Clark, Spencer-Palmer, and Woodwards' value of 3.63 percent, obtained with an alpha-ray analyzer. While there are some discrepancies between alpha-activity values, the author feels that since the values determined are dependent on the alpha-activities of both uranium 238 and normal uranium, which are in very close agreement with accepted values,[3, 6] the precisions quoted are reasonable.

This document is based on work performed for the Atomic Energy Commission by Carbide and Carbon Chemicals Corporation, at Oak Ridge, Tennessee.

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Comparison of Calculations on Cascade Theory*

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The diffusion equations of the cascade theory of electron showers are solved using improved mathematical methods. The results obtained agree essentially with those obtained earlier by Carlson and Oppenheimer, by Snyder, and by Serber, but do not agree with those obtained by Bhabha and Chakrabarty. Solutions are given corresponding to a single incident electron, a single incident γ-ray, and a $1/E$ spectrum of γ-rays. The total number of particles expected under a given thickness of material are given in tables for various incident energies and each of the above initial conditions. Other tables and formulas are included which enable one to calculate the spectrum of the particles and γ-rays for various initial conditions.

I. INTRODUCTION

SINCE the original papers on the theory of cascade showers by Bhabha and Heitler[1] and by Carlson and Oppenheimer[2] other contributions have been made by Snyder,[3] Landau and Rumer,[4] Serber,[5] Iyengar,[6] and Bhabha and Chakrabarty.[7] The work of Bhabha and Heitler was carried out without including ionization loss and is thus limited to the high energy portion of the spectrum. The calculations of Carlson and Oppenheimer did include ionization loss, but used simplified asymptotic forms for the high energy cross sections. They also replaced the integral equations of cascade theory by a simplifying differential equation. Their solution does not satisfy boundary conditions exactly, but provided the energy of the incident particle is sufficiently large, the error is insignificant. The work of Snyder was an extension of that of Carlson and Oppenheimer, using the same cross sections, but using the integral equations. This solution did not satisfy exact boundary conditions. A difficulty in this work which has been emphasized by Bhabha and Chakrabarty was that a certain function was computed only for integral values of its argument, the non-integral values being obtained by graphical interpolation. Soon after, Landau and Rumer showed that the exact forms for the asymptotic cross sections produced no essential complication. Serber then continued the series of calculations as they were begun by Carlson and Oppenheimer and extended by Snyder, but also with the reservation that the incident energy be large enough so that the errors in the boundary conditions are small. Still later, Iyengar gave a complete solution in which the exact Bethe-Heitler cross sections were used; a solution from which it is not easy to obtain numerical values according to Bhabha and Chakrabarty. At about this same time, Bhabha and Chakrabarty also gave in series forms, an exact solution of the diffusion equations using asymptotic forms for the cross sections.

One of the major difficulties in this work has been to express the solution in such a form that numerical values can be obtained with reasonable ease. This has been particularly true for the total number of particles present at a given thickness and for the energy spectrum of the particles and γ-rays at low energies. Although Bhabha and Chakrabarty have given a series solution which converges even for zero energy, the first two terms in their expansion account for only about seventy-five percent of the total energy dissipated in a shower. On the other hand, the forms of solution as given by Carlson and Oppenheimer, by Snyder and by Serber, give in a single term the total energy dissipated in a shower. The objective of the work being reported in this paper is: (a) to give exact representations for all values of its arguments of the function which had previously been computed only for integral values of its argument, (b) to give a solution of the diffusion equations which satisfies the correct boundary conditions,

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(c) to give formulas and tables by means of which the low energy spectrum of the particles and γ-rays may be computed, (d) to compare the results obtained with those of Bhabha and Chakrabarty.

II. THE DIFFUSION EQUATIONS

The physical basis of the diffusion equations is so well known that we will just reproduce them here. We denote by $P(E, t)$ the number of particles, by $\gamma(E, t)$ the number of quanta each per unit energy at energy E and at depth t. The ionization loss per unit of t we call β. Length t is measured in radiation units

$$\left[4\frac{Z^2N}{137}\left(\frac{e^2}{mc^2}\right)^2 \ln (191Z^{-1/3})\right]^{-1}.$$

The diffusion equations then are

$$\frac{\partial P(E,t)}{\partial t}-\beta\frac{\partial P(E,t)}{\partial E} = \lim_{\delta\to 0}\left[\int_{E+\delta}^{\infty} P(E',t)R(E',E'-E)\frac{E'-E}{E'^2}dE' - P(E,t)\int_{\delta}^{E} R(E,E')\frac{E'dE'}{E^2}\right] + 2\int_E^{\infty}\gamma(E',t)R(E,E')\frac{dE'}{E'} \quad (1)$$

$$\frac{\partial\gamma(E,t)}{\partial t}=\int_E^{\infty} P(E',t)R(E',E)\frac{EdE'}{E'^2} - \gamma(E,t)\int_0^{E} R(E',E)\frac{dE'}{E}\cdot \quad (2)$$

In these equations we take

$$R(E,E')=\left[1-\frac{4}{3}\frac{E}{E'}+\frac{4}{3}\left(\frac{E}{E'}\right)^2\right]\left[1+\frac{3}{4}\alpha\right]-\frac{3}{4}\alpha \quad (3)$$

$$\alpha=[9 \ln 191Z^{-\frac{1}{3}}]^{-1}\simeq\frac{1}{40}\cdot \quad (4)$$

Through the use of Eq. (3) we are limiting outselves to the asymptotic forms for the cross sections. We now look for solutions of (1) and (2) depending on a parameter, y, of the form

$$P_y(E,t)=\frac{1}{2\pi i\beta}\times\int_C\left(\frac{\beta}{E}\right)^{y+s+1}K_\mu(y,s)\Gamma(-s)\Gamma(y+s+1)e^{\mu t}ds, \quad (5)$$

$$\gamma_y(E,t)=\frac{1}{2\pi i\beta}\times\int_C\left(\frac{\beta}{E}\right)^{y+s+1}L_\mu(y,s)\Gamma(-s)\Gamma(y+s+1)e^{\mu t}ds. \quad (6)$$

