A Schematic Theory of Nuclear Cross Sections*

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In this paper a description of the energy dependence of nuclear cross sections, especially for neutrons, is given in terms of simple assumptions on the properties of the nuclei. The results are supposed to represent average values over individual fluctuations and resonances. The total cross section, the reaction cross section, and the transport cross section are calculated as functions of the neutron energy. Only two nuclear parameters occur in the expressions; the nuclear radius and the average kinetic energy of a nucleon inside the nucleus. Comparison of the theory with neutron experiments is made. New values of nuclear radii are computed from total cross sections at 14 and 25 Mev. Predictions of the inelastic cross sections at 14 Mev are in good agreement with the measurements. Experiments involving total and transport cross sections between 0.1 and 1.5 Mev are compared with theory and reasonably good agreement is found with most of these. However, for the elements, I, In, Sb, agreement in the context of the theory may be obtained only if the strength of resonance levels for these elements is assumed to be unusually low.

I. INTRODUCTION AND DISCUSSION OF RESULTS

THE cross sections for nuclear reactions are complicated functions of the energy of the incident particles. One observes resonances, maxima, and minima, and no two cross sections show exactly the same features. There seem to exist, however, certain general trends in the energy dependence of cross sections, especially if one averages over the individual fluctuations. It may be worth while, therefore, to try to understand the over-all behavior of nuclear cross sections on the basis of a few very general assumptions.

We are presenting here a schematic theory of nuclear cross sections which is based on an extremely simplified and rough picture of the nuclear structure. In fact, the only information used on the internal structure of the nuclei is the nuclear radius R and the average kinetic energy T of a nucleon within the nucleus.

The results obtained must be considered as a schematic picture of the actual conditions, trimmed of all individual fluctuations and deviations. It is the purpose of this approach to give a first orientation about the magnitude and energy dependence of cross sections without any claims to quantitative validity. On the other hand, the degree in which the predictions are found to be correct may be a measure of the validity of the underlying concepts.

We restrict ourselves to the study of the cross sections averaged over individual fluctuations. Resonances are not reproduced by this theory. If they occur, our theoretical results refer to the average value taken over an energy region containing several resonances. For sufficiently high energies, the resonances or fluctuations in the cross sections as a function of energy become very weak and unimportant. For it is expected that as the energy increases, the resonance levels become broader and finally merge into a continuous curve for the cross section in its dependence on energy. In this region our theoretical predictions should reproduce the actual cross sections without any averaging.

We shall show that the results herein join on continuously to those of an earlier paper[1] (to be designated henceforth as FPW) which apply to the resonance region. Both papers thus have a number of parameters in common. Of particular importance are the nuclear radius R and the wave number K of the incident particle within the nucleus. These are the only two parameters used here to represent the nucleus. Both parameters can only be defined approximately. The wave number K may be estimated by assuming that in the nuclear interior in the neighborhood of the nuclear surface, the dependence of the wave function on the coordinate r of the incident particle is periodic with a wave number corresponding to its kinetic energy within the nucleus. Hence it follows that $K^2=k^2+K_0^2$, where $k^2=2M\epsilon/\hbar^2$, M = mass of the incident particle, ϵ its energy outside the nucleus, and K_0 is the wave number K for zero incident energy. K_0 was estimated in FPW using a Fermi-Dirac model for the nucleus by

$$K_0^2=2M/\hbar^2[\epsilon_B+4\pi^2(3A/16\pi V)^{\frac{2}{3}}],$$

where A is the nuclear mass number, V the nuclear volume, and ϵ_B is that part of the binding energy of the added particle which is kinetic. We assume here, in contrast to FPW, that $\epsilon_B\approx 0$, for it is probable that a considerable fraction of the binding energy is potential in virtue of the exchange force between nucleons. For $\epsilon_0=0$, and $R=r_0A^{\frac{1}{3}}$, $r_0=1.5\times 10^{-13}$ cm, we find $K_0=1.0\times 10^{-13}$ cm^{-1}.

In addition to K_0 and R, the charge on both the incident and target nuclei are important parameters. We shall limit our discussion to neutron-initiated reactions, although formulas of Section II apply more generally. Charged particle reactions have been considered in more detail by J. K. Tyson.[2]

* This work was assisted by the Joint Program of the ONR and AEC.

[1] Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947).
[2] J. K. Tyson, Ph.D. thesis, M.I.T. (1948), issued as a Technical Report by the Laboratory of Nuclear Science and Engineer-

The following cross sections are calculated:

(1) *The "total" cross section*, σ_{tot}, which includes all processes that may be initiated by the incident neutrons such as transmutations, elastic, and inelastic scattering, etc. It is this cross section which determines the attenuation of a neutron beam in passing through matter.

(2) *The "reaction" cross section*, σ_r, which we define here, somewhat unconventionally, as the total cross section minus the elastic scattering cross section, $\sigma_r = \sigma_{tot} - \sigma_{el}$. The *elastic cross section*, σ_{el}, includes only those processes in which the quantum state of the target nucleus is left unchanged. Thus the reaction cross section, σ_r, includes all the processes in which the quantum state of the target nucleus is changed; i.e., it includes the processes in which the compound nucleus decays into a different channel than the one by which it was formed. Inelastic scattering, for example, is included in the reaction cross section.

(3) *The transport cross section*, σ_{tr}, which is defined by the integral

$$\sigma_{tr} = \sigma_{tot} - \int \sigma_{el}(\theta)\cos\theta d\Omega, \tag{1}$$

where $\sigma_{el}(\theta)d\Omega$ is the elastic scattering cross section into the solid angle $d\Omega$ in direction θ. The transport cross section determines the net loss of forward momentum of the incident beam.[a]

All three cross sections will be expressed in units of πR^2, where R is the nuclear radius:

$$\sigma_{tot,\,tr,\,r} = \pi R^2 \cdot F_{tot,\,tr,\,r}(x, X). \tag{2}$$

The three functions, F_{tot}, F_{tr}, F_r, will be shown to depend only on the dimensionless variables $x = kR$, and $X_0 = K_0 R$, where k is the wave number of the incoming neutron and K_0 is defined above. The following expression is useful for the computation of x:

$$x = 0.222R(\epsilon)^{\frac{1}{2}}, \tag{3}$$

where ϵ is the kinetic energy of the incoming neutron in Mev, and R is expressed in units of 10^{-13} cm.

The result of the calculation of the three functions F_{tot}, F_r, F_{tr} are plotted in Figs. 1 and 2 for different values of X_0 corresponding to different nuclear radii. ($X_0 = 5$ corresponds to chlorine $X_0 = 8$ to samarium.) The total cross section approaches $2\pi R^2$ for high energies as is well known. *However, the asymptotic value is reached only for extremely high values of x.* The behavior at higher energies is represented more faithfully by

$$\sigma_{tot} = 2\pi(R+\lambda\!\!\!^{-})^2 = 2\pi R^2(1+1/x)^2$$

which deviates from calculated values by less than 10 percent for $x > 4$, as is illustrated by the dotted curve in Fig. 1.

