On Infinite Relativistic Particle Matrices*

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Besides the known tensors of integral and half-odd-integral rank (spinors) there exists a new form of relativistic covariant entities with an infinite number of components, discovered in 1944 by Dirac. An especially simple type of them is generated by operations underlying a theory of the electron recently proposed by the present author. This way the representations are obtained immediately in matrix form, the matrices being of an unexpectedly simple type. Half of them have continuous spectra. They permit the setting up of wave equations having always positive energy.

* The present paper was completed at the end of 1947, but could not immediately be published. As far as it is concerned with the representations of the Lorentz-group, the matter has already been settled by V. Bargmann, Ann. of Math. **48**, 568 (1947) and Harish-Chandra, Proc. Roy. Soc. **A189**, 372 (1947), whose papers in the meantime came to the knowledge of the author. With regard to this work it should be pointed out more clearly, that these Lorentz-representations, i.e., the matrices M_k, Π_k in our terms, although mathematically the most interesting part, are only six of the sixteen elements of our theory. Our primary aims are not to construct these representations but (*a*) to translate the Poisson-brackets given in an earlier paper into commutation relations and (*b*) to find matrix representations for the κ_k and especially for the ι_k-matrices, that serve to set up the wave equation $\iota_k p^k = -m(\mathrm{I}, \mathrm{K})c$. The connection of our starting point (*a*) with Harish-Chandra's representation would be achieved, as far as the M_k, Π_k-matrices are concerned, by the unitary transformation that is indicated in its first steps by formula (37). This "normal representation" has been fully developed in the meantime and appears simultaneously in Zeits. f. Naturforschung **4a** (1949). The construction of the matrices mentioned under (*b*) needs a slight generalization of Harish-Chandra's method, because the six M_k, Π_k matrices behave like a subgroup in the 16-element-scheme. The connection between the invariants of the Lorentz-group, *viz.*, J^2, I (Harish-Chandra) resp. $-Q$, R (Bargmann) with our I, K is given by

$$-Q = J^2 = \mathrm{I}^2 - \mathrm{K}^2 - 1, \quad R = I = \mathrm{IK}.$$

The factorization of R resp. I is essential for the construction of irreducible representations of the whole 16 elements, because only I, not K commutes with all of them.

A short report of the present paper together with a first attack upon the mass problem appeared recently (Sommerfeld-Festheft). Zeits. f. Naturforschung **3a**, 559 (1948). In a simultaneous paper of Bopp (Zeits. f. Naturforschung **3a**, 564 (1948)) the underlying physical ideas have much been improved. The continuous spectrum of the ι-components has already been noticed by E. L. Hill, Phys. Rev. (2) **73**, 910 (1948), but it is a main result of our paper (see formula (23)) that ι^4 can be used in a discontinuous and even one side bounded form. Next to I it is just the key for the classification (see our Table III).

SOME years ago Dirac[1] discovered a new type of vector called "expansor" by him with an infinite number of components, which by a Lorentz-transformation undergo a *unitary* substitution. It is expected that expansors will be of great importance in particle-theory, as they provide a method of dealing with continuous entities more amenable to quantum theory than the usual methods of field theory, "quantization" not merely considered as an introduction of Planck's constant but essentially as the selection of an enumerable manifold from a continuous one. In its general form Dirac's theory looks rather complicated mainly by the particular behavior of the fourth world-coordinate. Meanwhile the author met with the same thing from another point of view, considering just radiation-reaction-forces. The method arrived at furnishes immediately the matrices of the representations and subsequently of the transformations from both a restricted and enlarged type of Dirac's. Starting with spinors rather than vectors of integral rank, we include integral as well as half-odd-integral representations. At the same time there is a wide restriction or ordering of the immense manifold of components by the existence of an invariant, which makes the representation split up into a number of non-combining ones, whose matrices can be written down in an unexpectedly simple way. Besides being Hermitian, as Dirac's theory implies, they have eigenvalues of a noteworthy type.

Representations of the Lorentz-transformation are induced[2] by a scheme of commutation-relations between six quantities, which by a suitable transformation may be combined into the components of a six-vector. In this form their commutation-rules generalize those of a moment of momentum and may be considered as the respective ones of the magnetic and relativistic electric momentum of a moving particle. For convenience we shall speak of the momentum components instead of the mathematical quantities. Denoting them by M_1, M_2, M_3, Π_1, Π_2, Π_3, we have[3]

$$M_1M_2 - M_2M_1 = iM_3, \quad \Pi_1\Pi_2 - \Pi_2\Pi_1 = -iM_3, \quad M_1\Pi_2 - \Pi_2M_1 = i\Pi_3, \tag{1}$$

together with the relations obtained from these by cyclic permutation of 123 and with

$$M_1\Pi_1 - \Pi_1M_1 = M_2\Pi_2 - \Pi_2M_2 = M_3\Pi_3 - \Pi_3M_3 = 0. \tag{2}$$

There is a very singular consequence of these equations, which looks almost like a contradiction between quantum-mechanics and relativity-theory. Of course,

[1] P. A. M. Dirac, Proc. Roy. Soc. London **A183**, 284 (1944).

[2] B. L. van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Verlag. Julius Springer, Berlin, 1932).

[3] See also P. A. M. Dirac, Proc. Roy. Soc. London **A155**, 447 (1936).

the relations are invariant under a Lorentz-transformation, i.e., they reproduce themselves, if one substitutes for the components their combinations relative to another system of reference. Now by the rules of quantum-mechanics the measurable values of any physical quantity are given by the eigenvalues of its representing matrix. However, because of the minus-sign on the second triplet of equations required by the relativistic invariance, the eigenvalues of the Π_k, $k=1, 2, 3$ are purely imaginary.