The contour of integration for s is taken to be a straight line parallel to the imaginary axis passing to the left of the origin and from $-i\infty-\delta$ to $i\infty-\delta$ with $\delta>0$. Substituting (5) and (6) into (1) and (2) and using (3) we get

$$\frac{1}{2\pi i\beta}\int_C ds\left(\frac{\beta}{E}\right)^{y+s+1}\Gamma(-s)\Gamma(y+s+1)\times\left\{[\mu+A(y+s)]K_\mu(y,s)-B(y+s)L_\mu(y,s)+(y+s+1)\frac{\beta}{E}K_\mu(y,s)\right\}e^{\mu t}=0, \quad (7)$$

and

$$\frac{1}{2\pi i\beta}\int_C ds\left(\frac{\beta}{E}\right)^{y+s+1}\Gamma(-s)\Gamma(y+s+1)\times\{[\mu+D]L_\mu(y,s)-C(y+s)K_\mu(y,s)\}e^{\mu t}=0, \quad (8)$$

in which

$$A(y)=\left(\frac{4}{3}+\alpha\right)\left(\frac{d\ln\Gamma(y+1)}{dy}+.5772\cdots-1+\frac{1}{y+1}\right)+\frac{1}{2}-\frac{1}{(y+1)(y+2)}, \quad (9)$$

$$B(y)=2\left(\frac{1}{y+1}-\right)\left(\frac{4}{3}+\alpha\right)\left(\frac{1}{(y+2)(y+3)}\right), \quad (10)$$

$$C(y)=\frac{1}{y+2}+\left(\frac{4}{3}+\alpha\right)\frac{1}{y(y+1)}, \quad (11)$$

$$D=7/9-1/6\alpha. \quad (12)$$

In order that (7) and (8) follow from (1) and (2), the real part of $y+s$ must be positive. Now, both terms in (8), and all except the last term in (7) have the same power of (β/E). If $K_\mu(y,s)$ is an analytic function of s in the strip from $-i\infty$ to $i\infty$ with real part between $-1-\delta$ and $-\delta$, the contour of integration for s may be displaced one unit to the left in the last term of (7). Under these conditions (5) and (6) will be a solution of

(1) and (2) provided

$$[\mu+A(y+s)]K_\mu(y,s)-B(y+s)L_\mu(y,s) = sK_\mu(y,s-1) \quad (13)$$

$$[\mu+D]L_\mu(y,s)-C(y+s)K_\mu(y,s)=0. \quad (14)$$

Eliminating $L_\mu(y,s)$ between Eqs. (13) and (14), we get

$$\{\mu^2+[A(y+s)+D]\mu+A(y+s)D - B(y+s)C(y+s)\}K_\mu(y,s) = s[\mu+D]K_\mu(y,s-1). \quad (15)$$

In order to make the dependence of μ definite, we take it to be a function of y satisfying the equation

$$\mu^2(y)+[A(y)+D]\mu(y)+A(y)D-B(y)C(y)=0. \quad (16)$$

The roots of (16) will now be called $\mu(y)$ and $\nu(y)$ with

$$\begin{aligned}\mu(y)&=-\tfrac{1}{2}[A(y)+D] +\tfrac{1}{2}\{[A(y)-D]^2+4B(y)C(y)\}^{\frac{1}{2}},\\ \nu(y)&=-\tfrac{1}{2}[A(y)+D] -\tfrac{1}{2}\{[A(y)-D]^2+4B(y)C(y)\}^{\frac{1}{2}}.\end{aligned} \quad (17)$$

By eliminating $\mu^2(y)$ between (16) and (15), we get

$$\{[A(y+s)-A(y)][\mu(y)+D] +B(y)C(y)-B(y+s)C(y+s)\}K_\mu(y,s) = s[\mu(y)+D]K_\mu(y,s-1), \quad (18)$$

or, to save writing later,

$$K_\mu(y,s)=g_\mu(y,s-1)K_\mu(y,s-1), \quad (19)$$

with

$$g_\mu(y,s-1) = \frac{s[\mu(y)+D]}{\left([A(y+s)-A(y)][\mu(y)+D] +B(y)C(y)-B(y+s)C(y+s)\right)}. \quad (20)$$

We shall show that a solution of (19) is

$$K_\mu(y,s+z)=K_\mu(y,s)[g_\mu(y,s)]^z \times\prod_{n=0}^{\infty}\left\{\frac{g_\mu(y,s+n)}{g_\mu(y,s+z+n)}\left(\frac{g_\mu(y,s+n+1)}{g_\mu(y,s+n)}\right)^z\right\}. \quad (21)$$

Using the asymptotic form for $A(y)$ it is easy to verify that the infinite product in (21) converges and that $\lim_{N\to\infty}[g_\mu(y,s+N)/g_\mu(y,N)]=1$, conditions which are necessary and sufficient for (21) to be a solution of (19). If (21) is a solution of (19) we should have (setting $s\to 0$, $z\to s$)

$$K_\mu(y,s)=\lim_{N\to\infty}[g_\mu(y,N+1)]^s \times\prod_{n=0}^{N}\frac{g_\mu(y,n)}{g_\mu(y,s+n)}, \quad (22)$$

if we take $K_\mu(y,0)=1$. Substituting (22) into (21) we get

$$\begin{aligned}K_\mu(y,s+z)&=\lim_{N\to\infty} g_\mu(y,N+1)^s\prod_{n=0}^{N}\frac{g_\mu(y,n)}{g_\mu(y,s+n)} \times\lim_{M\to\infty} g_\mu(y,s+M+1)^z\prod_{m=0}^{M}\frac{g_\mu(y,s+m)}{g_\mu(y,s+z+m)}\\ &=\lim_{N\to\infty}\left(\frac{g_\mu(y,s+N+1)}{g_\mu(y,N+1)}\right)^z g_\mu(y,N+1)^{s+z} \times\prod_{n=0}^{N}\frac{g_\mu(y,n)}{g_\mu(y,s+z+n)}.\end{aligned} \quad (23)$$

Since $\lim_{N\to\infty}\dfrac{g_\mu(y,s+N+1)}{g_\mu(y,N+1)}=1$ (23) gives

$$K_\mu(y,s+z)=\lim_{N\to\infty} g_\mu(y,N+1)^{s+z} \times\prod_{n=0}^{N}\frac{g_\mu(y,n)}{g_\mu(y,s+z+n)}, \quad (24)$$

which is (22) with s replaced by $s+z$. We thus see that the right-hand side of (21) depends only on $s+z$, as it must, if the left side is to depend only on $s+z$. Since (21) is valid for all values of z, it holds for $z=1$, in which case we get $K_\mu(y,s+1)=g_\mu(y,s)K_\mu(y,s)$ which is (19) with s replaced by $s+1$. Thus we see that (21) is a solution of (19).