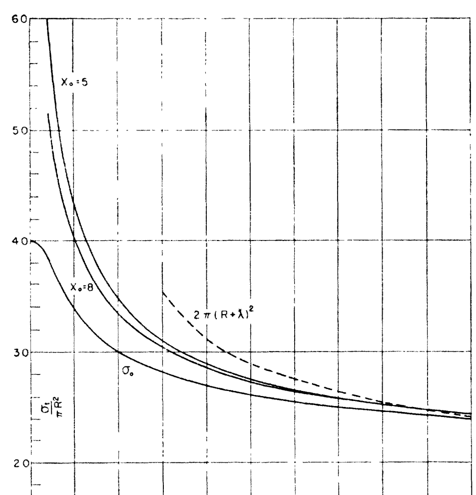

FIG. 1. Total cross sections as a function of $x = kR$, for values of $X_0 = 5$ and 8. σ_0 is the total cross section for the infinitely repulsive sphere of radius R. The dotted line gives the approximate behavior of all of these curves for high x.

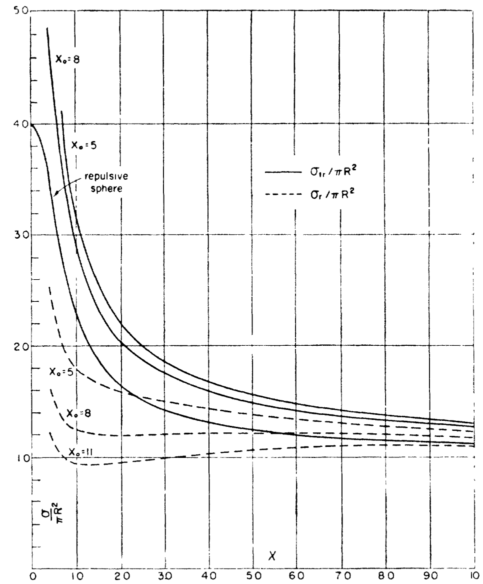

FIG. 2. Reaction and transport cross sections as a function of x for various values of X_0. The full line curves give the transport cross sections including the results for an infinitely repulsive sphere. The broken line curves give the reaction cross sections.

It is interesting to compare σ_{tot} with the cross section σ_0 of an infinitely repulsive (totally reflecting) sphere of radius R. The close agreement of σ_{tot} and σ_0 shows that the value of the total cross section for sufficiently large x is not very sensitive to the choice of the boundary conditions satisfied by the neutron wave at the surface of the nucleus.

The reaction cross section approaches πR^2 at high energies and shows a $1/v$ dependence at low energies.

ing, M.I.T. An article by this author will be submitted shortly to Phys. Rev.

[a] It is assumed here that inelastic scattering is spherically symmetric.

The behavior of this function can be understood from the following qualitative considerations. From the behavior of σ_{tot} discussed, we may infer that the cross section for striking the nucleus is given by $\pi(R+\lambdabar)^2$ which may be understood by giving the incident neutron a "size" $\lambdabar$. To initiate a reaction, however, the neutron must penetrate through the nuclear surface, which may be represented by a sharp drop of potential, corresponding to sudden change of wave number from k to $K=(K_0^2+k^2)^{\frac{1}{2}}$. This sudden change causes a partial reflection of the wave. An elementary calculation shows that the transmission T through the plane at which this change of wave number occurs is given by $T=4kK/(k+K)^2$. The reaction cross section should be the product of the cross section for striking the nuclear surface and the transmission T:

$$\sigma_r \approx \pi(R+\lambdabar)^2 T. \tag{4}$$

For small $k(k \ll R^{-1})$, this becomes $4\pi/kK_0$ and exhibits the $1/v$ law with the same scale factor as given by the precise calculation. In the intermediate energy range, the formula (4) is not a very good approximation to σ_r, because of complications introduced by the contribution of the partial waves with higher values of the angular momentum. It should also be apparent now why σ_{tot} has a $1/v$ dependence for small x, since it is composed of σ_r and σ_{el}, the latter being finite at zero energy.

Among the three sets of curves the strongest dependence on X_0 is found in σ_r. In both σ_{tot} and σ_{tr} the increased penetration for smaller X_0 is offset by a correspondingly lower scattering.

The expressions for σ_{tot} and σ_{tr} should be valid over the entire energy range. At low energies they signify average values over resonances. The expression for the reaction cross section σ_r, however, should be applied only within a limited energy range. It is calculated with the assumption that, once the neutron has penetrated through the nuclear surface, it will not leave the nucleus again by the same channel, i.e., in such a way that the nucleus is left in the same quantum state in which it was before the reaction. Thus, our expression for σ_r should not be applied if the main process induced by the neutron is elastic scattering. Therefore, the curve for σ_r is valid only at high energies (above 2 Mev for all nuclei except the lightest ones), where we expect strong inelastic scattering, and at very low energies if the main process is neutron capture. In the low energy region σ_r should represent the average over resonances. There is, however, a large region, between a few kilovolts and a few million volts, in which σ_r is not represented by our curves.

A critical remark is necessary in regard to the nuclear radius R which enters into the evaluation of the predicted cross sections, since they are proportional to πR^2. The nuclear radius is defined in this theory as the distance of the neutron from the center of the nucleus at which the wave number of the neutron changes from k (the value outside) to K (the value inside). Actually, this occurs gradually over a region of the order $1/K$. We expect therefore that R should be uncertain by this amount, so that R may deviate from the values of any other radius determination by as much as $\pm 1\times 10^{-13}$ cm. Very probably R may vary within these limits for different energies of the incident neutron. The validity of this theory is based upon the smallness of $1/K$ compared to R. Actually $(KR)^{-1}$ is not a very small number, so that the results of the theory must be considered as rough approximations only.

II. CALCULATION OF THE CROSS SECTIONS

The derivation of our results follows different lines in the high energy region and in the low energy region. The upper region is defined by the validity of the following condition:

(a) The energy of the incident neutron must be high enough so that it is energetically possible for the compound nucleus to decay into many channels; for example, the residual nucleus may be left in many different excited states. Thus the high energy region includes energies ϵ which are considerably larger than the first excitation energy of the target nucleus, say $\epsilon > 2$ Mev.

We start with the calculation of the cross sections in the high energy region. The guiding principle employed is akin to the Bohr hypothesis of the compound nucleus. We assume that once the particle has penetrated to within the boundaries of the nucleus, it will not leave it by the same channel in which it entered. This assumption is valid if condition (a) holds and also if the following condition is fulfilled:

(b) The interaction between the incoming neutron and the nucleons of the target nucleus must be strong enough so that the probability of decay into other channels is high.

Both factors (a) and (b) give rise to a large level width in the compound nucleus. In fact, it will make it considerably larger than the level spacing. Condition (b) sets an upper energy limit to the applicability of our method: If the energy is too high, (b) is not satisfied, because of the decrease of the interaction cross section between nucleons of high energy. The nuclei thus become transparent to incident neutrons of sufficiently great energy.[3] This will happen if ϵ exceeds about 70 Mev.