For proof introduce the quantities $\Pi=\Pi_1+i\Pi_2$ and $M=M_1+iM_2$, whose commutation with Π_3 gives

$$\Pi\Pi_3-\Pi_3\Pi=M, \quad M\Pi_3-\Pi_3M=-\Pi. \tag{3}$$

Bringing Π_3 into diagonal form and labelling the Π, M with its eigenvalues π', π'', $\cdots$, say, one has

$$(\pi''-\pi')\Pi_{\pi'\pi''}=M_{\pi'\pi''}, \quad (\pi''-\pi')M_{\pi'\pi''}=-\Pi_{\pi'\pi''}. \tag{4}$$

There may be more indices of the Π, M, say I, J, $\cdots$, a certain manifold of π's belonging to each of them. This does not prevent the conclusion, that for non-vanishing $\pi'-\pi''$

$$\Pi_{\pi'\pi''}=\frac{M_{\pi'\pi''}}{\pi''-\pi'} \tag{5}$$

for every I, J, $\cdots$, and in consequence, by the second Eq. (4), $M_{\pi'\pi''}=0$ or

$$(\pi''-\pi')^2=-1. \tag{6}$$

Thus the π are purely imaginary (save perhaps for a common additive constant, which may be shown actually to disappear).

In Dirac's theory of the electron this difficulty does not interfere, because his commutation-rules are of an entirely different ("anti-commutative") character. Moreover, there is no *intrinsic* magnetic and electric momentum at all. Different authors already have attempted to introduce one, as has Breit,[4] on the basis of experimental evidence. If one does not insist on the scheme (1), this may be realized without difficulty, since the Dirac-current can be decomposed in a well-known manner into a convective and a polarization parts. But now the author met with the same necessity from classical considerations[5] culminating in a system of Poisson-brackets for the momentum-components to be converted into commutation-rules. These Poisson-brackets are of the form (1) and by no means of an "anti-commutative" character, and so the question arose: Are there hermitian matrices, which fulfill the relations (1) or, in mathematical terms, are there unitary representations of the Lorentz-group?

At first sight the conclusion (4)–(5), which forbids them, looks quite stringent. There is only one way to escape it: The division by the factor $\pi''-\pi'$ is impossible, if the π-values are continuous (because then the left-hand sides of (4) become integrals with δ-functions) and this seems almost absurd. One might not expect, by "reasons of four-dimensional symmetry," the eigenvalues of the Π to be different from those of the M (we shall so term henceforth the Π_k, M_k, $k=1, 2, 3$, for brevity), which according to the first triple-set of Eqs. (1) are always discontinuous. But really one can only conclude that the Π-components among themselves and the M-components among themselves must have equal eigenvalues, because they can be completely exchanged by a rotation of the coordinate-system. A Lorentz-transformation never completely exchanges an M and a Π; either the transformation reproduces them or it yields a linear combination of two non-commuting ones. Now the eigenvalues of non-commuting matrices are not additive, and so the eigenvalues of these linear combinations need not be of a mixed character. The same thing comes in play at rotations, where it prevents the continuous alteration of quantized observables. There is another question, whether the sum of a matrix with a discontinuous spectrum and a second matrix with a continuous one may have discontinuous eigenvalues. A similar question arises in connection with the second triple-set of Eqs. (1), where two continuous-valued Π-components by mere multiplication and subtraction have to yield the discrete-valued M-components. But there is an elementary example of a matrix with a continuous spectrum, which even by simple multiplication with itself gets a discontinuous one. Imagine a particle reflected along a straight line between two rigid walls, represented by a potential-hole of suitable steepness, a "reflection-oscillator," so to say. It is just the case assumed to be realized, in first approximation, by the electrons inside a metallic conductor. The momentum (p) of such an oscillator, due to Heisenberg's uncertainty principle, must have a continuity of measurable values, its coordinate-values being bounded. Indeed the correspondence-principle shows that the "reflection-oscillator" will have a fully developed series-spectrum, in spite of its seemingly uniform motion. At the same time, evidently, its energy-spectrum is quantized, and so p^2 will have a discrete spectrum, whereas p has a continuous one.

After all, there is even a strong reason to guess that the eigenvalues of the Π may not be exhausted by their known finite representations. Confining oneself to the M one has $\mathbf{M}^2=M_1^2+M_2^2+M_3^2$ as a quantity which commutes with all the M_k, $k=1, 2, 3$. Therefore every value of $\mathbf{M}^2$ yields an independent representation of the M_k, which is necessarily finite, as $\mathbf{M}^2$ is definite. But $\mathbf{M}^2$ is no longer commutable with the Π_k-components, and so it may not serve to reduce the M, Π-system as a whole. However, there are now two invariants, viz. $\mathbf{M}^2-\mathbf{\Pi}^2$ and the scalar product $\mathbf{M\Pi}$, which commute with all M_k and Π_k, and these now are indefinite. So

[4] G. Breit, Phys. Rev. 72, 984 (1947).

[5] W. Wessel, FIAT Report No. 1131, henceforth referred to as l.c. Appeared also Zeits. f. Naturforschung 1, 622 (1946), with different numeration of formulas.

TABLE I. Commutation table.