The analytic character of $K_\mu(y,s)$ is most readily seen from Eq. (22). From the expression for $g_\mu(y,s)$ as determined from Eq. (20), and the values of $A(y)$, $B(y)$, and $C(y)$, one can see that $g_\mu(y,s)$ is finite and non-vanishing for all finite values of s except for zeros at $s=-y-n$, $n=1,2,3\cdots$. As a consequence of this property and the fact that (21) converges, $K_\mu(y,s)$ is analytic and non-vanishing for all finite values of s except for isolated poles at the points $s=-y-n$ with $n=1,2,3\cdots$. From the fact obtained earlier that real part $(y+s)$ must be greater than zero, we now see that the condition given earlier that the function $K_\mu(y,s)$ must be analytic in s in the strip $-i\infty$ to $i\infty$ with real part of s between $-1-\delta$ and $-\delta$ is in fact satisfied by (21).

The infinite product given by (21) does not converge very rapidly, so it is not very useful for numerical calculations. However, other expressions which converge to the same value, but which converge much more rapidly can be found. The sequence of functions which was used for the numerical calculations is

$$K_\mu(y, s+z)=K_\mu(y, s)\lim_{N\to\infty}[g_\mu(y, s+N-1)]^{\alpha(z)}\times[g_\mu(y, s+N)]^{\beta(z)}[g_\mu(y, s+N+1)]^{\gamma(z)}\times[g_\mu(y, s+N+2)]^{\delta(z)}\prod_{n=0}^{N}\frac{g_\mu(y, s+n)}{g_\mu(y, s+z+n)} \quad (25)$$

with

$$\begin{aligned}\alpha(z)&=-1/24\, z(z+1)(z-1)(z-2),\\ \beta(z)&=\ \ 1/24\, z(z-1)(z-2)(3z+7),\\ \gamma(z)&=-1/24\, z(z+1)(z+2)(3z-7),\\ \delta(z)&=\ \ 1/24\, z(z-1)(z+1)(z+2).\end{aligned}$$

The polynomials $\alpha(z)$, $\beta(z)$, $\gamma(z)$, $\delta(z)$ were chosen so that (25) gives exact results for all N if $z=0, \pm1, \pm2$. Also, $\alpha(z)+\beta(z)+\gamma(z)+\delta(z)=z$ which insures that (25) converges for the same values as (21).

However, for a given value of N, the number of terms in the product, and for small z (25) give results better by several significant figures than does (21).

III. THE BOUNDARY VALUE PROBLEM

In Section II we have given two solutions for each value of a certain parameter, y, of the diffusion equations of cascade theory. Our problem in this section will be to combine these solutions in such a manner that the resulting solution will represent either a single charged particle or a single photon of energy E_0, incident on the top of the material. To do this we write

$$P(E, t)=-\frac{1}{(2\pi)^2\beta}\sum_{n=0}^{\infty}\int_D dy\int_C ds\times\frac{\Gamma(-s)\Gamma(y+s+n+1)}{\Gamma(y+n+1)[\mu(y+n)-\nu(y+n)]}\times\left(\frac{E_0}{\beta}\right)^y\left(\frac{\beta}{E}\right)^{y+s+n+1}\times\{A_n(y+n)[\mu(y+n)+D]K_\mu(y+n, s)e^{\mu(y+n)t}-B_n(y+n)[\nu(y+n)+D]K_\nu(y+n, s)e^{\nu(y+n)t}\}, \quad (26)$$

$$\gamma(E, t)=-\frac{1}{(2\pi)^2\beta}\sum_{n=0}^{\infty}\int_D dy\int_C ds\times\frac{\Gamma(-s)\Gamma(y+s+n+1)}{\Gamma(y+n+1)[\mu(y+n)-\nu(y+n)]}\times\left(\frac{E_0}{\beta}\right)^y\left(\frac{\beta}{E}\right)^{y+s+n+1}C(y+s+n)\times\{A_n(y+n)K_\mu(y+n, s)e^{\mu(y+n)t}-B_n(y+n)K_\nu(y+n, s)e^{\nu(y+n)t}\}. \quad (27)$$

Equations (26) and (27) are simply linear combinations of (5) and (6), respectively, using both roots of Eq. (16) as given in Eq. (17). The function $K_\nu(y, s)$ is determined by using $\nu(y)$ instead of $\mu(y)$ in (21). In Eq. (27) we have used relation (14) to eliminate $L_\mu(y, s)$ and $L_\nu(y, s)$. Our problem is to determine the functions $A_n(y+n)$ and $B_n(y+n)$ to fit the given boundary conditions. The contour of integration for y is taken to be a straight line from $-i\infty+\epsilon$ to $i\infty+\epsilon$ and with real part of $y+s$ greater than zero.

If we now evaluate the integral over s in $P(E, t)$ in terms of the residues of the integrand at $s=0, 1, 2, \cdots$, we can write the result in the form

$$P(E, t)=\frac{1}{2\pi i\beta}\sum_{N=0}^{\infty}\sum_{k=0}^{N}\int_D\frac{dy\Gamma(y+N+1)(-1)^k\left(\frac{E_0}{E}\right)^y\left(\frac{\beta}{E}\right)^{N+1}}{\Gamma(k+1)\Gamma(y+N-k+1)[\mu(y+N-k)-\nu(y+N-k)]}\times\{A_{N-k}(y+N-k)[\mu(y+N-k)+D]K_\mu(y+N-k, k)e^{\mu(y+N-k)t}-B_{N-k}(y+N-k)[\nu(y+N-k)+D]K_\nu(y+N-k, k)e^{\nu(y+N-k)t}\}. \quad (28)$$

An entirely analogous expression could have been written for $\gamma(E, t)$.

