Two-Component Wave Equations

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In a recent note Jehle[1] has considered the system

$$\gamma^k(\partial_k - i\varphi_k)\psi = \mu\psi^* \tag{1}$$

involving a two-component wave function ψ and its complex conjugate ψ^*; the notation $\partial_k(k=0, 1, 2, 3)$ is used for $\partial/\partial x_k$; φ_k is a real external four-potential, μ a real constant; the four matrices γ^k and their complex conjugates γ^{k*} obey the rules

$$\tfrac{1}{2}(\gamma^{k*}\gamma^\lambda + \gamma^{\lambda*}\gamma^k) = -g^{k\lambda}I, \tag{2}$$

where I denotes the unit matrix, and

$$g^{00}=1, \quad g^{11}=g^{22}=g^{33}=-1, \quad g^{k\lambda}=0 \quad \text{if} \quad k\neq\lambda.$$

Kilmister[2] has investigated an aspect of the relationship of such two-component equations to Dirac's system.

The object of this letter is to show that Jehle's equations may be used to describe Majorana particles[3] interacting with a pseudovector field.

We first split ψ and γ^k into their real and imaginary parts, putting

$$\psi = \psi_R + i\psi_I, \tag{3}$$

$$\gamma^k = \gamma_R^k + i\gamma_I^k. \tag{4}$$

Since ψ_R and ψ_I have two components we may write the wave function as a four-row real matrix

$$\Psi = \begin{vmatrix} \psi_R \\ \psi_I \end{vmatrix}. \tag{5}$$

In this way the system (1) may be written

$$\Gamma^k(\partial_k - \Gamma\varphi_k)\Psi = \mu\Psi, \tag{6}$$

where

$$\Gamma^k = \begin{vmatrix} \gamma_R^k & -\gamma_I^k \\ -\gamma_I^k & -\gamma_R^k \end{vmatrix}, \qquad \Gamma = \begin{vmatrix} \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{vmatrix}. \tag{7}$$

Using Eqs. (7) and (2), we find that the Γ^k obey Dirac's anticommutation relations, i.e.:

$$\tfrac{1}{2}(\Gamma^k\Gamma^\lambda + \Gamma^\lambda\Gamma^k) = -g^{k\lambda}I. \tag{8}$$

The Γ^k are real. Moreover it can be seen on the examples given by Jehle that Γ^1, Γ^2, Γ^3 and $i\Gamma^0$ may be taken as Hermitian; they give then a Majorana representation of Dirac's matrices. We also see that Γ anticommutes with the four Γ^k and that $\Gamma^2=-1$. Thus

$$\Gamma = \pm i\Gamma^5, \tag{9}$$

where

$$\Gamma^5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3. \tag{10}$$

Both signs in Eq. (9) may occur, as can be seen directly on the two special examples given by Jehle.

Finally, Jehle's system becomes

$$\Gamma^k(\partial_k \pm i\Gamma^5\varphi_k)\Psi = \mu\Psi. \tag{11}$$

This equation, with Ψ quantized, may describe Majorana particles of "charge" ± 1 interacting with a pseudovector field φ_k. Considered from this point of view, it is covariant for the complete Lorentz group including reflections.

Jehle's formulation of Majorana's theory is easily extended to the case where there is an interaction with a real scalar field S and a real pseudoscalar field P. In this more general case, the wave equation can be written (assuming unit "charges")

$$\gamma^k(\partial_k - i\varphi_k)\psi = (\mu + S + iP)\psi^*. \tag{12}$$

[1] H. Jehle, Phys. Rev. **75**, 1609 (1949).
[2] C. W. Kilmister, Phys. Rev. **76**, 568 (1949).
[3] E. Majorana, Nuovo Cimento **14**, 171 (1937).

The Effect of Nuclear Shells upon the Pattern of the Atomic Species

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The principle of regularity and continuity of atomic species, as developed by the writer, continues the thorium radioactive series down through the stable species to helium 4, and the uranium series to neon 22 and on to hydrogen 2. Harkins and Popelka[1] have shown that all of the species of the uranium series from Pb 206 to H 2 are stable, with the exception of Sr 90, which is adjacent to, but not at the end of the 50-neutron shell. In the thorium-helium series 5 are unstable, but all of these are near, but not at, the end of the 50- or 82-neutron shells.

The purpose of the present communication is to exhibit other effects of these "magic numbers" upon the pattern of the stable atomic species. It is shown that the *neutron magic numbers* have a much greater effect than the corresponding proton numbers.

Figure 1 gives the pattern in such a form as to emphasize the limits of the valley of stability, which, on account of the existence of series, are somewhat irregular. The "magic" neutron numbers are represented by lines which are somewhat curved. The lower limit of stability is of interest. Up to MN 20 it lies on the x axis $N/P=1$, or the isotopic number $I=0$. Above that no stable species lies below the straight line from ${}_{20}{}^{20}\mathrm{Ca}_0{}^{40}$ to ${}_{80}{}^{116}\mathrm{Hg}_{36}{}^{196}$, where the values represent ${}_P{}^N E_I{}^A$, where E is the element in question, P is the proton or atomic number, I, the isotopic number, or number of extra neutrons and A the atomic mass.

Figures 2a and 2b present the simplest plot. This gives all of the stable species. Also the naturally radioactive species of the U and Th series above ${}_{81}\mathrm{Tl}$. The most abundant species are designated by a line around the symbol, and those below or slightly above 1 percent of the element by an open circle. Above $P=32$ these rare isotopes have the lowest neutron content, below this often the highest.

Very marked is the fact that for neutron $MN=50$ there are 6 species and 3 of these are the most abundant for the element. For neutron $MN-82$ (MN=magic number) there are 7 species with 5 the most abundant for the element.

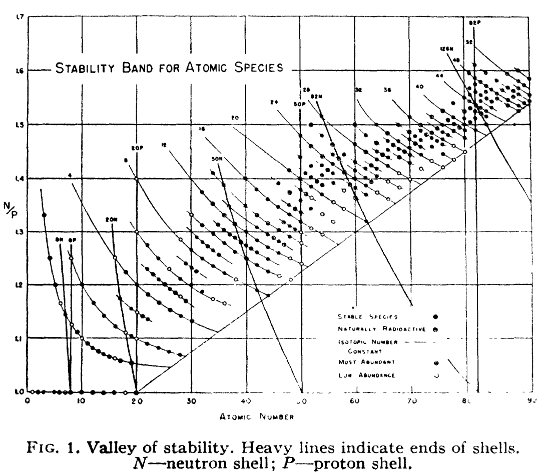

Fig. 1. Valley of stability. Heavy lines indicate ends of shells. N—neutron shell; P—proton shell.