	I	K	ι_1	ι_2	ι_3	ι_4	κ_1	κ_2	κ_3	κ_4	M_1	M_2	M_3	Π_1	Π_2	Π_3
I	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
K	0	0	κ_1	κ_2	κ_3	κ_4	ι_1	ι_2	ι_3	ι_4	0	0	0	0	0	0
ι_1	0	$-\kappa_1$	0	$-M_3$	M_2	$-\Pi_1$	$-K$	0	0	0	0	ι_3	$-\iota_2$	$-\iota_4$	0	0
ι_2	0	$-\kappa_2$	M_3	0	$-M_1$	$-\Pi_2$	0	$-K$	0	0	$-\iota_3$	0	ι_1	0	$-\iota_4$	0
ι_3	0	$-\kappa_3$	$-M_2$	M_1	0	$-\Pi_3$	0	0	$-K$	0	ι_2	$-\iota_1$	0	0	0	$-\iota_4$
ι_4	0	$-\kappa_4$	Π_1	Π_2	Π_3	0	0	0	0	K	0	0	0	$-\iota_1$	$-\iota_2$	$-\iota_3$
κ_1	0	$-\iota_1$	K	0	0	0	0	M_3	$-M_2$	Π_1	0	κ_3	$-\kappa_2$	$-\kappa_4$	0	0
κ_2	0	$-\iota_2$	0	K	0	0	$-M_3$	0	M_1	Π_2	$-\kappa_3$	0	κ_1	0	$-\kappa_4$	0
κ_3	0	$-\iota_3$	0	0	K	0	M_2	$-M_1$	0	Π_3	κ_2	$-\kappa_1$	0	0	0	$-\kappa_4$
κ_4	0	$-\iota_4$	0	0	0	$-K$	$-\Pi_1$	$-\Pi_2$	$-\Pi_3$	0	0	0	0	$-\kappa_1$	$-\kappa_2$	$-\kappa_3$
M_1	0	0	0	ι_3	$-\iota_2$	0	0	κ_3	$-\kappa_2$	0	0	M_3	$-M_2$	0	Π_3	$-\Pi_2$
M_2	0	0	$-\iota_3$	0	ι_1	0	$-\kappa_3$	0	κ_1	0	$-M_3$	0	M_1	$-\Pi_3$	0	Π_1
M_3	0	0	ι_2	$-\iota_1$	0	0	κ_2	$-\kappa_1$	0	0	M_2	$-M_1$	0	Π_2	$-\Pi_1$	0
Π_1	0	0	ι_4	0	0	ι_1	κ_4	0	0	κ_1	0	Π_3	$-\Pi_2$	0	$-M_3$	M_2
Π_2	0	0	0	ι_4	0	ι_2	0	κ_4	0	κ_2	$-\Pi_3$	0	Π_1	M_3	0	$-M_1$
Π_3	0	0	0	0	ι_4	ι_3	0	0	ι_4	κ_3	Π_2	$-\Pi_1$	0	$-M_2$	M_1	0

they may yield non-combining systems, being finite for themselves, but not limiting the Π_k- and M_k-values.

Indeed there are such representations, but one may not easily find them proceeding from the system (1), because the continuous character of the Π is unfavorable to their diagonal transformation, extending as it does also to the invariants. On the other hand reduction goes almost by itself, if one introduces the group of Eqs. (1) as a subgroup into the system the author arrived at in the above mentioned paper. One has only to interpret the Poisson-brackets as commutation-relations.

We shall not begin here with the somewhat remote starting point of our former considerations, but rather at once with the complete scheme of commutation-rules. Our electron, in addition to its magnetic momentum, will have a spin-angular momentum and a velocity to be discerned, following Dirac, from the velocity of its center of gravity. Let ι be a four-vector proportional to the four-velocity, but not bounded by the condition $\iota_\lambda \iota^\lambda = -1$ $(\lambda = 1 \cdots 4)$, and similarly let κ be a vector, whose spacial components are proportional to the spin-angular momentum, also unbounded and both dimensionless. Further let I and K be two invariants connected with the invariants of M, Π by

$$I^2 - K^2 = \mathbf{M}^2 - \mathbf{\Pi}^2, \quad IK = \tfrac{1}{2}(\mathbf{M\Pi} + \mathbf{\Pi M}). \tag{7}$$

Then there exists a group-like system of commutation-rules between these quantities of such a nature that a combination of two of them always gives a third one, and a Lorentz-transformation of all the components reproduces the whole system. The invariance is proved[5] for the Poisson-brackets and persists in the re-interpretation. For convenience, the system will be given in the form of a "group-table," the rows labeled with the first factor (a) and the columns with the second factor (b) of the operation

$$[ab] = ab - ba \tag{8}$$

and the value of $[ab]/i$ inserted at their crossing. (See Table I.)

The system is not a group in the mathematical sense (for lack of a unity element), being not even a ring (for failure of associativity), but one may speak quite well of subgroups, central, etc. The invariant 1 is the "central" and in consequence will split up the representation generally. Secondly, of course, one will choose one of the M-components, M_3 say, as a diagonal matrix, being certain that its eigenvalues will be discrete ones. In the third place, limiting oneself to the "subgroup" of Eqs. (1) and (2) expressed at the right lower corner of the present scheme, one might think of the introduction of K and Π_3 as quantities commuting with both I and M_3 and with one another, but by the reasons just mentioned this is impracticable. Going through the "group-table" one now sees that together with I, K, and Π_3 also ι_3, ι_4 and κ_3, κ_4 commute with M_3. None of them commutes with K, so that this quantity may be dropped. Left with I, M_3, ι_3, ι_4, κ_3, κ_4, one finds that they form two sets of mutually commuting elements, namely I, M_3, ι_3, κ_4 and I, M_3, κ_3, ι_4. Thus, the elements of one of these sets may be brought simultaneously in diagonal form and their eigenvalues may be used to label the rows and columns of all matrices. It will be shown that ι_3 and κ_4 have continuous spectra too, whereas κ_3 and ι_4 have discrete ones. Therefore, we will have a I, M_3, κ_3, ι_4-representation or, as I commutes with all elements, a set of non-combining M_3, κ_3, ι_4 representations.