Since the functions $A_n(y+n)$ and $B_n(y+n)$ are not uniquely determined at this point, we subject them to the relationships

$$\sum_{k=0}^{N}\frac{(-1)^k[\mu(y+N-k)-\nu(y+N-k)]^{-1}}{\Gamma(k+1)\Gamma(y+N-k+1)}\{A_{N-k}(y+N-k)[\mu(y+N-k)+D]K_\mu(y+N-k, k)-B_{N-k}(y+N-k)[\nu(y+N-k)+D]K_\nu(y+N-k, k)\}=0, \quad (29)$$

$$\sum_{k=0}^{N}\frac{(-1)^k[\mu(y+N-k)-\nu(y+N-k)]^{-1}}{\Gamma(k+1)\Gamma(y+N-k+1)}\{A_{N-k}(y+N-k)K_\mu(y+N-k, k)-B_{N-k}(y+N-k, k)K_\nu(y+N-k, k)\}=0, \quad (30)$$

for $N=1, 2, 3, \cdots$. The relations (29) and (30) were chosen so that only the terms with $A_0(y)$ and $B_0(y)$ survive at $t=0$ in (27) and (28). If we use relationships (29) and (30), we find for $t=0$ the values

$$P(E,0)=\frac{1}{2\pi iE_0}\int_D dy\left(\frac{E_0}{E}\right)^{y+1}\times\left\{\frac{A_0(y)[\mu(y)+D]-B_0(y)[\nu(y)+D]}{\mu(y)-\nu(y)}\right\}, \quad (31)$$

$$\gamma(E,0)=\frac{1}{2\pi iE_0}\int_D dy\left(\frac{E_0}{E}\right)^{y+1}\times\left\{\frac{A_0(y)-B_0(y)}{\mu(y)-\nu(y)}\right\}C(y). \quad (32)$$

If the incident radiation is a single charged particle of energy E_0, then from (31) and (32) we obtain

$$A_0(y)=B_0(y)=1. \quad (33)$$

If the incident radiation is a single γ-ray of energy E_0, we obtain

$$A_0\dagger(y)=-\frac{[\nu(y)+D]}{C(y)},\quad B_0\dagger(y)=-\frac{[\mu(y)+D]}{C(y)}. \quad (34a)$$

If the incident radiation contains no charged particles and has a $1/E$ distribution of γ-rays up to the energy E_0,

$$A_0{}^*(y)=-\frac{[\nu(y)+D]}{yC(y)},\quad B_0{}^*(y)=-\frac{[\mu(y)+D]}{yC(y)}. \quad (34b)$$

Note the fact that we have placed an asterisk on the $A_0(y)$ to indicate these particular boundary conditions. We have also placed a dagger on the $A_0\dagger(y)$ and $B_0\dagger(y)$ to indicate the particular boundary condition as given by (34a). Throughout the remainder of this paper an asterisk or dagger will be placed on various quantities whenever it is necessary to distinguish the initial conditions (34a) or (34b) from (33).

We now give the solution of Eqs. (29) and (30) for $A_N(y+N)$ and $B_N(y+N)$, namely,

$$A_N(y+N)=\sum_{k=1}^{N}\frac{(-1)^{k+1}\Gamma(y+N+1)}{\Gamma(y+N-k+1)[\mu(y+N-k)-\nu(y+N-k)]\Gamma(k+1)}\times\{[\mu(y+N-k)-\nu(y+N)]K_\mu(y+N-k,k)A_{N-k}(y+N-k)-[\nu(y+N-k)-\nu(y+N)]K_\nu(y+N-k,k)B_{N-k}(y+N-k)\}. \quad (35)$$

$$B_N(y+N)=\sum_{k=1}^{N}\frac{(-1)^{k+1}\Gamma(y+N+1)}{\Gamma(y+N-k+1)[\mu(y+N-k)-\nu(y+N-k)]\Gamma(k+1)}\times\{[\mu(y+N-k)-\mu(y+N)]K_\mu(y+N-k,k)A_{N-k}(y+N-k)-[\nu(y+N-k)-\mu(y+N)]K_\nu(y+N-k,k)B_{N-k}(y+N-k)\}. \quad (36)$$

Through the recursion relationships (35) and (36), $A_N(y+N)$ and $B_N(y+N)$ can ultimately be expressed in terms of certain rational functions of y, $\mu(y+N)$, $\nu(y+N)$ through the values of $A_0(y)$ and $B_0(y)$ as given by either (33) or (34).

We note here that if we define

$$f_{N+1}(y+1,t)=\sum_{k=0}^{N}\frac{(-1)^{k+N}\Gamma(y+1)}{\Gamma(k+1)\Gamma(y+N-k+1)[\mu(y+N-k)-\nu(y+N-k)]}\times\{A_{N-k}(y+N-k)[\mu(y+N-k)+D]K_\mu(y+N-k,k)e^{\mu(y+N-k)t}-B_{N-k}(y+N-k)[\nu(y+N-k)+D]K_\nu(y+N-k,k)e^{\nu(y+N-k)t},$$

then we can write (28) in the form

$$P(E,t)=\frac{1}{2\pi iE_0}\int_D dy\left(\frac{E_0}{E}\right)^{y+1}\sum_{N=0}^{\infty}f_N(y+1,t). \quad (37)$$

One can prove, except for notation, if we use $A_0(y)=B_0(y)=1$, that the functions $f_N(y,t)$ are identical with those given by Bhabha and Chakrabarty in their Eqs. (11); thus our solution is, as it must be, identical with their solution.

In order to express the solution in a form which is most useful for obtaining the total number of particles in the shower, and the low energy spectrum of the particles and the γ-rays, we examine the analytic character of the functions $A_N(y)$ and $B_N(y)$. To do this we note that the functions $\mu(y)$ and $\nu(y)$ have branch points at $y=0$ and in the negative half-plane. As one passes around a branch point the functions $\mu(y)$ and $\nu(y)$ interchange. Exactly the same properties are true of the functions $K_\mu(y,s)$ and $K_\nu(y,s)$. This property

Table I.

