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The Energy Level Density and Partition Function of Nuclei

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In this paper the nuclear energy level densities given by van Lier and Uhlenbeck, and by Bohr and Kalckar are compared with those given by Wergeland. It is shown that Wergeland's formula agrees slightly better with observational data. Wergeland's energy level density is then used to calculate the partition function of nuclei at a temperature of 1 Mev since this partition function enters into a discussion of the equilibrium theory of the origin of the chemical elements.

In many problems it is necessary to have a formula for the density of the energy levels of a nucleus. As examples we may mention the statistics of nuclear reactions[1] or the discussion of the equilibrium theory of the origin of the chemical elements.[2] It is therefore important to have as accurate an expression for the level density as possible. Bethe[3] assumed that the nucleus might be treated as a Fermi-Dirac gas of Z protons and N neutrons in a spherical box, the volume of which is the volume of the nucleus, and proportional to $A=N+Z$. Van Lier and Uhlenbeck[4] have generalized Bethe's procedure by considering a gas of Fermi-Dirac particles, where the density of the energy levels of the *individual* particles is left undefined. It turns out that the level density of the nucleus is given by

$$\rho(A,E)dE=[1/E(48)^{\frac{1}{2}}]\exp[\pi(2E/3\Delta)^{\frac{1}{2}}]dE, \quad (1)$$

where E is the excitation energy, $\rho(A,E)$ the density of the levels for a nucleus of atomic weight A in the neighborhood of the energy E, and Δ the spacing of the individual energy levels at the top of the Fermi distribution. It is seen that ρ depends on A and the density of the individual levels only through Δ. It is therefore not surprising that Bohr and Kalckar[5] obtained the same formula under the assumption of equidistant individual levels. For Δ, it seems a good approximation to use

$$\Delta=10/A \text{ Mev.} \quad (2)$$

Recently, Wergeland[6] has attacked the problem from a different point of view. Instead of using the approximation that the nucleons are free, he uses the liquid drop model of the nucleus. The total energy of the nucleus is no longer the sum of the individual energies of the constituent nucleons but equal to the sum of the energies of the excited proper vibrations of the drop. In this way, Wergeland obtains for the level density:

$$\rho(A,E)dE=1.14A^{1/7}E^{-5/7}\exp(0.9A^{2/7}E^{4/7})dE, \quad (3)$$

where all energies are expressed in Mev.

In order to compare the two formulas, we have calculated the level spacings for various values of A and E. The results are given in Table I. The first number given is that obtained from Eq. (1), and the second value that obtained from Eq. (3).

It is seen from Table I that Wergeland's formula (3) gives a better estimate for the energy level distance at about 10 Mev than formula (1) since from experimental data (slow neutrons) follows that the level distance should be of the order of magnitude of a few ev.[7] This is not surprising since one should expect better results from the liquid drop model than from the free particles model.

Since the level density following from formula (1) is so much larger than that given by formula (3), it is of interest to repeat the calculations of Beskow and Treffenberg[2] of the partition function of a nucleus at a temperature $T=\theta/k$, where $\theta=1$ Mev. The partition function is given by the formula

$$\Gamma=\sum g\exp(-E/\theta), \quad (4)$$

where the summation must be extended over all excited states and g is the statistical weight of the level.

TABLE I. Energy level distances in ev for different values of A and E.

E A	0.1 Mev	0.5 Mev	1 Mev	5 Mev	10 Mev	50 Mev	100 Mev
50	10^5	5.10^4	2.10^4	100	1	10^{-9}	10^{-16}
	5.10^4	5.10^4	3.10^4	1000	100	10^{-4}	10^{-9}
100	5.10^4	10^4	2000	0.5	5.10^{-4}	10^{-16}	10^{-27}
	4.10^4	3.10^4	2.10^4	300	10	10^{-7}	10^{-13}
150	3.10^4	3000	300	10^{-2}	10^{-6}	10^{-22}	10^{-34}
	3.10^4	2.10^4	10^4	100	2	10^{-8}	10^{-16}
200	2.10^4	1000	100	10^{-4}	10^{-8}	10^{-27}	10^{-41}
	2.10^4	2.10^4	6000	40	0.5	10^{-10}	10^{-17}
250	10^4	400	20	10^{-5}	10^{-10}	10^{-31}	10^{-47}
	2.10^4	10^4	5000	20	0.2	10^{-11}	10^{-19}

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[*] This paper was written while the author was temporarily at Yerkes Observatory, Williams Bay, Wisconsin.