At this point it seems no longer possible to avoid introducing the spinors of the previous paper. These spinors, although otherwise disliked for their entangled connection with world-vectors, are here in their proper place. They express the sixteen elements through eight real quantities (four complex spinors) in either bilinear or quadratic homogeneous forms, which exhibit at once their spectral character. The reduction of the elements to half their number involves a set of supplementary conditions studied explicitly l.c.,[5] but unimportant for the present purpose except for its compatibility with the scheme of Table I.[6] To be brief, we shall now refer repeatedly to the foregoing paper and, concerning the spinor-calculus, to the paper of Laporte and Uhlenbeck.[7] Replace the factor Γ of l.c. (5.10) by i and the round brackets by the square brackets of formula (8). They read then,

$$[\psi_\mu \chi^\nu] = \delta_\mu{}^\nu, \quad [\psi_{\dot\mu} \chi^{\dot\nu}] = -\delta_{\dot\mu}{}^{\dot\nu} \tag{9}$$

and are compatible in this form. Indeed, if an ordinary spinor ψ_μ is expressed in terms of two real quantities, say, a, b, in the form $\psi_\mu = a+ib$, its $\psi_{\dot\mu}$ is obtained by change of i into $-i$. With a, b as Hermitian matrices, ψ will transform correctly if one also understands $\psi_{\dot\mu}$ as $a-ib$. In other words, denoting as usual by a † the process of changing the rows and columns of a matrix and substituting all their i by $-i$, the matrix $\psi_{\dot\mu}$ is to be considered as $\psi_\mu{}^\dagger$, and so

$$[\psi_\mu{}^\dagger \chi^{\nu\dagger}] = [\chi^\nu \psi_\mu]^\dagger = -\delta_{\dot\mu}{}^{\dot\nu}. \tag{10}$$

Introduce now the real and imaginary parts in the form

$$p_1 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_{\dot 1}), \quad q_1 = \frac{1}{i\sqrt{2}}(\chi^1 - \chi^{\dot 1}),$$
$$p_2 = \frac{1}{i\sqrt{2}}(\psi_1 - \psi_{\dot 1}), \quad q_2 = \frac{1}{\sqrt{2}}(\chi^1 + \chi^{\dot 1}). \tag{11}$$

Then by (9) the p, q are canonical in the quantum-mechanical sense:

$$[p_m q_n] = \frac{1}{i}\delta_{mn}, \quad m, n = 1, 2. \tag{12}$$

Having exhausted only the spinors with the index 1, a second system p_3, p_4, q_3, q_4 will be introduced, also transforming the spinors with the index 2. Note that index 4 does not play a singular role, the p_k, q_k not being the components of a world-vector. With these p_k, q_k, $k = 1 \cdots 4$, the I, K, M, Π, ι, κ are now to be expressed as the I, J, $\mathfrak{M}$, $\mathfrak{P}$, j, k of the former paper, Greek type indicating dimensionless quantities. For instance M_3 takes the form (note $\psi^1 = \psi_2$, $\psi^2 = -\psi_1$)

$$M_3 = M_{12} = -\frac{1}{2i}(m_{12} - m_{\dot 1\dot 2}), \qquad \text{l.c. (4.11)}$$
$$= -\tfrac{1}{2}(\psi_1\chi^1 - \psi_2\chi^2 + \psi_{\dot 1}\chi^{\dot 1} - \psi_{\dot 2}\chi^{\dot 2}), \qquad \text{l.c. (4.7)}$$
$$= -\tfrac{1}{2}(p_1q_2 - p_2q_1) + \tfrac{1}{2}(p_3q_4 - p_4q_3). \tag{13}$$

[6] Beside the eight conditions indicated l.c. there is a supernumerary one, the number of independent elements being seven, as has been shown by W. Kofink, Ann. d. Physik (V) **38**, 421 (1940). Indeed Eq. (3.26) resp. (4.26) l.c. has a dual one to be formed by jM^* instead of jM.

[7] O. Laporte and G. E. Uhlenbeck, Phys. Rev. **37**, 1380, 1552 (1931).

