the approximate and exact forms agree rather well [see text after formula (7.10)].

still, we notice that $v(r)$ is numerically small compared to $g(v)$ (which is unity at the origin). Since $w(r) = g(r) - W(r)v(r)$ is the important quantity in evaluating T, and $W(r)$ is of order unity, the smallness of $v(r)$ relative to $g(r)$ insures that the values of T computed with the approximate $v(r)$ will agree closely with the exact values of T. (See later.)

We now return to Eq. (7.3) for $k \cot\delta$, to terms in k^4 inclusive, to examine the significance of the coefficient T. *The term Tk^4 in the expansion of $k \cot\delta$ gives the first indication of a shape for the well.* For example, T is negative for a square well, positive for a Yukawa well. Conversely, two wells with the same values of α, r_0, and T are indistinguishable within the accuracy of formula (7.3) (and this means indistinguishable with the present experimental data).

We shall assume that α and r_0 have been determined from the experimental data. The dimensionless coefficient $P = Tr_0^{-3}$ can be written as a function of the measured dimensionless quantity (αr_0). We then conclude that *two wells with the same value of P for the measured value of (αr_0) are indistinguishable as to shape within the accuracy of this analysis.* Curves of P vs. (αr_0) are given in Fig. 10 for the four usual well shapes. We see that the four usual well shapes *can* be distinguished from one another if P is known to sufficient accuracy.

In order to understand the coefficient T a little better, and to give an estimate of the error of the integral equation iteration method, the following calculation has been performed. We start with a square well $W(r)$ of range b and depth adjusted to resonance ($s = 1$). We then add a lump of potential $\epsilon/b\delta(r - r')$ at the position $r' > b$; the dimensionless constant ϵ is considered small. This change has several results: (1) the effective well depth is deeper; (2) the intrinsic range is longer; (3) the coefficient T is changed.

We now decrease the depth of the square well by an amount of order ϵ until we are again at resonance. We shrink the scale of length until the intrinsic range is again equal to b. We then have a modified well which is identical with the original square well up to terms of order k^2 inclusive; i.e., the modified well has the same scattering length $(a = \alpha^{-1} = \infty)$ and the same effective range (which in this case just equals the intrinsic range) as the original square well. It is then reasonable to ask for the change in the well shape parameter $P = T/b^3$. $(P = P^*$ since $b = r_0$ here.) The calculation is very tedious but elementary. The result is, to terms of order ϵ

$$P = -0.03271 + \epsilon P_1(r'),$$
$$P_1(r') = -0.0398 - 0.1018x + 0.2494x^2$$
$$+ \tfrac{2}{3}x^3 + \tfrac{1}{3}x^4, \quad (7.11)$$

where $x = (r' - b)/b$. (7.11) was obtained by the use of the integral equation iteration method (7.6). If one uses the exact iteration method (7.8), (7.11) is replaced by

$$P = -0.03267 + \epsilon P_1'(r'),$$
$$P_1'(r') = -0.0397 - 0.1013x + 0.2500x^2$$
$$+ \tfrac{2}{3}x^3 + \tfrac{1}{3}x^4. \quad (7.12)$$

We see that (7.11) and (7.12) differ for the square well ($\epsilon = 0$) by less than $\frac{1}{2}$ percent. Furthermore, the polynomials $P_1(r')$ and $P_1'(r')$ have the same coefficients within $\frac{1}{2}$ percent, the coefficients of x^4 and x^3 being actually identical. A similar statement is true if the extra lump of potential is added *inside* the square well (i.e., if $r' < b$). (The corresponding expressions are more complicated and will not be written down here.) We conclude that *the values of P (and hence of T) obtained by the integral equation iteration method (7.6) are accurate to better than 1 percent for all reasonable well shapes.* Considering the fact that the values of T turn out to be too small for an experimental determination to be possible, we feel that this accuracy is altogether

adequate. An exact differential equation iteration method (analogous to (7.8)) will be used in the paper on proton-proton scattering.