y	$\mu(y)$	$a(y)$	$b(y)$	$H(y)$	$M(y)$	$A_1(y)$	$\alpha(y)$	$\beta(y)$	$A_0^*(y)$	$A_1^*(y)$	$\alpha^*(y)$	$\beta^*(y)$	$f(y)$
0.1	3.7875	2.4997	2.718	0.187	0.100	−0.0019	0.4892	0.4894	3.0711	0.00872	1.0354	0.8103	3.600
0.2	2.2793	1.8904	2.415	0.275	0.200	−0.0099	0.5002	0.4447	2.0975	0.0365	1.0516	0.9388	3.385
0.3	1.5686	1.6239	2.185	0.3295	0.300	−0.021	0.5460	0.3663	1.6800	0.0813	1.0864	0.9487	3.140
0.4	1.1253	1.4614	2.055	0.3645	0.410	−0.041	0.6220	0.2603	1.4424	0.1410	1.1385	0.9096	2.890
0.5	0.8122	1.3458	1.875	0.3830	0.515	−0.054	0.7320	0.1811	1.2900	0.2220	1.2138	0.8616	2.635
0.6	0.5751	1.2548	1.762	0.3940	0.620	−0.061	0.8440	0.0937	1.1859	0.2980	1.2818	0.8225	2.375
0.7	0.3876	1.1785	1.680	0.4005	0.730	−0.063	0.9700	0.0918	1.1128	0.4030	1.3558	0.8684	2.095
0.8	0.2347	1.1110	1.635	0.4021	0.841	−0.052	1.093	0.148	1.0609	0.5153	1.4196	0.9712	1.705
0.9	0.1075	1.0490	1.599	0.4000	0.960	−0.034	1.205	0.250	1.0245	0.6250	1.4681	1.1114	1.530
1.0	0	0.9905	1.562	0.3951	1.074	0	1.304	0.379	1.0000	0.7360	1.4991	1.2687	1.284
1.1	−0.0918	0.9344	1.545	0.3875	1.200	+0.0418	1.390	0.521	0.9848	0.8594	1.5146	1.4261	1.085
1.2	−0.1706	0.8797	1.537	0.3775	1.315	0.087	1.463	0.664	0.9772	0.9870	1.5157	1.5772	0.810
1.3	−0.2390	0.8291	1.520	0.3640	1.435	0.148	1.524	0.796	0.9758	1.123	1.5083	1.6935	0.750
1.4	−0.3001	0.7690	1.500	0.3490	1.550	0.213	1.575	0.916	0.9778	1.272	1.4821	1.7925	0.615
1.5	−0.3500	0.7237	1.475	0.3335	1.670	0.289	1.618	1.024	0.9877	1.430	1.4645	1.8471	0.480
1.6	−0.3952	0.6747	1.445	0.3195	1.790	0.370	1.654	1.121	0.9996	1.590	1.4388	1.8789	0.360
1.7	−0.4346	0.6276	1.420	0.3050	1.910	0.454	1.686	1.208	1.0145	1.759	1.4122	1.8766	0.250
1.8	−0.4692	0.5826	1.375	0.2915	2.035	0.546	1.712	1.279	1.0321	1.950	1.3854	1.8486	0.155
1.9	−0.4996	0.5400	1.335	0.2785	2.150	0.641	1.734	1.342	1.0517	2.140	1.3608	1.8030	0.055
2.0	−0.5262	0.5001	1.276	0.2673	2.272	0.746	1.753	1.389	1.0727	2.333	1.3382	1.7565	−0.034
2.1	−0.5497	0.4625	1.215	0.2555	2.395	0.845	1.770	1.446	1.0962	2.545	1.3185	1.7104	−0.135
2.2	−0.5704	0.4277	1.150	0.2445	2.530	0.950	1.784	1.490	1.1203	2.762	1.3015	1.6455	−0.230
2.3	−0.5887	0.3956	1.090	0.2335	2.655	1.061	1.797	1.522	1.1453	2.984	1.2870	1.5781	−0.310
2.4	−0.6049	0.3659	1.040	0.2240	2.790	1.173	1.808	1.554	1.1710	3.229	1.2752	1.5180	−0.405
2.5	−0.6193	0.3386	0.985	0.2140	2.930	1.286	1.818	1.582	1.1972	3.482	1.2659	1.4592	−0.490
2.6	−0.6322	0.3136	0.935	0.2055	3.060	1.400	1.827	1.609	1.2237	3.740	1.2597	1.4075	−0.565
2.7	−0.6435	0.2908	0.880	0.1975	3.210	1.513	1.835	1.632	1.2506	4.025	1.2551	1.3603	−0.650
2.8	−0.6537	0.2699	0.830	0.1905	3.355	1.633	1.842	1.651	1.2775	4.312	1.2517	1.3154	−0.730
2.9	−0.6628	0.2508	0.780	0.1845	3.510	1.752	1.848	1.666	1.3044	4.601	1.2493	1.2721	−0.805
3.0	−0.6710	0.2335	0.733	0.1799	3.688	1.889	1.854	1.687	1.3307	4.888	1.2501	1.2405	−0.881

also holds true for the $A_0(y)$, and $B_0(y)$, redundantly in case $A_0(y)$ and $B_0(y)$ are given by (33), and also if $A_0(y)$ and $B_0(y)$ are given by (34). By examination of (35) and (36) one can see by an induction argument that the functions $A_n(y)$ and $B_n(y)$ are analytic for real part of y greater than zero, and that they have branch points only where $\mu(y)$ and $\nu(y)$ have branch points, and that if we pass around a branch point the functions $A_N(y)$, and $B_N(y)$ interchange. The analytic property of $A_N(y)$ and $B_N(y)$ for real part of y greater than zero enables us to displace the contour of integration, D,n units to the left in the n^{th} terms of Eqs. (26) and (27). We then obtain

$$P(E,t)=-\frac{1}{(2\pi)^2\beta}\sum_{n=0}^{\infty}\left(\frac{\beta}{E_0}\right)^n\int_D dy\int_C ds \times\frac{\Gamma(-s)\Gamma(y+s+1)}{\Gamma(y+1)[\mu(y)-\nu(y)]}\left(\frac{E_0}{\beta}\right)^y\left(\frac{\beta}{E}\right)^{y+s+1} \times\{A_n(y)[\mu(y)+D]K_\mu(y,s)e^{\mu(y)t} -B_n(y)[\nu(y)+D]K_\nu(y,s)e^{\nu(y)t}\},\tag{38}$$

$$\gamma(E,t)=-\frac{1}{(2\pi)^2\beta}\sum_{n=0}^{\infty}\left(\frac{\beta}{E_0}\right)^n\int_D dy\int_C ds \times\frac{\Gamma(-s)\Gamma(y+s+1)}{\Gamma(y+1)[\mu(y)-\nu(y)]}\left(\frac{E_0}{\beta}\right)^y\left(\frac{\beta}{E}\right)^{y+s+1}C(y+s) \times\{A_n(y)K_\mu(y,s)e^{\mu(y)t}-B_n(y)K_\nu(y,s)e^{\nu(y)t}\}.\tag{39}$$

It is the first term in the expansion of the solution in the form (38), (39) in powers of β/E_0 that has been used previously by Snyder and Serber, and as one can see from (38) and (39) this is adequate for all values of the energy and thickness provided (β/E_0) is sufficiently small. We observed here for reference that the integrands of (38) and (39) do not have branch points as functions of y and s.