We shall now obtain a mathematical formulation of the principal assumption in such a form as to permit the calculation of the various cross sections in the high energy region. Essentially we assume that there is a vanishingly small probability that any incident particle which enters the nucleus will be re-emitted without any loss of coherence with the initial beam. Thus the target nucleus acts as a "sink" for the incident particles. We express this assumption in the following mathematical form: the wave function ψ for the neutron within the nucleus cannot be given as a function of $\mathbf{r}$ (its dis-

[3] Fernbach, Serber, and Taylor, Phys. Rev. 75, 1352 (1949).

tance to the center of the nucleus) alone; however, its dependence on r may be approximated by a *converging* wave

$$r\psi \sim e^{-iKr},$$

expressing the fact that the entering particle does not return. K is the wave number which corresponds to the average kinetic energy inside the nucleus.[b] This can be expressed by a boundary condition: At the nuclear surface the logarithmic derivative of $r\psi$ equals:

$$f = R[\partial(r\psi)/\partial r/r\psi]_{r=R} = -iKR = -iX. \tag{5}$$

We recall that $K^2 = K_0^2 + k^2$; $K_0 = 1.0\times 10^{-13}$ cm^{-1} as estimated in FPW. Boundary condition (5) determines the behavior of the wave function outside the nucleus; i.e., the entire nucleus is represented by the parameters X_0 in addition to the nuclear radius R. (An infinitely repulsive sphere corresponds to $\psi=0$ at $r=R$ and therefore $X_0=\infty$.)

We follow FPW in determining the cross sections as determined by boundary condition (5). For the region $r>R$, the wave function $\varphi = r\psi$ for a neutron (the following treatment applies as well to charged incident particles) with an angular momentum l in units of $\hbar$ may be written

$$\varphi = u_l + \eta v_l, \tag{6}$$

where

$$\begin{aligned} u_l &\underset{r\to\infty}{\longrightarrow} e^{-i(kr - l\pi/2)} \\ v_l &\underset{r\to\infty}{\longrightarrow} e^{+i(kr - l\pi/2)}, \end{aligned} \tag{7}$$

u_l and v_l are known functions of x, and form a conjugate complex pair.

The reaction, elastic, and total cross sections for neutrons are then

$$\sigma_r^{(l)} = (\pi/k^2)(2l+1)(1-|\eta|^2) \tag{7a}$$

$$\sigma_{el}^{(l)} = (\pi/k^2)(2l+1)|1+\eta|^2 \tag{7b}$$

$$\sigma_{tot}^{(l)} = (\pi/k^2)(2l+1)2(1+Re\eta). \tag{7c}$$

η_l can be adjusted so that (6) fulfills the boundary condition (5):

$$\eta_l = -\frac{xu_l' + iXu_l}{xv_l' + iXv_l},$$

where u_l and v_l are the values, u_l' and v_l' the derivatives in respect to x, of the functions (7) evaluated at $x=kR$. We define the phases δ and δ' by $v=|v|e^{i\delta}$ and $v'=|v'|e^{i\delta'}$ and use the Wronskian relation $|vv'|\sin(\delta_l'-\delta_l)=1$. We then get[c]

[b] This approach is similar to that of H. Bethe (Phys. Rev. **57**, 1125 (1940)) to describe nuclear absorption.

[c] These two expressions follow directly from formulas (38) and (39) in FPW, if one puts $f=f_0-ih=-iX$ so that $f_0=0$, $h=X$.

TABLE I. Re-evaluation of nuclear radii.

Element	Observed $\sigma_{tot}\times 10^{24}$	Energy in Mev	$R\times 10^{+13}$ cm	r_0	Reference
Be	0.65	14	2.4	1.17	2
B	1.16	14	3.4	1.54	2
C	1.29	25	3.8	1.65	3
O	1.60	25	4.3	1.71	3
Mg	1.83	14	4.5	1.57	2
Al	1.92	14	4.6	1.53	2
Al	1.85	25	4.6	1.52	3
S	1.58	14	4.1	1.30	2
Cl	1.88	25	4.7	1.44	3
Fe	2.75	14	5.6	1.46	2
Cu	2.50	25	5.5	1.38	3
Zn	3.03	14	5.9	1.48	2
Se	3.35	14	6.3	1.46	2
Ag	3.82	14	6.8	1.44	2
Ag	3.70	25	6.9	1.46	3
Cd	4.25	14	7.2	1.48	2
Sn	4.52	14	7.4	1.52	2
Sb	4.35	14	7.3	1.46	2
Au	4.68	14	7.5	1.33	2
Hg	5.64	14	8.3	1.42	2
Hg	5.25	25	8.4	1.44	3
Pb	5.05	14	7.8	1.32	2
Bi	5.17	14	7.9	1.34	2

$$\sigma_r^{(l)} = \frac{4\pi}{k^2}(2l+1)\frac{xX}{x^2|v_l'|^2 + X^2|v_l|^2 + 2Xx}$$

$$\sigma_{el}^{(l)} = \frac{4\pi}{k^2}(2l+1)\left|\sin\delta_l i^{i\delta_l} + \frac{x/|v_l|^2}{i(x/|v_l|^2+X) + x|v'/v|\cos(\delta_l'-\delta)}\right|^2. \tag{8}$$

The sum of these two expressions can be written in the form:

$$\sigma_{tot}^{(l)} = (4\pi/k^2)(2l+1)\left[\sin\delta_l + \frac{\cos 2\delta_l(x/|v_l|^2+X) + \sin 2\delta_l(x|v'/v|\cos(\delta_l'-\delta_l))}{x^2|v_l'|^2 + X^2|v_l|^2 + 2Xx}\right]. \tag{9}$$

These cross sections are to be summed over l to obtain σ_r, σ_{el}, and σ_{tot}. Note that the $\sin^2\delta_l$ term in σ_{tot} is just the term which would arise from the scattering by an infinitely repulsive sphere of radius R. No resonances appear in any of these cross sections, as may be expected from the fact that the value of f in (5) is a pure imaginary. The value of $\sigma_r^{(0)}$ is $(4\pi/k^2)Xx/(X+x)^2$ which is just $4\pi/k^2$ times the transmission T discussed in the introduction.

Finally, σ_{tr} the transport cross section is computed. We need to evaluate the integral

$$J = \int \sigma_{el}(\theta)\cos\theta/d\Omega.$$

Writing $\sigma_{el} = |f(\theta)|^2$ and $f(\theta) = (1/k)\sum A_l P_l(\cos\theta)$ where

$$A_l = (2l+1)^{\frac{1}{2}} e^{-2i\delta_l}\left[e^{i\delta_l}\sin\delta_l + \frac{x/|v_l|^2}{x|v_l'/v_l|\cos(\delta_l'-\delta_l) + i(X+x/|v_l|^2)}\right]$$

TABLE II. Reaction cross sections at 14 Mev.