Returning now to the discussion of the result (7.11), we see that the effect of a little lump of attractive potential on the well-shape parameter P depends upon the position of this extra potential. If r' is just a little larger than b, $b < r' < 1.4b$, the extra potential will give a very slight *negative* contribution to P. On the other hand, a little lump of attractive potential placed anywhere farther out, at any $r' > 1.4b$, will give a *positive* contribution to P. For $r > 1.4b$, the effect of the extra potential on the well-shape parameter increases very rapidly with r', eventually going like $\epsilon/3[(r'-b)/b]^4$. Some numbers are given in Table III. It is seen that P_1 increases by an order of magnitude between $r' = 2b$ and $r' = 3b$ and by another order of magnitude between the latter value of r' and $r' = 5b$.

In spite of the large values of $P_1(r')$ for large x, the overall value of P is quite small for most reasonable wells. This is due to the fact that the commonly assumed forms for the well-shape simply don't have much of a tail. For example, an order-of-magnitude calculation using Table III shows that values of $r > 5b$ don't contribute any significant amount to the P of a Yukawa well; for this well shape the main contribution occurs between $r = 2b$ and $r = 3b$.

We mentioned before that we have also computed the effect of a little lump of attractive potential added *inside* the range of the square well. The resulting formula is rather long, so we merely show the result graphically in Fig. 11. For $r' < b$, $P_1(r')$ decreases in absolute value, reaches zero at $r' \cong 0.5b$, and then stays positive down to $r' = 0$. For $r' \ll b$, $P'(r')$ is proportional to $(r'/b)^2$. This shows that for well shapes which stay finite at the origin (e.g., the exponential well) the main contribution to P comes from the tail. Even for well shapes singular at the origin (e.g., the Yukawa well), the region

TABLE III. A small extra lump of potential at a point $r' > b$

$$\Delta W = -2m/\hbar^2(\Delta V) = (\epsilon/b)\delta(r-r')$$

is added to a square well of range b and depth adjusted to resonance at zero energy. The table gives the effect upon the well shape parameter P: $P = -0.0327 + \epsilon P_1(r')$.

r'/b	$x = (r'-b)/b$	$P_1(r')$
1.0	0.0	$-$ 0.040
1.2	0.2	$-$ 0.045
1.4	0.4	$+$ 0.011
1.6	0.6	$+$ 0.176
1.8	0.8	$+$ 0.516
2.0	1.0	$+$ 1.108
2.5	1.5	$+$ 4.31
3.0	2.0	$+$ 11.42
3.5	2.5	$+$ 24.7
4.0	3.0	$+$ 46.9
4.5	3.5	$+$ 81.3
5.0	4.0	$+$131.5

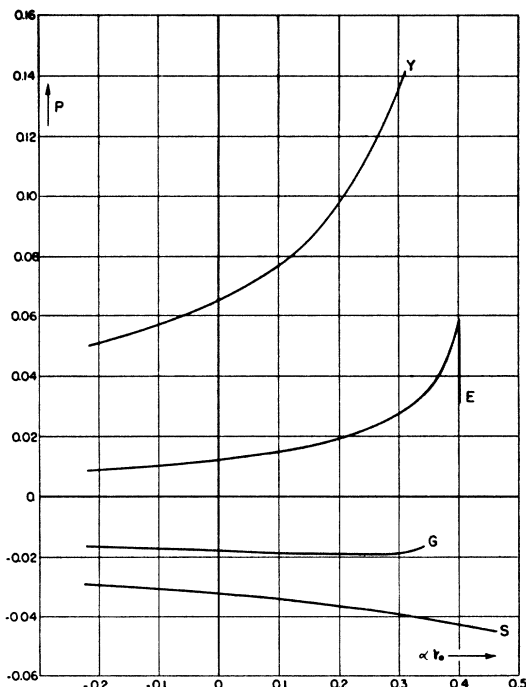
FIG. 10. Ordinate: P, Abscissa: αr_0. The coefficient $P = T r_0^{-3}$ of the k^4 term in the expansion of $k \cot\delta(= -1/a + \frac{1}{2}r_0 k^2 - P r_0^3 k^4 + \cdots)$ is shown as a function of αr_0 for the four well shapes. The coefficient P is the first indication of well shape in the expansion of $k \cot\delta$. P lies between $+0.16$ and -0.05 for the well shapes treated and the range of αr_0 covered. With present experimental data, $|P| \lesssim 0.1 - 0.2$ is indistinguishable from $P = 0$ for the nuclear ranges in question.

near the origin does not give a large contribution to P. The contribution from the "tail" (i.e., the main effect) is positive for an attractive potential. We conclude that for all ordinary attractive potentials P is not likely to fall much below its square-well value (~ -0.04). We do not know the shape-parameter for "velocity-dependent" forces, but it can be computed with our formalism. Since the computation would be quite laborious, we did not undertake to carry it through. It should be pointed out that the considerations of the additional lump of potential should be viewed as a qualitative argument only and cannot be expected to yield anything but order-of-magnitude values for the effects of the tail of a nuclear potential.