We now write (38) in the form

$$P(E,t)=\sum_{n=0}^{\infty}\left(\frac{\beta}{E_0}\right)^n P_n(E,t),$$

with

$$P_n(E,t)=-\frac{1}{(2\pi)^2\beta}\int_D dy\int_S ds \times\frac{\Gamma(-s)\Gamma(y+s+1)}{\Gamma(y+1)[\mu(y)-\nu(y)]}\left(\frac{E_0}{\beta}\right)^y\left(\frac{\beta}{E}\right)^{y+s+1} \times\{A_n(y)[\mu(y)+D]K_\mu(y,s)e^{\mu(y)t} -B_n(y)[\nu(y)+D]K_\nu(y,s)e^{\nu(y)t}\}.\tag{40}$$

The same type expansion holds for $\gamma(E,t)$. We remark here that the functions $P_n(E,t)$ and $\gamma_n(E,t)$ are themselves solutions of the basic diffusion equations.

IV. LOW ENERGY EXPANSIONS

As has been noted before, the functions $K_\mu(y,s)$ and $K_\nu(y,s)$, $\Gamma(y+s+1)$ and $C(y+s)$ have poles at $s=-y-n$. This makes it convenient for $E<\beta$ to evaluate the integrals over s as in (38) and (39) in terms of the residues at these poles. We now compute in this manner the total number of particles of energy greater than E,

$$N(E,t)=\sum_{n=0}^{\infty}(\beta/E_0)^n N_n(E,t)$$

with

$$N_n(E,t)=\int_E^{\infty}P_n(E',t)dE'$$

$$=\frac{1}{2\pi i}\int dy\frac{\left(\frac{E_0}{\beta}\right)^y}{\Gamma(y+1)[\mu(y)-\nu(y)]}$$

$$\times\sum_{k=0}^{\infty}\left[\frac{1}{\Gamma(k+1+a(k))}\frac{d^{k+a(k)}}{ds^{k+a(k)}}\left\{\frac{\Gamma(-s)\Gamma(y+s)}{(y+s+k)^{-k-1-a(k)}}\right.\right.$$

$$\times\left(\frac{\beta}{E_0}\right)^{y+s}[A_n(y)[\mu(y)+D]K_\mu(y,s)e^{\mu(y)t}$$

$$\left.\left.-B_n(y)[\nu(y)+D]K_\nu(y,s)e^{\nu(y)t}]\right\}\right]_{s=-y-k},\quad(41)$$

Table II. N.

$t \diagdown \epsilon$	2	3	4	5	6	7	8
0.5	2.38						
1	2.32	3.75	4.87	5.68	6.33	6.7	8.5
2	1.760	4.55	8.86	14.46	26.7	33.1	44.3
3	1.170	3.85	10.13	21.17	41.0	68.7	115.8
4	0.729	2.81	9.07	23.6	53.42	106.3	201.5
5		1.94	7.09	22.01	57.33	135.0	287.5
6		1.22	5.23	18.40	54.20	142.1	335.4
7		0.68	3.52	13.97	46.42	135.2	357.5
8		0.48	2.39	9.96	37.25	119.2	346.0
10			0.90	4.72	20.16	75.6	261.3
12			0.31	1.99	9.67	41.1	155.2
14				0.79	4.0	20.2	85.5
16				0.30	1.67	9.1	42.2
18				0.12	0.67	3.9	18.9
20					0.1	1.6	8.0

with

$$a(k)=0 \quad\text{for}\quad k=0,1,$$
$$a(k)=1 \quad\text{for}\quad k=2,$$
$$a(k)=2 \quad\text{for}\quad k=3,4,5\cdots.$$

The general form of the answer is readily discernible from (41) and is

$$N_n(E,t)=\frac{1}{2\pi i}\int_D dy\left(\frac{E_0}{\beta}\right)^y\left[\frac{A_n(y)[\mu(y)+D]K_\mu(y,-y)e^{\mu(y)t}-B_n(y)[\nu(y)+D]K_\nu(y,-y)e^{\nu(y)t}}{y[\mu(y)-\nu(y)]}\right.$$

$$-\frac{1}{[\mu(y)-\nu(y)]}\left\{A_n(y)[\mu(y)+D]K_\mu(y,-y)\left[\frac{y+s+1}{g_\mu(y,s)}\right]_{s=-y-1}e^{\mu(y)t}\left(\frac{E}{\beta}\right)\left(\ln\frac{\beta}{E}+f(y)\right)\right.$$

$$\left.\left.-B_n(y)[\nu(y)+D]K_\nu(y,-y)\left[\frac{y+s+1}{g_\nu(y,s)}\right]_{s=-y-1}e^{\nu(y)t}\left(\frac{E}{\beta}\right)\left(\ln\frac{\beta}{E}+g(y)\right)\right\}+0\left(\frac{E}{\beta}\right)^2\left\{\ln^3\frac{\beta}{E}+\cdots\right\}\right].\quad(42)$$

In (42) $f(y)$ and $g(y)$ may be evaluated from (41). The first term in (42) clearly determines the total number of particles. For $t>\frac{1}{2}$ the terms involving $\mu(y)$ are much larger than those involving $\nu(y)$. The terms involving $\mu(y)$ may be evaluated approximately by the saddle point method for $n=0$ in which case we get

$$N_0(E,t)\simeq\frac{e^{\mu(y)t+\epsilon y}}{[\beta(y)+b(y)t]^{\frac{1}{2}}}A_0(y)$$

$$\times\left\{H(y)-M(y)\frac{E}{\beta}\left[\ln\left(\frac{\beta}{E}\right)+f(y)\right]\right.$$

$$\left.+0\left(\frac{E}{\beta}\right)^2\left[\ln^3\frac{\beta}{E}+\cdots\right]\right\}.\quad(43)$$