TABLE II.

n_1	n_3	n_2	n_4
ι	0	ι	0
$\iota-1$	1	$\iota-1$	1
$\iota-2$	2	$\iota-2$	2
.	.	.	.
.	.	.	.
.	.	.	.
0	ι	0	ι

Having the form of angular momentum components, the brackets may be brought into quadratic forms by the contact-transformation

$$p_1 = \frac{1}{\sqrt{2}}(\bar p_1 - \bar q_2), \quad q_1 = \frac{1}{\sqrt{2}}(\bar p_2 + \bar q_1),$$
$$p_2 = \frac{1}{\sqrt{2}}(\bar p_2 - \bar q_1), \quad q_2 = \frac{1}{\sqrt{2}}(\bar p_1 + \bar q_2), \tag{14}$$
$$\text{etc.,}$$

generated by the function

$$S(q_1 \cdots q_4, \bar p_1 \cdots \bar p_4)$$
$$= \sqrt{2}\sum_{k=1}^{4} \bar p_k q_k - \bar p_1 \bar p_2 - \bar p_3 \bar p_4 - q_1 q_2 - q_3 q_4, \tag{15}$$

in the well-known manner ($p_k = \partial S/\partial q_k$, $\bar q_k = \partial S/\partial \bar p_k$). So we have

$$M_3 = -\tfrac{1}{4}(\bar p_1{}^2 + \bar q_1{}^2) + \tfrac{1}{4}(\bar p_2{}^2 + \bar q_2{}^2)$$
$$+ \tfrac{1}{4}(\bar p_3{}^2 + \bar q_3{}^2) - \tfrac{1}{4}(\bar p_4{}^2 + \bar q_4{}^2), \tag{16}$$

and it is seen at once that the eigenvalues of M_3 are all positive and negative integral and (note the factor $\frac{1}{4}$) half-odd-integral numbers. Owing to the change of signs there is no "zero-point" term, as of course must be expected from the first triple-set of (1). Likewise the other three representative quantities I, κ_3, and ι_4 just amount to quadratic forms. Writing for abbreviation

$$H_k = \tfrac{1}{2}(\bar p_k{}^2 + \bar q_k{}^2), \quad k = 1, 2, 3, 4, \tag{17}$$

one obtains ($\iota^4 = -\iota_4$)

$$M_3 = \tfrac{1}{2}(-H_1 + H_2 + H_3 - H_4),$$
$$I = \tfrac{1}{2}(H_1 - H_2 + H_3 - H_4),$$
$$\kappa_3 = \tfrac{1}{2}(H_1 + H_2 - H_3 - H_4), \tag{18}$$
$$\iota^4 = \tfrac{1}{2}(H_1 + H_2 + H_3 + H_4).$$

The fourth quantity is positive definite, which will be of both mathematical and physical importance. The other three exhaust the possibilities of two negative signs. Obviously all have discrete eigenvalues, and by reasons of rotational transformability M_1, M_2 and κ_1, κ_2 will be discrete-valued too.

Correspondingly, the Π_3, K, ι_3, κ^4 are continuous, re-establishing in this way the "relativistic symmetry." The quantity $\kappa^4 = -\kappa_4$ appears immediately, without the preceding transformation, in the form

$$\kappa^4=\tfrac{1}{4}\sum_{k=1}^{4}(p_k{}^2-q_k{}^2), \tag{19}$$

which evidently has no discrete eigenvalues, the term $-q^2$ corresponding to a potential-peak instead of a potential-hole. The invariant K and the component Π_3 become

$$\left.\begin{matrix}\mathrm{K}\\ \Pi_3\end{matrix}\right\}=\pm\tfrac{1}{2}(\bar{p}_1\bar{p}_2-\bar{q}_1\bar{q}_2)+(\bar{p}_3\bar{p}_4-\bar{q}_3\bar{q}_4), \tag{20}$$

and a transformation to principal axes shows that they are also composed of terms of the form p^2-q^2. The component $\iota_3=-\frac{1}{2}(\bar{p}_1\bar{q}_2+\bar{p}_2\bar{q}_1)+(\bar{p}_3\bar{q}_4+\bar{p}_4\bar{q}_3)$ is not so easily transformed. But

$$\iota_2=-\tfrac{1}{2}(\bar{p}_1\bar{p}_4-\bar{q}_1\bar{q}_4)+\tfrac{1}{2}(\bar{p}_2\bar{p}_3-\bar{q}_2\bar{q}_3) \tag{21}$$

has again the form (20), and from this the continuous character of the ι_1- and ι_3-components may be inferred.

Let us introduce the eigenvalues explicitly putting

$$H_k=n_k+\tfrac{1}{2},\quad n_k=0,1,2,\cdots,\quad k=1,2,3,4. \tag{22}$$

As the $\mathrm{M}_3\cdots\iota^4$ shall be used as indices themselves, we shall drop their subscripts, writing simply† μ, κ for M_3, κ_3 and, to spare the "zero-point" term, $\iota=\iota^4-1$. So we have from (18)

$$\left.\begin{aligned}\mu&=\tfrac{1}{2}(-n_1+n_2+n_3-n_4),\\ \mathrm{I}&=\tfrac{1}{2}(n_1-n_2+n_3-n_4),\\ \kappa&=\tfrac{1}{2}(n_1+n_2-n_3-n_4),\\ \iota&=\tfrac{1}{2}(n_1+n_2+n_3+n_4).\end{aligned}\right\} \tag{23}$$

Obviously the eigenvalues of I are all integral and half-odd-integral numbers, both positive and negative including zero. Every I by its character, integer or not, determines a set of ι, κ, μ of the same character. Furthermore, every given value of ι admits only a finite number of combinations of n_1, n_2, n_3, n_4. Thus the definiteness of ι^4 enables one to unfold every representation step by step. The value of ι may be written according to (23) in the forms $\iota=n_2+n_4+\mathrm{I}$ or $\iota=n_1+n_3-\mathrm{I}$, depending on the sign of I, showing that always

$$\iota\geqq|\mathrm{I}|. \tag{24}$$

For a more detailed study we may confine ourselves to the most simple case

$$\mathrm{I}=0. \tag{25}$$

Hence, $n_1+n_3=n_2+n_4$ and $\iota=n_1+n_3$, for instance. Given the value of ι one has the possible arrangements shown in Table II. Every left-side arrangement can be combined with every right-side one, there being $(\iota+1)^2$ combinations. Thus ι^4 is $(\iota+1)^2$ or $(\iota^4)^2$-fold degenerate.