The coefficients P and P^* are dimensionless quantities. They depend upon the well depth, of course. We can plot the dimensionless quantity P (or P^*) against the dimensionless well-depth parameter s. Such a plot is given in Fig. 12 for the square well, The Gaussian well, the exponential well and the Yukawa well.

The striking feature of these numerical results is the *small value of the coefficient P*. P lies between -0.05 and $+0.15$ for the wells calculated so far provided we restrict ourselves to reasonable values
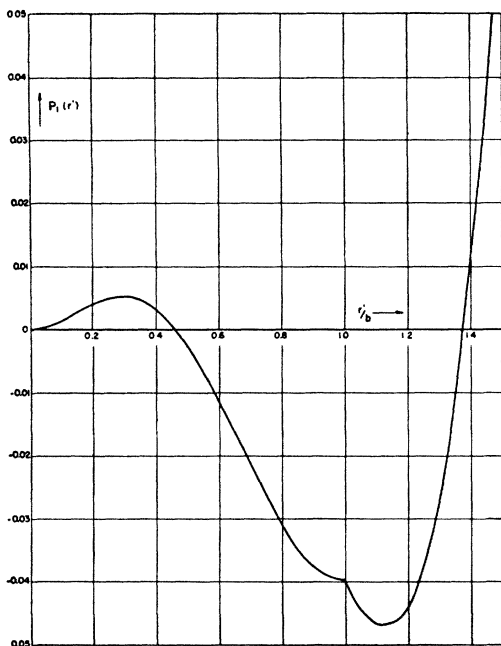
FIG. 11. Ordinate: $P_1(r')$; Abscissa: r'/b. The dependence of the addition $\epsilon P_1(r')$ to the well shape parameter $P[=-0.327 +\epsilon P_1(r')]$ caused by a small extra lump of potential $\Delta V = -(\hbar^2/2mb)\epsilon\delta(r-r')$ added to a square well of range b and depth adjusted to resonance at zero energy is shown as function of the position r' of the added lump. Table III gives values of $P_1(r')$ for $r'>b$. The magnitude of $P_1(r')$ is greater for $r'>b$ than for $r'<b$ and increases rapidly when $r'\gg b$. Any potential shape with a "tail" will have a value of P more positive than a square well (assuming the potential is attractive at all distances); the value of P will be more positive, the longer the "tail" of the potential.

of the range. The present experimental data are such that values of P less than 0.1 or 0.2 simply cannot be distinguished from $P=0$ for the nuclear ranges in question.

As an illustration of this point, we have plotted in Fig. 13 the triplet part of the n-p cross section, i.e., $\frac{3}{4}\sigma_t$, against energy for the square and Yukawa wells under the assumption that $a_t=5.220\times10^{-13}$ cm, exactly. These wells were chosen since they have the extreme values of P (see Fig. 12), and hence indicate extremes in well shape for the four conventional shapes. It is seen that an accuracy of 1 percent in the n-p cross section as a whole is not sufficient to resolve the difference even if we assume that the singlet part of the cross section is known exactly and does not introduce any uncertainty (which is a very unreasonable assumption, of course).

We might be tempted to conclude forthwith that the assumption $P=0$ made in reference 8 (i.e., the shape-independent approximation) is valid. However, this conclusion is premature. It is just barely possible that the presence of the tensor force will lead to a significant change in the order of

magnitude of P. We do not think so, but we do not have any proof at this time.