Element	$\sigma_r\times10^{24}$ cm² Exp.	Theor.	Reference
Al	0.90	1.0	5
Al	1.06		7
Fe	1.43	1.4	5
Fe	1.45		7
Cd	1.89	2.0	7
Au	2.51	2.2	7
Hg	2.47	2.5	5
Bi	2.56	2.3	7
Pb	2.22		5
Pb	2.29	2.3	7a
Pb	2.56		7

then

$$J=(4\pi/k^2)\sum_l(2l)/(4l^2-1)Re(A_l\delta A_{l-1}). \qquad (10)$$

The transport cross sections are given by $\sigma_{\text{tot}}-J$. The curves in Figs. 1, 2, and 3 were computed from formulas (8), (9), and (10).

We now turn to the calculation of the cross sections in the low energy region. This is the region in which the condition (a) is no longer fulfilled. We no longer can use the simplified boundary condition (5). It is expected that resonances will occur at low energies and it seems natural to fall back upon the more detailed expressions derived in FPW and average them over an energy region containing many resonances.

We will prove two points:

(A) The concepts of FPW join continuously into the ones of the present paper; the calculation of the cross sections in the high energy region is a generalization of the ideas used in FPW.

(B) The cross sections as calculated by FPW, and averaged over many resonances, yield the same expressions (9) and (10) for σ_{tot} and σ_{tr} for the low energy region also.

In order to demonstrate (A), we show that the boundary condition (5) follows from the more complicated boundary condition of FPW if the energy of the incident neutron is high. We compare (5) with the corresponding condition, Eqs. (17a) and (24) of FPW:

$$f(W)=f_0(W+i\Gamma_a/2)=-X\tan\frac{\pi}{D}\left(W+\frac{i\Gamma_a}{2}-E_r\right) \qquad (11)$$

where Γ_a was called the absorption width in FPW. In the terminology of the present paper Γ_a should be referred to as the reaction width since it is the partial width corresponding to the processes we have included in the reaction cross section. D is the level distance. Expression (11) becomes $f(W)=-iX$ (Eq. (5)) if[d]

$$\pi\Gamma_a/2D\gg1. \qquad (12)$$

Condition (12) requires that the reaction width be large compared to the level distance D. This is the region in which resonances disappear. Thus the boundary condition of FPW goes over into the one used here when the energy of the incident neutron is sufficiently high so that the width of the levels is larger than the distance between them.

We turn next to point (B). In FPW, the total cross section σ_{tot}, is the sum of σ_{sc} and σ_{abs}. The former σ_{sc}, is identical with the elastic scattering cross section σ_{el} of this paper, the latter, σ_{abs}, is the same as the reaction cross section of this paper. The total cross section averaged over many resonances for a partial wave of angular momentum l can be found by adding $\langle\sigma_{\text{abs}}\rangle_{\text{Av}}^{(l)}$ as given in FPW by the unnumbered equation on p. 154, preceding Eq. (47), and $\langle\sigma_{\text{sc}}\rangle_{\text{Av}}^{(l)}$ as given by the unnumbered equation on p. 156 preceding Eq. (49). One obtains

$$\langle\sigma_{\text{tot}}^{(l)}\rangle_{\text{Av}}=\frac{4\pi}{k^2}(2l+1)\left\{\sin^2\delta_l+\frac{\pi}{2}\frac{\Gamma_{nl}}{D}\cos2\delta_l\right\}\cdot \qquad (13)$$

This goes over into expression (9) derived above for the

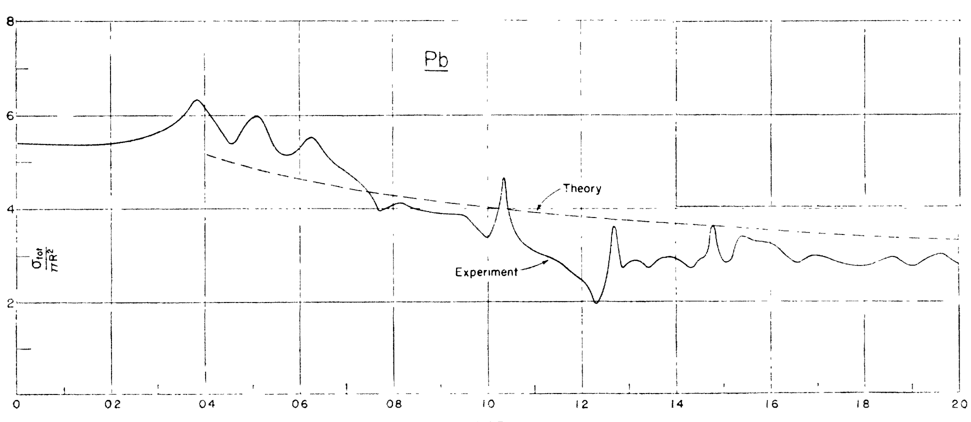

FIG. 3. σ_{tot} for Pb. The full line curve is drawn through the experimental points as given in reference 10. Some fluctuations appearing in this curve are interpreted as resonances, others are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 7.8×10^{-13} cm.

[d] This has been pointed out first by J. K. Tyson in reference 2.

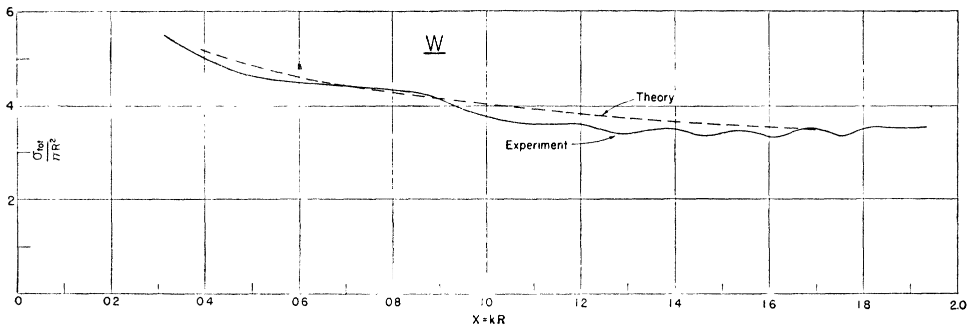

FIG. 4. σ_{tot} for W. The full line curve is drawn through the experimental points of yet unpublished measurements of H. Barschall. We are grateful to the author for giving us his results before publication. The broken line curve gives the theoretical values for a radius of 7.8×10^{-13} cm.

high energy region if $x\ll X$ and if the relation

$$\Gamma_{nl}=\frac{2x}{X}\frac{1}{|v_l|^2}\frac{D}{\pi} \tag{14}$$

is used. Formula (14) was derived in FPW as a qualitative relation between neutron width and level density. It has been found to hold reasonably well in a recent analysis of the experimental data.[4] The condition $x\ll X$ is always fulfilled in the resonance region for there $\Gamma_{nl}\ll D$. Thus we may conclude that the total cross sections given in Fig. 1 hold in the low energy region if the experimental data are averaged over many resonance levels.