† In the forthcoming paper Zeits. f. Naturforschung **4a** the letter κ denotes the maximum value of M_3.

Eliminating n_3 and n_4, for instance, one gets from (23)

$$\kappa=n_1+n_2-\iota,\quad \mu=-n_1+n_2. \tag{26}$$

Accordingly κ will have its maximum value if n_1 and n_2 have theirs (and inversely), or

$$-\iota\leqq\kappa\leqq\iota \tag{27}$$

according to Table II. The terminal values are realizable just once, with $\mu=0$. Furthermore, still following Table II, one sees that $\kappa=\iota-1$ and $\kappa=1-\iota$ are both realizable twofold with $\mu=-1$, 1, or $\kappa=\iota-2$ and $\kappa=2-\iota$ threefold with $\mu=-2$, 0, 2, etc., down to $\kappa=0$ with $\mu=-\iota$, $-\iota+2$, $-\iota+4$, $\cdots$, $\iota-2$, ι, so that the μ-values attached to a value of κ may be written

$$|\kappa|-\iota,\ |\kappa|-\iota+2,\ \cdots,\ \iota-|\kappa|-2,\ \iota-|\kappa|. \tag{28}$$

Briefly stated: a multiplication of two matrices a and b in the ι, κ, μ-representation will be accomplished by the process

$$(\iota'\kappa'\mu'|ab|\iota''\kappa''\mu'')=\sum_{0}^{\infty}\iota\sum_{-\iota}^{\iota}\kappa\sum_{|\kappa|-\iota}^{\iota-|\kappa|}\mu(2)\times(\iota'\kappa'\mu'|a|\iota\kappa\mu)(\iota\kappa\mu|b|\iota''\kappa''\mu''), \tag{29}$$

the index (2) indicating steps of two units in the index. Similarly in the case $\mathrm{I}=\frac{1}{2}$:

$$\sum_{n_1-n_2+n_3-n_4=1}=\sum_{\frac{1}{2}}^{\infty}\iota\left\{\sum_{\frac{1}{2}}^{\iota}\kappa\sum_{-\iota+\kappa-\frac{1}{2}}^{\iota-\kappa-\frac{1}{2}}\mu(2)+\sum_{-\frac{1}{2}}^{-\iota}\kappa\sum_{-\iota-\kappa+\frac{1}{2}}^{\iota+\kappa+\frac{1}{2}}\mu(2)\right\}. \tag{30}$$

In the preceding expressions exchange of κ and μ is possible. To write down the matrices we make the usual transformation

$$\bar{p}=\frac{1}{\sqrt{2}}(P+Q),\quad \bar{q}=\frac{i}{\sqrt{2}}(P-Q), \tag{31}$$

for every k. Any H, formula (22), with eigenvalues m, n will then be made diagonal by

$$P_{mn}=(m+1)^{\frac{1}{2}}\delta_{m+1,n},\quad Q_{mn}=(m)^{\frac{1}{2}}\delta_{m-1,n}, \tag{32}$$

being different H_k to be distinguished by different m_k, n_k. For convenience we shall give the six representative matrices explicitly:

$$\left.\begin{aligned}\mathrm{M}_1&=\tfrac{1}{2}(-P_1Q_3-P_3Q_1+P_2Q_4+P_4Q_2),\\ \mathrm{M}_2&=(i/2)(P_1Q_3-P_3Q_1+P_2Q_4-P_4Q_2),\\ \mathrm{M}_3&=\tfrac{1}{2}(-P_1Q_1+P_2Q_2+P_3Q_3-P_4Q_4).\\ \Pi_1&=-\tfrac{1}{2}(P_1P_4+Q_1Q_4+P_2P_3+Q_2Q_3),\\ \Pi_2&=(i/2)(P_1P_4-Q_1Q_4-P_2P_3+Q_2Q_3),\\ \Pi_3&=-\tfrac{1}{2}(P_1P_2+Q_1Q_2-P_3P_4-Q_3Q_4).\end{aligned}\right\} \tag{33}$$

Note that $P^\dagger=Q$, $Q^\dagger=P$, guaranteeing the hermitian character of the M, Π. The arrangement of the factors is arbitrary.

Inserting the P, Q from (32) one has for instance

$$\Pi_1=-\tfrac{1}{2}\{[(m_1+1)(m_4+1)]^{\frac{1}{2}}\delta_{m_1+1,n_1}\,\delta_{m_2n_2}\times\delta_{m_3n_3}\delta_{m_4+1,n_4}+\cdots\}, \tag{34}$$

TABLE III.