VIII. THE GROUND STATE OF THE DEUTERON

We now proceed to discuss the modifications which our present more accurate expression for $k\cot\delta$ introduces into the determination of the ground state energy of the deuteron. The argument of Section 5 which led to formula (5.2) is unchanged. However, we now substitute the result (7.3) to get

$$\gamma=\alpha+\tfrac{1}{2}r_0\gamma^2+T\gamma^4. \tag{8.1}$$

We can again solve this by successive approximations, getting

$$\gamma=\alpha[1+\tfrac{1}{2}(\alpha r_0)+\tfrac{1}{2}(\alpha r_0)^2+(P+\tfrac{5}{8})(\alpha r_0)^3 \\ +(3P+\tfrac{7}{8})(\alpha r_0)^4+\cdots]. \tag{8.2}$$

Since this formula is accurate to order $(\alpha r_0)^4$ only, we replace P as a function of (αr_0) (see Fig. 10) by the linear approximation afforded by the first two terms of a Taylor's expansion around $\alpha r_0=0$. These linear approximations are collected in Table IV for the well shapes which have been treated numerically so far.

The relation (8.2) enables us, in the case of the triplet state, to convert the dimensionless figures of Section 2 into dimensional plots. For any given well shape we get four relations between the five quantities, s, b, α_t, r_t, and T_t (the first two are the well parameters, the latter three are variational
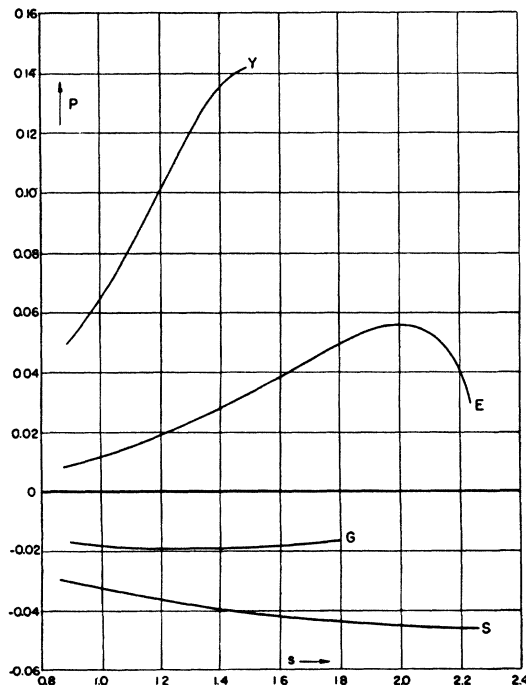

FIG. 12. Ordinate: P; Abscissa: s. The coefficient P is shown as a function of the well depth parameter s for the four well shapes.

TABLE IV. Linear approximation for P as a function of (αr_0) for various well shapes.

Shape	$P = P(0) + P'(0)(\alpha r_0) + \cdots$
Square well	$-0.0327 - 0.0184(\alpha r_0)$
Gaussian well	$-0.0183 - 0.0060(\alpha r_0)$
Exponential well	$+0.0119 + 0.0190(\alpha r_0)$
Yukawa well	$+0.0648 + 0.0942(\alpha r_0)$

parameters) so that only one of them can be varied independently.

There are various ways of showing the results graphically. We have decided upon two kinds of graphs: (1) the intrinsic range b_t in the triplet state vs. the triplet scattering length a_t for the various wells; the values of r_t are indicated parametrically on each curve; (2) the well depth parameter s for the triplet well as a function of the intrinsic range b for the various well shapes; again r_t is indicated parametrically along the curves.

These graphs are used as follows: the experimentally known a_t is used to find the corresponding intrinsic range b for whatever well shape is assumed (type 1 plot). Then the well depth s implied by this value of the intrinsic range (and the binding energy of the deuteron) is found from the type 2 plot.
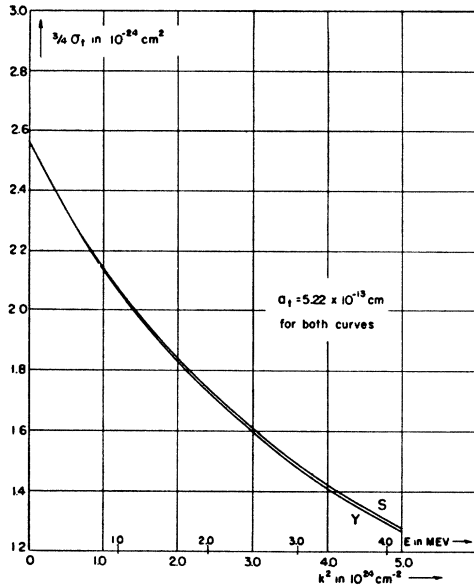