Here

$$\epsilon=\ln(E_0/\beta),\quad t=(\epsilon y-\alpha(y))/a(y)$$

$$a(y)=-y\frac{d\mu(y)}{dy},\quad b(y)=y^2\frac{d^2\mu(y)}{dy^2}$$

$$H(y)=\frac{[\mu(y)+D]K_\mu(y,-y)}{(2\pi)^{\frac{1}{2}}[\mu(y)-\nu(y)]}$$

$$M(y)=2(4/3+\alpha)DH(y)/[\mu(y)+D].$$

The functions $f(y)$ and $\alpha(y)$, $\beta(y)$ are too complicated to reproduce here. The values of $\mu(y)$, $a(y)$, $b(y)$, $f(y)$, $H(y)$, $M(y)$, $A_1(y)$ and $A_0^*(y)$, $A_1^*(y)$, $\alpha(y)$, $\beta(y)$, $\alpha^*(y)$, $\beta^*(y)$, are given in Table I. Of course, we have $A_0(y)=1$ for an incident electron. Also we have $A_0\dagger(y)=yA_0^*(y)$, $A_1\dagger(y)=(y-1)A_1^*(y)$, $\alpha\dagger(y)=\alpha^*(y)-1$ and $\beta\dagger(y)=\beta^*(y)-1$.

Table III. N^*.

$t \diagdown \epsilon$	2	3	4	5	6	7	8
0.5	2.23						
1	2.18	3.96	6.00	8.77	11.67	13.2	30.1
2	1.75	4.29	8.96	15.90	27.16	40.0	65.5
3	1.22	3.66	9.60	21.27	43.67	77.0	131.5
4	0.79	2.71	8.42	22.49	52.8	109.9	213.9
5	0.66	1.92	6.58	20.48	55.6	130.9	291.2
6		1.28	4.99	17.34	51.7	141.0	332.0
7		0.76	3.56	13.01	46.5	129.8	350.5
8		0.51	2.33	10.01	34.67	114.2	328.0
10			1.02	4.81	19.2	70.9	248.8
12			0.36	1.80	9.2	40.6	147.5
14				0.70	4.2	20.9	85.5
16				0.28	1.7	9.7	43.0
18				0.12	0.67	4.2	18.9
20					0.1	1.5	8.0

TABLE IV. $N\dagger$.

$t \backslash \epsilon$	2	3	4	5	6	7	8
0.5	0.935						
1	1.61	1.88	2.27	3.1	3.2	3.6	8.5
2	1.76	3.59	5.97	8.85	13.33	13.9	24.4
3	1.39	3.86	8.51	16.19	27.2	46.0	71.0
4	0.990	3.34	9.02	20.40	42.5	78.9	140.2
5	0.632	2.49	8.04	21.60	51.2	109.0	218.2
6	0.412	1.75	6.51	19.96	53.5	130.1	285.2
7		1.12	4.85	16.93	49.9	133.9	326.0
8		0.80	3.48	13.20	43.0	126.5	342.1
10		0.29	1.60	7.40	26.5	93.5	291.4
12			0.65	3 28	13.8	54.5	201.2
14				1.32	6.8	29.4	113.5
16				0.52	3.5	13.5	61.2
18				0.19	1.9	5.8	29.9
20					0.5	2.6	9.9

In a manner quite similar to that in which (43) was obtained we obtain, for $\gamma(E, t)$, the value

$$\gamma(E, t) = (4/3+\alpha)\frac{N(0, t)}{E[\mu(y)+D]} + 0\left(\frac{E}{\beta}\right)\left[\ln^3\frac{\beta}{E}+\cdots\right] \quad (44)$$

with the saddle point relationship between t and y as given above. If we compare the γ-ray distribution as given by (44) with the γ-ray distribution as given by Carlson and Oppenheimer in their Eq. (34) and the total number of particles as given by their Eq. (36), we see that the relation between the γ-ray distribution and the total number of particles as given by them as compared with (44) differs only by the factor $(4/3+\alpha)$. One can also see that the differentiation of (43) with respect to the energy gives a particle distribution of the same logarithmic form as was found by Carlson and Oppenheimer in their Eq. (34). We find

$$P(E, t) = \frac{2D(4/3+\alpha)N(0, t)}{\beta[\mu(y)+D]}\left\{\ln\frac{\beta}{E}+f(y)-1\right\}+\cdots$$
$$= 2D\frac{E}{\beta}\gamma(E, t)\left\{\ln\frac{\beta}{E}+f(y)-1\right\}+\cdots. \quad (45)$$

On the other hand, the answer we find for $P(E, t)$ does not agree with that given by Bhabha and Chakrabarty in their Eq. (37). Of course, if all the terms in their expansion (35) had been used, the results would have to agree. The expansions which we have given here are particularly useful for $E<\beta$, while the expansions given by Bhabha and Chakrabarty are most useful for larger values of the energy. We also note here that if $\beta/E_0>1$, the saddle point that was used to obtain (43) and (44) does not exist. In general, formulas (43) and (44) give the proper dependence of the answer on t only if β/E_0 is small. However, the type of energy dependence is still correctly given for diffusion Eqs. (1) and (2) as in (42), since it depends only on the location and order of the poles in the integrands of Eqs. (5) and (6) and not on the thickness and initial conditions. It is possible to evaluate the integrals over y in (42) in terms of residues of various poles in the left-hand plane. This leads to an expansion in powers of t, which is unfortunately not very useful for numerical calculations unless t is very small.

There is one check which may be made on the expansions which have been given and on the saddle point method of integration: the total initial energy must be absorbed by ionization. This leads to the relation

$$\int_0^\infty N(0, t)dt = E_0/\beta. \quad (46)$$

If we use the value of $N_n(E, t)$ as given by (42) and those of $A_0(y)$ and $B_0(y)$ as given by (33) or (34a) or (34b) for the different initial conditions, one can show that

$$\int_0^\infty N_n(0, t)dt = 0; \quad n>0$$
$$E_0/\beta; \quad n=0$$
$$\int_0^\infty N_n\dagger(0, t)dt = 0; \quad n>0$$
$$E_0/\beta; \quad n=0$$
$$\int_0^\infty N^*(0, t)dt = 0; \quad n>1$$
$$(E_0/\beta)/(4/3+\alpha); \quad n=1$$
$$E_0/\beta - 1/(4/3+\alpha); \quad n=0. \quad (47)$$