The same can be shown to be valid for the transport cross section by employing expression (10) for J. A_l is given by expression (39) in FPW which may be conveniently written as the sum of two terms:

$$A_l=(A_l)_{\mathrm{res}}+(A_l)_{\mathrm{pot}}$$

$$(A_l)_{\mathrm{res}}=(2l+1)^{\frac{1}{2}}e^{-2i\delta_l}\times\frac{x/|v_l|^2}{i(x/|v_l|^2+h)-[f_0-x|v_l'/v_l|\cos(\delta_l'-\delta_l)]}$$

$$(A_l)_{\mathrm{pot}}=e^{-i\delta_l}\sin\delta_l.$$

In averaging J over resonances, we make use of the fact that resonances occur at different energies for different l's. Therefore, $Re(A_l{}^*A_{l-1})$ averaged over a resonance equals

$$\langle Re(A_l{}^*A_{l-1})\rangle_{\mathrm{Av}}=Re\left(\left[(A_l{}^*)_{\mathrm{pot}}(A_{l-1})_{\mathrm{pot}}+\frac{1}{\Delta}\int(A_l{}^*)_{\mathrm{res}}(A_{l-1})_{\mathrm{pot}}d\epsilon+\frac{1}{\Delta}\int(A_l{}^*)_{\mathrm{pot}}(A_{l-1})_{\mathrm{res}}d\epsilon\right]\right),$$

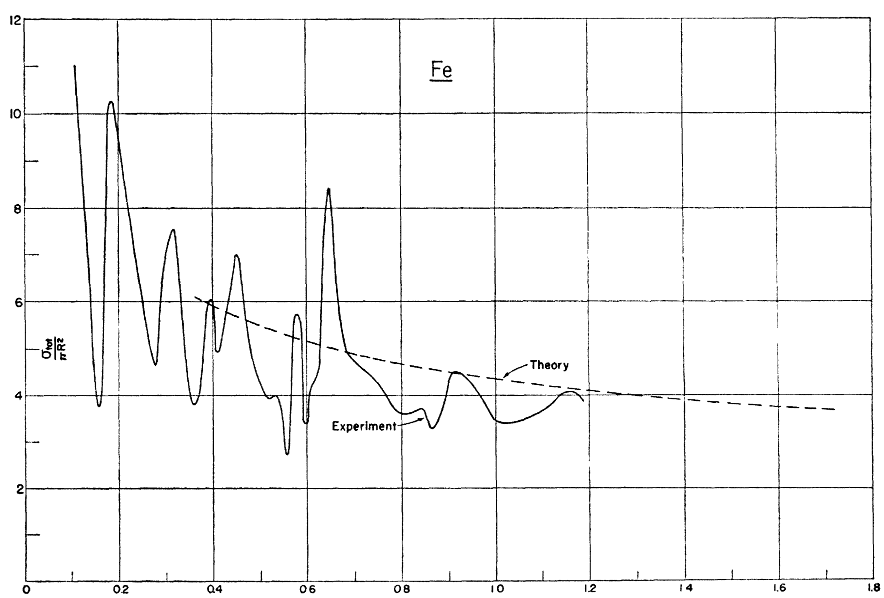

FIG. 5. σ_{tot} for Fe. The full line curve is drawn through the experimental points as given in reference 8. Some fluctuations appearing in this curve are interpreted as resonances, others are within the statistical errors of the measurement. The broken line curve gives the theoretical values for a radius of 4.6×10^{-13} cm.

[4] E. Wigner, Am. J. Phys. 17, 107 (1949).

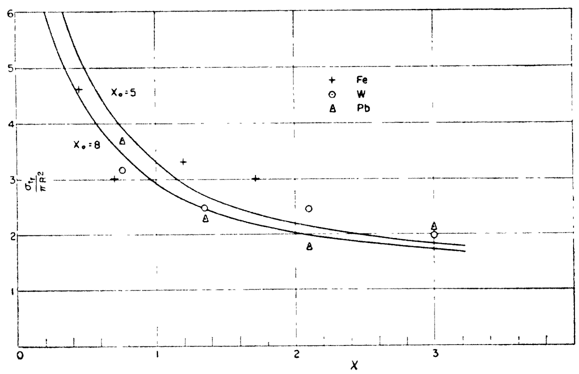

FIG. 6. σ_{tr} for Pb, W, Fe. The full line curves give the theoretical values. The upper curve ($X_0=5$) should be valid for Fe, the lower one ($X_0=8$) for W and Pb.

where the integrals extend over a small energy interval, $\Delta \ll D$, around a characteristic resonance. The result of this calculation is again identical with (10) if we place $x \ll X$ and employ relation (14).

One would not expect the expression (8) for the reaction cross section to hold at the lower energies. Following the derivation, it is clear that σ_r may also be described as the cross section σ_c for the formation of the compound nucleus, since it was calculated by asserting that the neutron is "absorbed" after entering the nucleus. At high energies (and sometimes at very low energies) σ_c and σ_r are identical since the compound nucleus almost always decays into a channel other than the entrance channel. At intermediate energies the compound nucleus will decay with appreciable probability into the entrance channel so that σ_c and σ_r will differ considerably. σ_c will also contain the elastic resonance scattering (which is that decay of the compound nucleus in which the residual nucleus is the target nucleus in its original state). We may expect that the reaction cross section will be lower in the resonance energy region than that shown in Fig. 2.

III. APPLICATION TO EXPERIMENTAL DATA

A. Nuclear Radii

It has been commonly assumed in the evaluation of fast neutron cross sections that $\sigma_{\text{tot}}=2\pi R^2$. Our curves (Fig. 1) indicate that this asymptotic value is reached only at extremely high energies at which the concept of a completely absorbing nucleus is no longer applicable. Thus the determination of nuclear radii from the measured cross sections by Amaldi *et al.*,[5] and Sherr[6] must be revised since, at the energies of 14 Mev and 25 Mev, the theoretical values are considerably larger than $2\pi R^2$. Table I gives the result of this re-evaluation on the basis of Fig. 1; the table also includes the value of r_0 required to satisfy the relation

$$R=r_0A^{\frac{1}{3}}.$$

The discrepancy between nuclear radii calculated from 14 and 25 Mev noted in the literature is absent. We should emphasize once more that the value of R determined in this fashion may differ from other evaluations at different energies by as much as 1 or 2×10^{-13} cm.

B. Cross-Section Measurements

A number of neutron cross sections have been measured at an initial neutron energy of 14 Mev which can be interpreted as reaction cross sections. The inelastic scattering cross sections measured by Amaldi *et al.*[5] can be considered as reaction cross sections since inelastic scattering is the principal reaction at that energy. Recently D. D. Phillips and R. W. Davis[7] and also

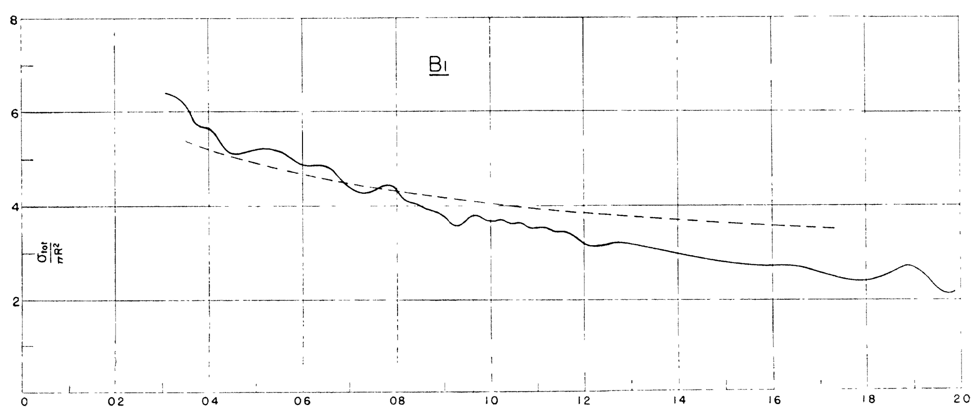

FIG. 7. σ_{tot} for Bi. The full line curve is drawn through the experimental points as given in reference 10. The fluctuations appearing in this curve are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 7.8×10^{-13} cm.