ι'	$\kappa'\ \mu'$ \ $\kappa''\ \mu''$	$\iota''=1$	$\iota''=2$				$\iota''=3$									$\iota''=4$															
		0 0	1 0	0 1	0 -1	-1 0	2 0	1 1	1 -1	0 2	0 0	0 -2	-1 1	-1 -1	-2 0	3 0	2 1	2 -1	1 2	1 0	1 -2	0 3	0 1	0 -1	0 -3	-1 2	-1 0	-1 -2	-2 1	-2 -1	-3 0
$\iota'=1$	0 0			1	1																										
$\iota'=2$	1 0							$\sqrt{2}$	$\sqrt{2}$																						
	0 1	$-i$								2	1																				
	0 -1	i									1	2																			
	-1 0												$\sqrt{2}$	$\sqrt{2}$																	
$\iota'=3$	2 0																$\sqrt{3}$	$\sqrt{3}$													
	1 1		$-i\sqrt{2}$																$\sqrt{6}$	$\sqrt{2}$											
	1 -1		$i\sqrt{2}$																	$\sqrt{2}$	$\sqrt{6}$										
	0 2			$-2i$																		3	1								
	0 0			i	$-i$																		2	2							
	0 -2				$2i$																			1	3						
	-1 1					$-i\sqrt{2}$																				$\sqrt{6}$	$\sqrt{2}$				
	-1 -1					$i\sqrt{2}$																					$\sqrt{2}$	$\sqrt{6}$			
	-2 0																												$\sqrt{3}$	$\sqrt{3}$	
$\iota'=4$	3 0																														
	2 1						$-i\sqrt{3}$																								
	2 -1						$i\sqrt{3}$																								
	1 2							$-i\sqrt{6}$																							
	1 0							$i\sqrt{2}$	$-i\sqrt{2}$																						
	1 -2								$i\sqrt{6}$																						
	0 3									$-3i$																					
	0 1									i	$-2i$																				
	0 -1										$2i$	$-i$																			
	0 -3											$3i$																			
	-1 2												$-i\sqrt{6}$																		
	-1 0												$i\sqrt{2}$	$-i\sqrt{2}$																	
	-1 -2													$i\sqrt{6}$																	
	-2 1														$-i\sqrt{3}$																
	-2 -1														$i\sqrt{3}$																
	-3 0																														

the . . . indicating three similar terms. Here the m_k, n_k are to be substituted by the I, ι, κ, μ according to (23). Inverting these formulas we have, putting $\mathrm{I}=0$ and indicating m and n by one and two accents respectively:

$$\begin{aligned} m_1&=\tfrac{1}{2}(\iota'+\kappa'-\mu'), & m_2&=\tfrac{1}{2}(\iota'+\kappa'+\mu'),\\ m_3&=\tfrac{1}{2}(\iota'-\kappa'+\mu'), & m_4&=\tfrac{1}{2}(\iota'-\kappa'-\mu'),\\ n_1&=\tfrac{1}{2}(\iota''+\kappa''-\mu''), & n_2&=\tfrac{1}{2}(\iota''+\kappa''+\mu''),\\ n_3&=\tfrac{1}{2}(\iota''-\kappa''+\mu''), & n_4&=\tfrac{1}{2}(\iota''-\kappa''-\mu''). \end{aligned} \tag{35}$$

The rather extended terms are reduced to half their number of factors by the commutability of Π_1 and κ (see Table I), involving κ-diagonal character of Π_1 at least for all non-vanishing values of κ, and actually throughout. Thus Π_1 attains the form

$$\Pi_1=-\tfrac{1}{4}\delta_{\kappa'\kappa''}\{[(\iota'+\kappa'-\mu'+2)(\iota'-\kappa'-\mu'+2)]^{\frac{1}{2}} \times\delta_{\iota'-\mu'+2,\iota''-\mu''}\delta_{\iota'+\mu',\iota''+\mu''}+\cdots\} \tag{36}$$

and may easily be written down row by row, the root being constant in every row. The structure of the Π-matrices will become clear from Table III, where the upper triangle refers to $-2\cdot\Pi_1$ and the lower to $-2\cdot\Pi_2$, the missing triangles being the Hermitian complements. The ι-, κ- and μ-matrices, of course, are immediately given by either the head or the left margin of the Π-representations. A complete representation may now be gained simply from Table I, if one disposes of Π_3 or one further M-component. There is some interest in considering M_1, for instance. As it commutes with ι, only its quadratic submatrices are different from zero. Table IV gives $2\cdot\mathrm{M}_1$ with the same meaning of rows and columns as Table III. Evidently the M-submatrices are further reducible, finite representations of the rotation group (as they are) being characterized by matrices of rank $2j+1$, $j=0, 1, 2, \cdots$ instead of $(\iota+1)^2$, $\iota=0, 1, 2, \cdots$. Thus e.g. $(1\kappa'\mu'|\mathrm{M}_1|1\kappa''\mu'')$ will be decomposed by a unitary transformation $U^{\dagger}\cdot\mathrm{M}_1U$

TABLE IV.

	-1	1	
-1			1
1			-1
	1	-1	

	$-\sqrt{2}$	$\sqrt{2}$						
$-\sqrt{2}$			$-\sqrt{2}$	$\sqrt{2}$				
$\sqrt{2}$				$-\sqrt{2}$	$\sqrt{2}$			
	$-\sqrt{2}$					$\sqrt{2}$		
	$\sqrt{2}$	$-\sqrt{2}$				$-\sqrt{2}$	$\sqrt{2}$	
		$\sqrt{2}$					$-\sqrt{2}$	
			$\sqrt{2}$	$-\sqrt{2}$				$\sqrt{2}$
				$\sqrt{2}$	$-\sqrt{2}$			$-\sqrt{2}$
						$\sqrt{2}$	$-\sqrt{2}$	