From the small values of $A_1(y)$ as shown in Table I, it is evident that $N_0(E, t)$ will be quite close to $N(E, t)$ even for $\beta/E_0\sim.1$. And, we see from (47) that the higher terms $N_n(E, t)$ and $N_n\dagger(E, t)$ can only change the shape of the $N(E, t)$ and $N\dagger(E, t)$ curves slightly. For $N(E, t)$

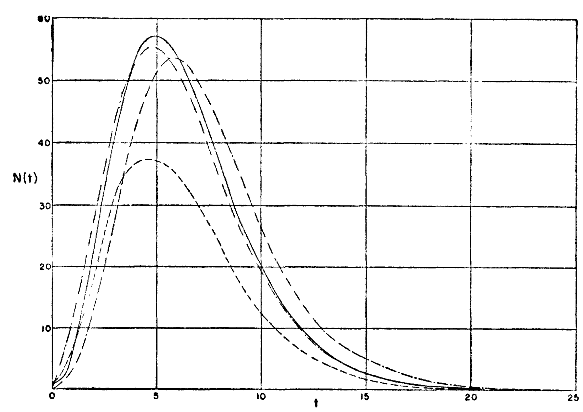

FIG. 1. Graph of $N(t)$ (solid curve, ——), $N^*(t)$ (dash double dot curve, ——··), $N\dagger(t)$ (dash dot curve, —·), $N_{BC}(t)$ (dashed curve — —) for $\epsilon=6$.

this is actually a small decrease in $N(E, t)$ for small t and a small increase in $N(E, t)$ for large t. Thus, in our case, the numerical evaluation of the integrals in (47) essentially gives a gross check on the saddle point method of evaluating $N_0(t)$, $N_0\dagger(t)$, $N_0^*(t)$ and $N_1^*(t)$. One can also show, using (42), that $N_0(0,0)=1$ and $N_0\dagger(0,0)=0$. We also wish to observe at this point that the saddle point method can be used for the approximate evaluation of only $N_0(t)$, $N_0^*(t)$, $N_0\dagger(t)$ and $N_1^*(t)$. For the other $N_n(t)$, etc., the functions $A_n(y)$ have zeros at $y=1, 2, 3\cdots n$ which zeros prevent the application of the saddle point method.

In Tables II, III, and IV we give the values of $N(t)$, $N^*(t)$ and $N\dagger(t)$, respectively, for $\epsilon=2, 3, 4, 5, 6, 7$, and 8. These values were calculated using formulas (43). Since the values of t and $N(t)$ are determined parametrically by assigning values of y, these values of t and $N(t)$ were plotted with $N(t)$ as a function of t and the values of $N(t)$ etc., as given in Tables II, III, and IV were obtained by reading the graphs. The values of $N(t)$ and $N\dagger(t)$ were calculated using $N(t)=N_0(t)$ and $N\dagger(t)=N_0\dagger(t)$; however, we used $N^*(t)=N_0^*(t)+(\beta/E_0)N_1^*(t)$. It is not practical to compute higher terms in the series for $N(t)$, $N\dagger(t)$, and $N^*(t)$ since we cannot use the saddle point method for their computation. In addition, relations (47) insure that the areas under these curves will be correct to the accuracy of the saddle point method.

In Table V we give the values for various ϵ of $\int_0^\infty N(t)dt$, $\int_0^\infty N^*(t)dt$, $\int_0^\infty N\dagger(t)dt$, and $\int_0^\infty N_{BC}(t)dt$ as numerically evaluated by plotting curves for $N(t)$ $N^*(t)$ and $N\dagger(t)$ as determined by using Eqs. (43), and for $N_{BC}(t)$ the values as given by Bhabha and Chakrabarty[7] in their Table III, and then measuring the area under the curves, using a planimeter. These areas should, of course, be equal to E_0/β which is listed in column 2 of this table. It is evident on inspection of this table that the agreement of columns 3, 4, and 5 with column 2 is much better than is that of column 6.

Figure 1 is a graph of $N(t)$, $N^*(t)$, $N\dagger(t)$, and $N_{BC}(t)$ for $\epsilon=6$. It is evident from these curves that the numerical values obtained by Bhabha and Chakrabarty are in error by about 35 percent near the maximum of the shower curve.

I wish here to thank Miss Jean Snover who is primarily responsible for the numerical work and preparation of the graph and the tables.

TABLE V.

ϵ	$e^\epsilon=\frac{E_0}{\beta}$	$\int_0^\infty N dt$	$\int_0^\infty N^* dt$	$\int_0^\infty N\dagger dt$	$\int_0^\infty N_{BC} dt$
2	7.389	7.22	7.52	7.26	5.64
3	20.086	20.4	20.8	20.3	14.29
4	54.598	55.1	55.2	55.5	38.27
5	148.41	149.4	149.2	149.7	102.8
6	403.43	406.4	406.3	407.4	279.8
7	1096.6	1106.8	1111.5	1103.1	736.3
8	2981.0	2997	3001	2998	2001

Note added after completion of manuscript.—The results of this paper have recently been obtained by W. T. Scott,[8] starting with a Laplace transform in t, and Mellin transforms in both E and E_0. The resulting homogeneous difference equation is analogous to Eq. (19) and was solved in a similar way. This approach yields explicit formulas for A_n and B_n, namely:

$$A_n(y)=\frac{\Gamma(y+1)}{\Gamma(y-n+1)}(-1)^n[\mu(y)+D]^n \times\prod_{j=1}^{n}\frac{1}{[\mu(y)-\mu(y-j)][\mu(y)-\nu(y-j)]}$$

$$B_n(y)=\frac{\Gamma(y+1)}{\Gamma(y-n+1)}(-1)^n[\nu(y)+D]^n \times\prod_{j=1}^{n}\frac{1}{[\nu(y)-\nu(y-j)][\nu(y)-\mu(y-j)]}.$$

Furthermore,

$$A_n\dagger(y)=(y-n)A_n^*(y)=\frac{B(y-n)}{\mu(y)+D}A_n(y)$$

$$B_n\dagger(y)=(y-n)B_n^*(y)=\frac{B(y-n)}{\nu(y)+D}B_n(y).$$

From these explicit expressions for $A_n(y)$ and $B_n(y)$, one can directly verify the statements made earlier in this paper covering the analytic properties of these functions, together with the fact that they satisfy Eqs. (29) and (30).

[8] Privated communication. The method is outlined in W. T. Scott's unpublished thesis (University of Michigan, 1941).