[5] Amaldi, Bocciarelli, Cacciapuoto, and Trabacchi, Nuovo Cimento **3**, 203 (1946).
[6] R. Sherr, Phys. Rev. **68**, 240 (1945).
[7] These measurements were made at the Los Alamos Scientific Laboratory. We are grateful to the authors for the communications of these results before publication.

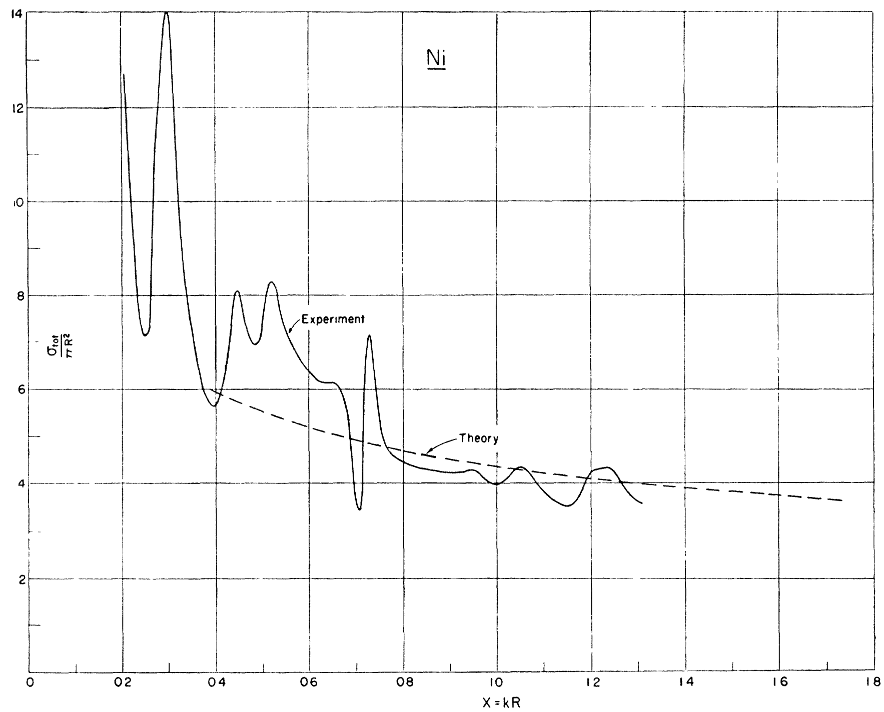

FIG. 8. σ_{tot} for Ni. The full line curve is drawn through the experimental points as given in reference 8. Some fluctuations appearing in this curve are interpreted as resonances, others are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 4.6×10^{-13} cm.

Gittings *et al.*[7a] have measured the reaction cross section of neutrons at 14 Mev in several elements at determining the cross section for those processes in which the neutron is removed from the beam and is not re-emitted at energies higher than 12 Mev. We do not use the inelastic cross sections given in reference 9 as a measure of the reaction cross section, since they are measured with neutron energies below 3 Mev, where the elastic re-emission of the neutron is not yet negligible. Table II shows the measured values together with the theoretical reaction cross sections calculated with the nuclear radii of Table I. This comparison is a rather sensitive test of the theory, since the reaction cross section is a measure of how many neutrons have penetrated into the nucleus. It checks our basic assumptions, expressed by the boundary condition (5), as to the number of particles which get into the inside of the nucleus and initiate a nuclear reaction. It is also quite sensitive to the value of K_0. The total cross sections are much less sensitive to these assumptions for any change in penetration into the inside of the nucleus is offset by an opposite change in the elastic scattering.

Extensive measurements of total neutron cross sections as functions of the neutron energy have been

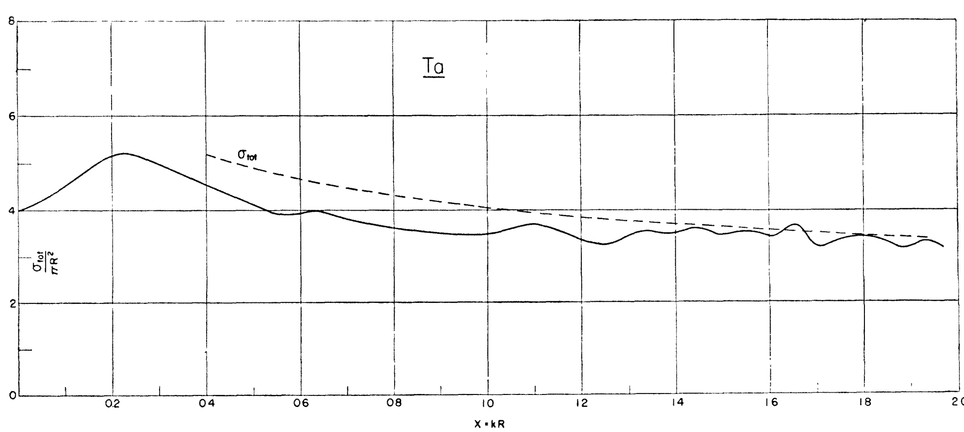

FIG. 9. σ_{tot} for Ta. The full line curve is drawn through the experimental points as given in reference 8a. The fluctuations appearing in this curve are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 8.5×10^{-13} cm.

[7a] Gittings, Barschall, and Everhart, Phys. Rev. **75**, 1610 (1949).

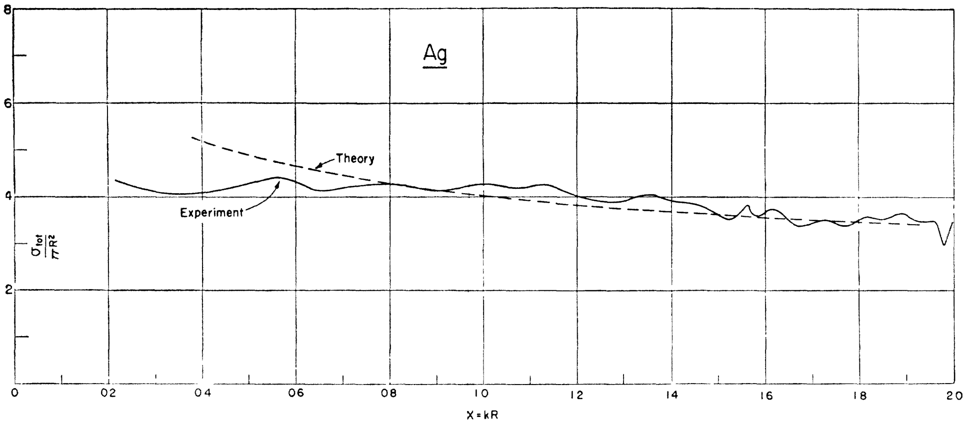

FIG. 10. σ_{tot} for Ag. The full line curve is drawn through the experimental points as given in reference 8a. The fluctuations appearing in this curve are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 7.5×10^{-13} cm.