FIG. 13. Ordinate: $\frac{3}{4}\sigma_t$; Abscissa: k^2, E. The triplet part of the incoherent scattering cross section is shown as a function of energy for the two extremes in well shapes treated, the square and the Yukawa wells. The expansion $k \cot\delta = -1/a_t + \frac{1}{2}r_t k^2 - Pr_t^3 k^4 + \cdots$ was used to evaluate $\sigma_t (= (4\pi/k^2) \sin^2\delta)$ with the value of a_t determined from Fig. 7 ($a_t = 5.22 \times 10^{-13}$ cm). The smallness of the difference in cross section (only about 0.01 barns at 4 Mev) is due to the smallness of the coefficients Pr_0^3. It implies that in order to determine anything about the potential shape from incoherent scattering data the experiments of 2-6 Mev must be accurate to much better than one percent, assuming that the singlet cross section is known exactly (a very unreasonable assumption).
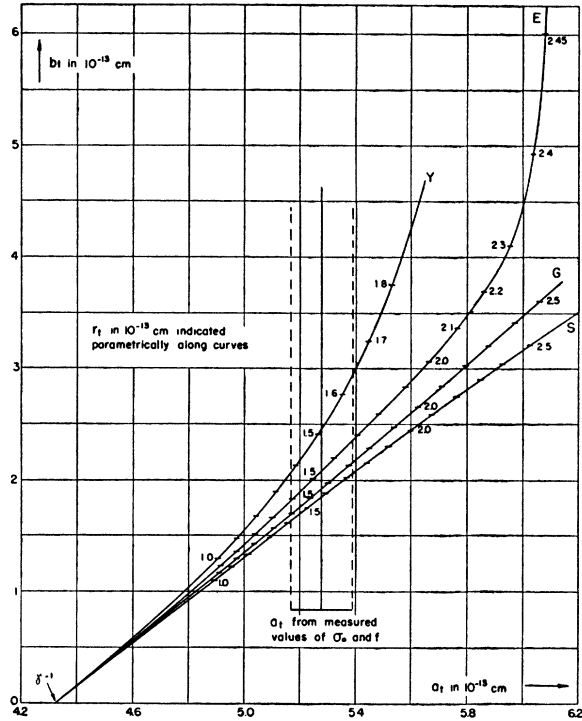

FIG. 14. Ordinate: b_t; Abscissa: a_t. The intrinsic range in the triplet state b_t is shown versus the triplet scattering length a_t for the four well shapes. The values of the triplet effective range r_t for each well shape are indicated parametrically along the curves. The experimental value of a_t (from Fig. 7) is shown. The value of the effective range r_t consistent with this a_t is about the same for all four well shapes; $r_t \cong 1.5 - 1.6 \times 10^{-13}$ cm. The "best" intrinsic ranges can be read off the figure.

The effective range r_t in the triplet state can be read off from either plot.

Plots of type (1) and (2) are shown as Figs. 14 and 15 respectively. We have indicated on Fig. 14 the present experimental value of a_t with an estimate of error. This value comes from measurements of the incoherent and coherent n-p cross sections at epithermal energies.[28-30] It is seen that the effective range r_t implied by this value at a_t lies within the limits (6.9) for all the well shapes considered. The fact that we get nearly the same r_t for all well shapes is a consequence of the smallness of the coefficient P for these wells. The residual amount of variation of r_t with different well shapes for the same experimental a_t indicates the effect of P.