[1] V. Weisskopf, Phys. Rev. **52**, 295 (1937).

[2] G. Beskow and L. Treffenberg, Arkiv f. Mat. Astr. o. Fys. 34A, No. 13 (1947).

[3] H. A. Bethe, Phys. Rev. **50**, 332 (1936).

[4] C. van Lier and G. E. Uhlenbeck, Physica **4**, 531 (1937).

[5] N. Bohr and F. Kalckar, Kgl. Danske Vid. Sels. Mat.-fys. Medd. **14**, No. 10 (1937).

[6] H. Wergeland, Fra Fysik. Verden, 223 (1945).

[7] See e.g., H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).

TABLE II. Values of the partition function for different values of A.

A	50	100	150	200	250
$\log_{10}\Gamma$	2.6	6.1	9.6	13.1	16.5
$\log_{10}\Gamma'$	2.2	2.6	2.9	3.2	3.5
B & T	3	6.5	10	13.5	17

Formula (4) can be written in the form:[2]

$$\Gamma \cong 1+\int_{\Delta/2}^{\infty} \rho(A, E) \exp(-E/\theta)dE, \tag{5}$$

where the 1 can be neglected in all practical instances.

Using expression (1) for $\rho(A, E)$, Γ is given by

$$\Gamma=\int_{\Delta/2}^{\infty} (1/E(48)^{\frac{1}{2}}) \exp[\pi(2E/3\Delta)^{\frac{1}{2}}-E/\theta]dE. \tag{6}$$

The integrand varies very little up to its maximum which occurs for

$$E_{\max}=\pi^2\theta^2/6\Delta, \tag{7}$$

and the integral is therefore in good approximation given by

$$\{(1/E(48)^{\frac{1}{2}}) \exp[\pi(2E/3\Delta)^{\frac{1}{2}}-E/\theta]\}_{E=E_{\max}} \times(E_{\max}-\Delta/2),$$

or, using Eqs. (2) and (7) and substituting numerical values:

$$\Gamma \cong 0.14 \exp(0.16A), \tag{8}$$

which agrees very closely with the values plotted by Beskow and Treffenberg[2] (compare also Table II).

Using formula (3) on the other hand, we have

$$\Gamma'=\int_{\Delta/2}^{\infty} 1.14A^{1/7}E^{-5/7} \exp[0.9A^{2/7}E^{4/7}-E/\theta]dE. \tag{9}$$

The maximum now occurs for

$$E_{\max}'=0.5A^{2/3}\theta^{7/3} \text{ Mev}, \tag{10}$$

and the integral is approximately equal to

$$\Gamma' \cong 0.9A^{\frac{1}{3}} \exp(0.1A^{\frac{2}{3}}), \tag{11}$$

which is much smaller than the Γ of Eq. (8), due to the fact that formula (3) gives a wider spacing of the levels. In Table II we have given the values of $\log_{10}\Gamma$ and $\log_{10}\Gamma'$ as calculated from Eqs. (8) and (11) for different values of A; in the last row, we have inserted the values of $\log_{10}\Gamma$ as given by Beskow and Treffenberg.[2]

From Table II it is immediately seen that Beskow and Treffenberg have grossly overestimated the influence of the excited states of the nucleus in calculating their partition function, using formula (1) for the energy level density. It remains to be seen whether this overestimation will invalidate their conclusions as to the relative abundances of the chemical elements in the stellar models discussed by them,[8] but it might be worth while to reinvestigate this point and to recalculate their abundance curve, using Γ' instead of Γ, and perhaps also using a different value for θ.

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[8] G. Beskow and L. Treffenberg, Arkiv f. Mat. Astr. o. Fys. 34A, No. 17 (1947).