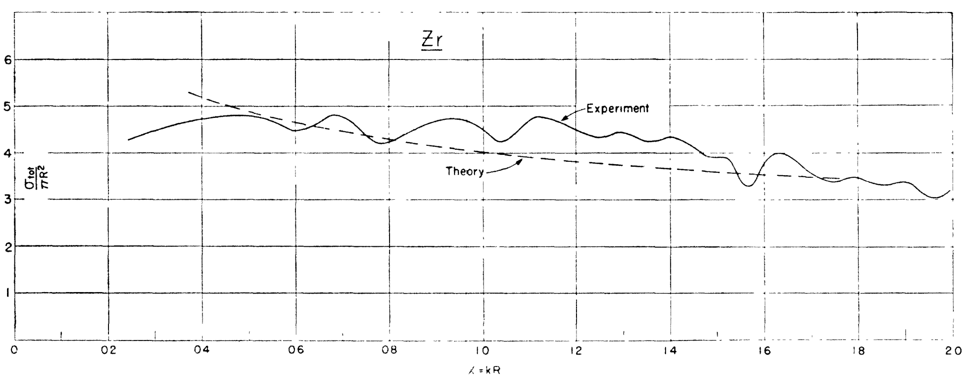

FIG. 11. σ_{tot} for Zr. The full line curve is drawn through the experimental points as given in reference 8a. The fluctuations appearing in this curve are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 7.8×10^{-13} cm.

made by Barschall *et al.*[8, 8a] on a series of nuclei in the energy region between 50 and 1400 kev. Transport cross sections have been measured at several energies by Barschall *et al.*[9] on a number of elements, especially Fe, W, and Pb. Figures 3–5 show σ_{tot} for these elements. In these figures, and in the following ones, the experimental values are plotted as functions of x and are compared with the theoretical expectation taken from Figs. 1 and 2. It should be noted that the theoretical curve is supposed to represent the average over individual fluctuations. The observed curves lie reasonably close to the theoretical curve computed from Fig. 1. The radius for each of the elements Pb, Bi, W, was taken to be 7.8×10^{-13} cm in order to obtain best agreement. Bi, however, shows a tendency toward lower radii at the higher energies. The value of 7.8×10^{-13} agrees with the radius of Pb as determined by Amaldi *et al.*,[5] which is perhaps fortuitous since the measurements are made in quite different energy regions. The radius taken for Fe, 4.8×10^{-13} cm, does not agree as well with the Amaldi value of 5.4×10^{-13} cm; the difference is within the expected range of fluctuation, however.

The transport cross sections have been measured[8] in a similar energy region for Pb, W, Fe and are plotted in Fig. 6 and compared with theory employing the same radii as were chosen in the calculation of σ_{tot}. It is possible to fit both σ_{tot} and σ_{tr} by the same choice of radius.

It is of interest to note that σ_{tot} for Pb and Bi has the expected theoretical behavior although it is well known that both elements show an exceptionally small σ_{capture} for neutrons. This behavior does not contradict our theories in view of the fact that the distance between resonance levels has been found by Barschall[10] for Pb to be large, of the order of 100 kev. This is in contrast to

[8] Barschall, Bockelmann, and Seagondollar, Phys. Rev. **73**, 659 (1948). A number of additional measurements were given to the authors by Barschall prior to publication. We express our appreciation to Dr. Barschall for his communications.

[8a] Bockelmann, Peterson, Adair, and Barschall, Phys. Rev. **76**, 277 (1949).

[9] Barschall, Battat, Bright, Graves, Jorgensen, and Manley, Phys. Rev. **72**, 881 (1947).

[10] Barschall, Bockelman, Peterson, and Adair, Phys. Rev. **76**, 1146 (1949).

most of the other heavy elements investigated so far. In this respect Pb and Bi are similar to light elements such as Fe and Ni. It then follows that the radiation width is much smaller than the neutron width (Eq. (14)), since the neutron width is proportional to the level distance. This has the consequence that neutron re-emission is more probable than neutron capture, leading to low values of σ_{capture}.

We now turn to elements for which only σ_{tot} is measured. Figures 7 to 11 show σ_{tot} for the elements Bi, Ni, Ta, Ag, Zr, between 100 and 1500 kev. We have excluded the region below 100 kev for Ni because it is difficult to average over the strong resonances which occur there. The agreement is fair in Ag, Bi, and Ni, although Bi shows the same tendency toward lower radii at higher energies as Pb. Ta seems to lie below the theoretical curve at low energies. This phenomenon appears more strongly in the following group of elements where it will be discussed. The total cross section for Zr agrees well with theory if a radius of 7.8×10^{-13} is chosen. This value is 1×10^{-13} cm larger than that predicted by $R=1.5\times10^{-13}$ $A^{\frac{1}{3}}$ and is still a possible fluctuation of the radius as defined in this theory.

The total cross-section measurements for the elements Sb, I, and In show one common characteristic feature. They are all almost constant in the energy region between 50 and 1400 kev. This behavior is in striking contrast to the elements previously discussed and also to the predictions of our theory. Figures 12, 13, and 14 show that the theoretical curve deviates by almost a factor of 2 if the nuclear radius is chosen so as to fit the high energy end. Only part of this can be explained by assuming variation of nuclear radius with energy. It may be possible to interpret this behavior by assuming an abnormal level distribution such that there is a lack of resonance levels in the low energy region, or that the resonance levels in this region are abnormally weak. We call a level a weak one if its width is much smaller than the relation (14) would predict. The weakness of a level depends therefore upon the level distance. If this is the case, the σ_{tot} is given by σ_0 rather than by σ_{tot} as seen from Eq. (13). We therefore, in Figs. 12–14, have also included the curves for σ_0 which seem to lie closer to the experimental points.