One can also look at formula (8.1) from a different point of view. We know γ from the binding energy of the deuteron, and we know the triplet state $\alpha = \alpha_t$ from the cross section (coherent and incoherent) near zero energy. We then see that (8.1) gives us a linear relation between r_t and T_t. This linear relation is illustrated in Fig. 16 for some representative values of α_t (γ is known with sufficient accuracy so that the error due to it can be neglected). Also shown on Fig. 16 are curves of
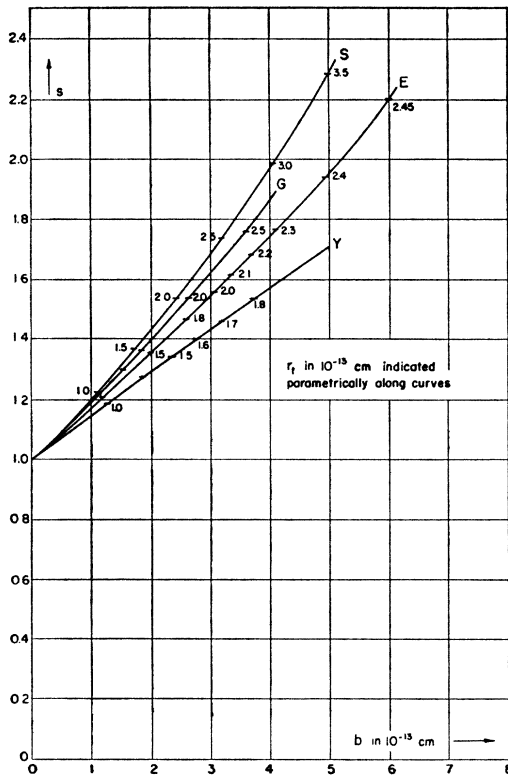
Fig. 15. Ordinate: s; Abscissa: b. The well depth parameter s is given as a function of the intrinsic range b for the four well shapes. Values of the triplet effective range r_t are indicated parametrically along each curve.

not know the values of P for some representative "velocity-dependent" forces.

While an analysis of the experimental data to find P in the triplet and singlet states is possible in principle, we have concluded that the small values of P which we can expect to find and the large experimental errors with which we are faced make such an analysis unprofitable at this time. More accurate cross section measurements would help here, but the required accuracy is probably beyond the limits of the present techniques.

We conclude this section by collecting for reference purposes for the four well shapes treated the following interpolation formulas for the well depth parameter s:

(1) s as a function of (αr_0),
(2) s as a function of (αb),
(3) s as a function of (γb).

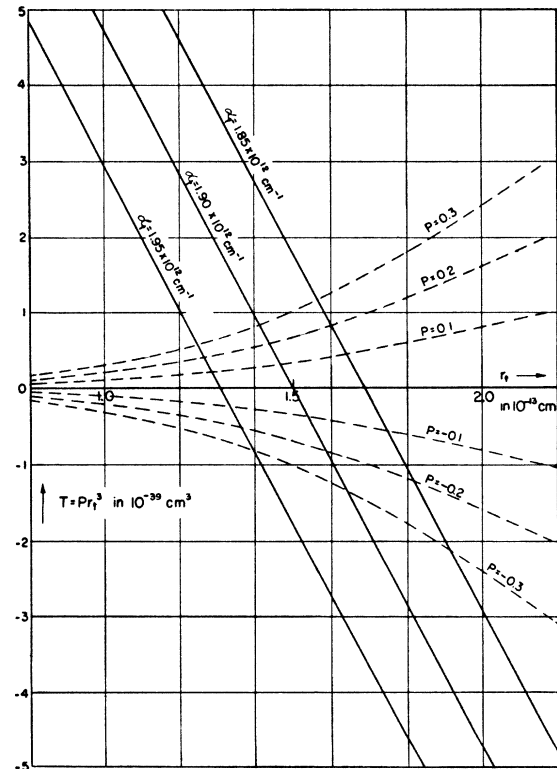These are collected in Table V. The square well and the exponential well can, of course, be solved

constant P. We recall that for the wells calculated so far, P lies between -0.04 and $+0.15$ for the region of (αr_0) involved here. Furthermore, P is positive for the exponential and Yukawa wells, negative for the square and Gaussian wells. The analysis involving an extra lump of potential (see formula (7.11) and the discussion there) leads us to believe that attractive wells with a reasonably large tail will have positive values of P. Putting these two pieces of information together, a look at Fig. 16 indicates that the estimate (6.8) of the triplet effective range is not affected appreciably by the well shape, within the experimental error. Three cautions are in place here, however: (1) the values of P were obtained without tensor forces. Until the calculation with tensor force is completed, the result we have just stated must be considered not quite certain; (2) we have assumed that the nuclear force in the triplet state is attractive at all distances. Our analysis shows that a lump of repulsive potential placed some distance out beyond the main range b would give a negative contribution to P. This possibility seems unlikely but it cannot be excluded on the basis of our present knowledge regarding the nature of nuclear forces; (3) We do

Fig. 16. Ordinate: T; Abscissa: r_t. The coefficient $T(=Pr_t^3)$ of the k^4 term in the expansion of $k \cot\delta$ for the triplet state is shown *versus* the triplet effective range r_t. Equation (8.1) determines a linear relation between r_t and T once a value of $\alpha_t(=a_t^{-1})$ is specified (the value of γ being well known). This linear relation for representative values of α_t is plotted on the figure. Also given are curves of constant P. For values of $|P| < 0.2$ (which is the case for the four well shapes considered), the value of r_t implied by a chosen value of α_t is not significantly different from the value implied by $P = 0$ (see also Fig. 14).