	$-\sqrt{3}$	$\sqrt{3}$													
$-\sqrt{3}$			-2	$\sqrt{3}$											
$\sqrt{3}$				$-\sqrt{3}$	2										
	-2					$-\sqrt{3}$	$\sqrt{3}$								
	$\sqrt{3}$	$-\sqrt{3}$					-2	2							
		2						$-\sqrt{3}$	$\sqrt{3}$						
			$-\sqrt{3}$							$\sqrt{3}$					
			$\sqrt{3}$	-2						$-\sqrt{3}$	2				
				2	$-\sqrt{3}$						-2	$\sqrt{3}$			
					$\sqrt{3}$							$-\sqrt{3}$			
						$\sqrt{3}$	$-\sqrt{3}$						2		
							2	-2					$-\sqrt{3}$	$\sqrt{3}$	
								$\sqrt{3}$	$-\sqrt{3}$					-2	
										2	$-\sqrt{3}$				$\sqrt{3}$
											$\sqrt{3}$	-2			$-\sqrt{3}$
													$\sqrt{3}$	$-\sqrt{3}$	

with

$$U=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad (37)$$

into (0) and

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

By the same transformation $(1\kappa'\mu'|\mathrm{M}_3|1\kappa''\mu'')$ goes over into (0), (1 0 −1). In this way every submatrix can be decomposed again into a step-matrix, labeled by μ and its maximum value. Yet for the present purpose it seems more interesting to have evidence of the connection between μ and κ, and so this transformation may first be set apart. Physically it may appear strange to find magnetic and angular momenta combined in the way indicated by these matrices, but in the case $\mathrm{I}=\frac{1}{2}$ one will have $\kappa=(\frac{1}{2}, -\frac{1}{2})$ together with $\mu=(-\frac{1}{2}, \frac{1}{2})$, and this will be correct for a negative electron.

Given the representations of the Lorentz-transformation, explicit transformation formulas may easily be worked out. Let us begin with an infinitesimal transformation (ϵ) to a system of reference moving in direction 1,

$$\mathrm{M}_3'=U^\dagger\mathrm{M}_3U=\mathrm{M}_3+\epsilon\Pi_2, \quad (38)$$

say. Putting

$$U=1+\epsilon u \quad (39)$$

one has after a multiplication with $U^\dagger$

$$\mathrm{M}_3u=u\mathrm{M}_3+\Pi_2. \quad (40)$$

According to Table I this equation will be obeyed by

$$u=-i\Pi_1. \quad (41)$$

A finite transformation with $\varphi=N\epsilon$ will now be effectuated by an N-fold repetition of this process. So we have at the limit $\epsilon=0$, $N=\infty$:

$$U=\lim_{N=\infty}\left(1-i\frac{\varphi}{N}\Pi_1\right)^N=\exp(-i\varphi\Pi_1). \quad (42)$$

Similarly a rotation will be obtained through M instead of Π. The exponential matrices being infinite (and actually containing an infinite number of elements in every row and column) will transform an "expansor" $A\mathrm{I}\iota\kappa\mu$ of Dirac's type according to

$$A'=\exp(-i\Pi)A \quad (43)$$

in a unitary way.

At present we must restrict ourselves to these mathematical preliminaries, as every application in the sense proposed l.c. will need rather laborious work. As is well known Dirac's theory agrees excellently with experience, and every attempt to improve it can be tested only by very subtle effects, e.g., the term-shift recently studied by Bethe,[8] and related phenomena. Only two general advantages of the present apparatus may be emphasized. First, as mentioned in the introductory remarks, the continuous spectrum shown by half of our matrices seems to account for the essentially continuous, non-quantum character of the radiation-damping-process and its natural line-breadth in a more intrinsic manner than does the usual procedure of Hohlraum quantization. It is satisfactory, that κ_k, $k=1, 2, 3$, corresponding to the spin-angular momentum, is quantized, whereas ι_k, $k=1, 2, 3$, corresponding to the four-velocity, is continuous. Very probably the ordinary velocity, to be represented by the Hermitian real part of $(\iota^4)^{-1}\iota_k$, $k=1, 2, 3$ will still be continuous. This should be more comprehensible than the puzzling behavior of Dirac's velocity matrices, which always have the eigenvalues $\pm c$ (=velocity of light).

Moreover our matrix I allows one to set up Hamilton-functions (wave-equations) with definite energy. In the rest-system of the center of gravity ($p=0$) the energy will have the form $E=(u^4)^{-1}m_0c^2$ (m_0=rest-mass, u^4=4-component of four-velocity). Now, in Dirac's theory one is provided with two operators to be connected with the mass, both having eigenvalues $+1$ and -1 and non-commuting with the Hamilton-operator as a whole. Thus, E can be positive of negative, and transitions may occur in external fields. Here we have also two invariants, I and K, of which I commutes with ι^4. Thus $E=(\iota^4)^{-1}m_0c^2$ with $m_0=m_0(\mathrm{I})$ will be hermitian and of correct transformation character. Owing to the definiteness of ι^4, however, any positive even function $m_0(\mathrm{I})$ will now always yield a positive E. With odd m_0 there may be $E>0$ or $E<0$; yet by the exclusive character of I there will be no transitions in any case, as long as m_0 does not depend on K. Of course this simple example is only an informative one and does not solve the difficult mass-problem. Indeed the physical point of view arrived at in the last section of the former paper would make one expect a term with $(\iota^4)^{-1}(\mathrm{I}^2+\mathrm{K}^2)^{\frac{1}{2}}$ rather than $(\iota^4)^{-1}$.

[8] H. Bethe, Phys. Rev. **72**, 339 (1947).