It may be of interest to discuss in this connection the cross section σ_a of radiative capture. This cross section is part of the reaction cross section. It is expected that at very low energies σ_a may approach the full reaction cross section, since the neutron width Γ_n becomes very small (see (13)). Let us call ϵ_0 the energy at which the neutron width is equal to the radiation width Γ_a. For

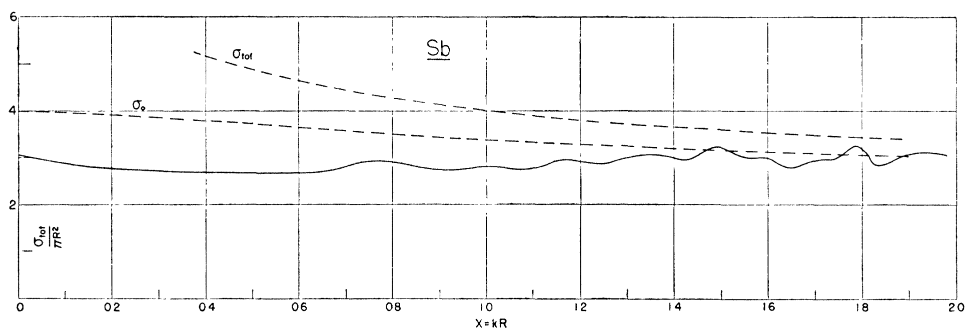

FIG. 12. σ_{tot} for Sb. The full line curve is drawn through the experimental points as given in reference 8a. The fluctuations appearing in this curve are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 8.5×10^{-13} cm. σ_0 the cross section due to an infinitely repulsive sphere is included.

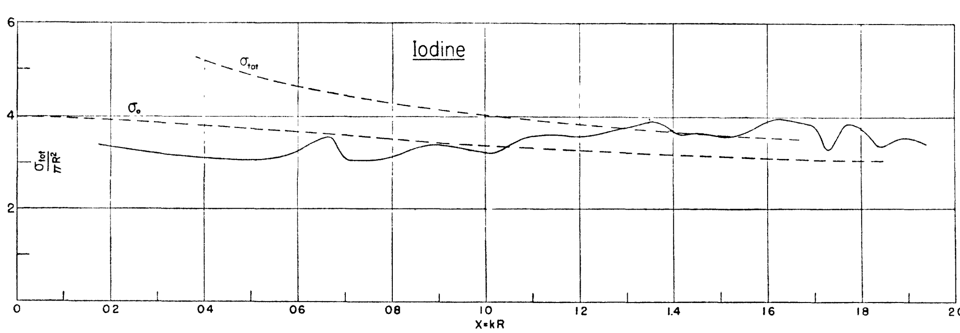

FIG. 13. σ_{tot} for I. The full line curve is drawn through the experimental points as given in reference 8a. The fluctuations appearing in this curve are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 7.5×10^{-13} cm. σ_0 the cross section due to an infinitely repulsive sphere is included.

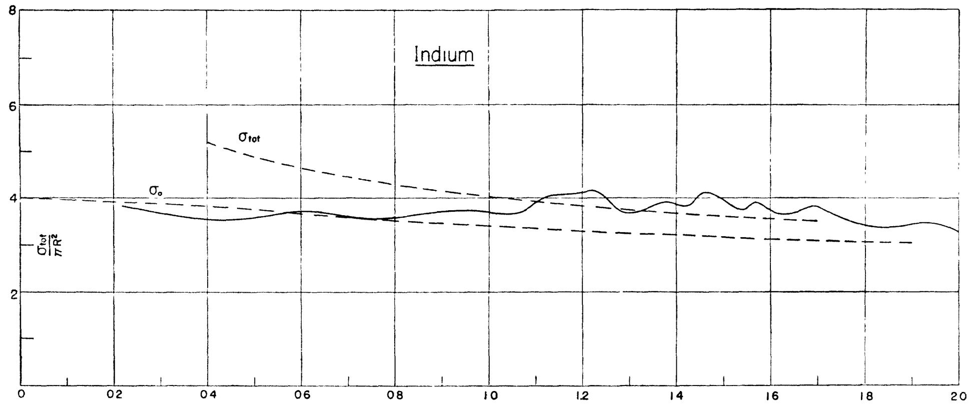

FIG. 14. σ_{tot} for In. The full line curve is drawn through the experimental points as given in reference 8a. The fluctuations appearing in this curve are within the statistical errors of the measurements. The broken line curve gives the theoretical values for a radius of 7.5×10^{-13} cm. σ_0 the cross section due to an infinitely repulsive sphere is included.

energies $\epsilon \ll \epsilon_0$ the re-emission of the neutron is much less likely than the capture, so that we may assume $\sigma_a \cong \sigma_r$, since no other process than capture can occur. In heavier elements ($A>100$) the radiation width Γ_a was found to be of the order of 0.1 ev, and independent of the neutron energy. An approximate average expression for the neutron widths in these nuclei is $\Gamma_n \sim 10^{-3}\epsilon^{\frac{1}{2}}$, where ϵ is expressed in ev. This leads to $\epsilon_0 \sim 10{,}000$ ev, and one should expect that σ_a should be equal to the reaction cross section σ_r for energies below ϵ_0 after averaging over resonances. Our expression for the reaction cross section at these energies can be approximated by

$$\sigma_r \sim 4\pi/kK_0 \cong 500/\epsilon^{\frac{1}{2}} \quad (\epsilon \text{ is expressed in ev}). \tag{15}$$

It would be of interest to measure the absorption cross section averaged over resonances for nuclei $A>100$ in the region below 10^4 ev. The $1/v$ dependence has been established in some nuclei by Lichtenberger *et al.*[11] The value of the constant of the $1/v$ law would be a test of our assumptions concerning Γ_n. Recent measurements by I. H. Dearnley[12] at the G. E. Laboratories have shown some indications that this constant is somewhat smaller than indicated in relation (15). This can be attributed to a number of reasons. A series of "weak" levels, as assumed before to explain the behavior of the total cross section in some elements, would depress the constant in (15). The constant would be decreased also if strong variations of Γ_n occur from level to level. In this case there would be some levels even below ϵ_0 in which $\Gamma_n > \Gamma_r$ and these levels would not lead to capture. Their existence would therefore lower the constant of the $1/v$ law in (15). Some levels of this type have been found[13] in heavy nuclei as W^{186} and Sm^{152}. The existence of these levels, however, has no influence upon the average total cross section as long as the average of Γ_n over many levels remains equal to (14). It is irrelevant for the value of the total cross section whether a level leads to capture or re-emission.

The measurement of σ_a at higher energies ($\epsilon \gg \epsilon_0$) contributes much less information. In fact, as Wigner[4] has pointed out, it is essentially independent of Γ_n as soon as $\Gamma_n \gg \Gamma_r$. This can be seen most simply in the $l=0$ contribution to σ_a whose average over resonances is given by

$$\sigma_a^{(0)} = \frac{2\pi^2}{k^2}\frac{\Gamma_n\Gamma_r}{D(\Gamma_n+\Gamma_r)}.$$

If $\Gamma_n \gg \Gamma_r$, σ_a depends only upon Γ_r/D and cannot be used to check our assumptions regarding Γ_n.

[11] Lichtenberger, Nobles, Monk, Kubitschek, and Dancoff, Phys. Rev. **72**, 164 (1947).

[12] We are very much indebted to Dr. Dearnley for communicating this result prior to publication.

[13] S. Harris and C. O. Muehlhause, Phys. Rev. **76**, 189 (1949); A. W. Sunyar and M. Goldhaber, Phys. Rev. **76**, 189 (1949).