TABLE V. Interpolation formulas for the well-depth parameter s as a function of (αr_0), (αb), (γb) for various well shapes.

I. s versus (αr_0)	
Shape	s
Square well	$s = [1 - 1.574(\alpha r_0) + 0.353(\alpha r_0)^2] \times [1 - 2.384(\alpha r_0) + 1.30(\alpha r_0)^2]^{-1}$
Gaussian well	$s = [1 - 2.030(\alpha r_0) + 0.587(\alpha r_0)^2] \times [1 - 2.810(\alpha r_0) + 1.805(\alpha r_0)^2]^{-1}$
Exponential well	$s = [1 - 2.9739(\alpha r_0) + 1.3602(\alpha r_0)^2] \times [1 - 3.7134(\alpha r_0) + 3.1177(\alpha r_0)^2]^{-1}$
Yukawa well	$s = [1 - 3.5093(\alpha r_0) + 2.0354(\alpha r_0)^2] \times [1 - 4.1348(\alpha r_0) + 3.7072(\alpha r_0)^2]^{-1}$

II. s versus (αb)	
Shape	s
Square well	$s = 1.0 + 0.8098(\alpha b) + 0.6516(\alpha b)^2 + 0.5229(\alpha b)^3 + 0.185(\alpha b)^4 + 0.747(\alpha b)^5$
Gaussian well	$s = 1.0 + 0.7811(\alpha b) + 0.5687(\alpha b)^2 + 0.352(\alpha b)^3 + 0.262(\alpha b)^4 + 0.020(\alpha b)^5$
Exponential well	$s = 1.0 + 0.7109(\alpha b) + 0.4411(\alpha b)^2 + 0.4658(\alpha b)^3 - 0.3877(\alpha b)^4$
Yukawa well	$s = 1.0 + 0.6361(\alpha b) + 0.2837(\alpha b)^2 + 0.0338(\alpha b)^3 - 0.150(\alpha b)^4 - 0.031(\alpha b)^5$

III. s versus (γb)	
Shape	s
Square well	$s = 1.0000 + 0.8104(\gamma b) + 0.2426(\gamma b)^2 + 0.0184(\gamma b)^3 + 0.0041(\gamma b)^4 + \cdots$
Gaussian well	$s = 1.0000 + 0.7806(\gamma b) + 0.1773(\gamma b)^2 + \cdots$
Exponential well	$s = 1.0000 + 0.7517(\gamma b) - 0.0046(\gamma b)^2 + 0.0657(\gamma b)^3 - \cdots$
Yukawa well	$s = 1.0000 + 0.6364(\gamma b) - 0.0340(\gamma b)^2 + 0.0067(\gamma b)^3 + 0.0139(\gamma b)^4 - \cdots$

exactly so that the numerical formulas are unnecessary. They are retained only for the sake of uniformity and completeness. A curve of αb vs. s is plotted in Fig. 4. Since, for some purposes, more accuracy is desired than can be obtained from a graph, the numerical formula (2) may be of some use. Figure 15 shows s in terms of b. To plot this graph, the value $\gamma^{-1} = 4.332 \times 10^{-13}$ cm was used to convert values of γb to corresponding values of b. In view of the experimental uncertainty in γ, it is useful to have a numerical relation between s and (γb) since such a relation will be independent of the experimental value of γ.

The ranges of validity of the formulas in Table V are as follows for 0.02 percent accuracy in s:

(1) $-0.2 < \alpha r_0 < 0.4$,
(2) $-0.2 < \alpha b < 0.6$,
(3) $0 < \gamma b < 0.6$.

With these formulas one can determine quite rapidly the well depth for a given well shape upon assuming a value of the intrinsic range, for example, and knowing the binding energy of the deuteron (from 3). The formulas agree quite closely with those obtained by variational methods applied to the calculation of s directly.

ACKNOWLEDGMENTS