## III. CONCLUSIONS

The preliminary results from photographic plate detection at 40 degrees, 60 degrees, and 80 degrees using a 10 Mev beam[8] agree with the 10.8 Mev data within the limits of error. Over the angular region of the measurements the scattering is almost entirely nuclear. Calculated coulomb scattering at 35 degrees amounts to 4.7 percent of the total differential scattering cross section for 10.8 Mev. For comparison the $d$–$d$ scattering at 3.5 Mev is included in Fig. 2.[2] The differential cross section is somewhat lower for the higher energy data; at 90 degrees the ratio of the cross section at 10.8 Mev to that at 3.5 Mev is 0.523. For 10.8 Mev scattering the cross section remains more nearly constant over a wider angular region near 90 degrees than for 3.5 Mev scattering.

[8] Rosen, Tallmadge, and Williams, Phys. Rev. **75**, 1632 (1949).

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# On the Production of Π-Mesons by Nucleon-Nucleon Collisions

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Besides the doubts concerning the actual meson theories, and the use of a perturbation method for the study of strongly coupled fields, doubts have been raised as to the validity of some approximations usually made in the computation of the cross section for meson production by nucleon-nucleon collision. This cross section is computed here in a relativistically invariant manner for the pseudoscalar meson (pseudoscalar coupling) with the charged and the charge symmetrical theory. The results are valid for all energies of the incoming particle. The energy distribution and the total cross section derived from the invariant statistical factors are given as a point of comparison. Some computational tools are presented: a projection and a permutation operator and a method for the calculation of the density of states. In the Appendix two different ways of considering the meson production (mesonic analog to the bremsstrahlung or third-order process) are discussed.

THE problem of the production of mesons has interest for the origin of mesons in cosmic rays and for the artificial production of mesons. Many papers have been published on the production of mesons by collision between nucleons.[1–9] In the actual state of the theory, there exists no satisfactory method to compute the cross sections for the production of mesons:

(1) The equation of propagation of the meson field and the form of the interaction between the nucleon field and the meson field are not known.

(2) The perturbation method is not adapted to the study of strongly coupled fields. Attempts to circumvent the inadequacy of the perturbation method for problems involving mesons have not yielded entirely reliable methods. In the absence of an invariant relativistic method which gives the solution to radiative scattering problems without making an expansion in terms of powers of the coupling constant, the study of the multiplicity of the mesons produced in each collision is not really possible.

Besides these basic difficulties, illegitimate approximations have often been made in the course of the calculations. Taking advantage of the new computational techniques, the cross section for the production of a meson is computed here in a relativistically invariant manner. Hence, the result shares the large uncertainty of the present meson theories and of the use of the perturbation method, but it is not impaired by undue and unnecessary approximations.

We compute here the cross section for the production of a pseudoscalar meson with pseudoscalar coupling

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[1] L. W. Nordheim and G. Nordheim, Phys. Rev. **54**, 254 (1938). E. Strick, Phys. Rev. **76**, 190 (1949). (Scalar mesons—Born approximation.)

[2] H. S. W. Massey and H. C. Corben, Proc. Camb. Phil. Soc. **35**, 84 (1939). (Vector mesons. Born approximation—non-relativistic approximation. Mesonic analog to bremsstrahlung, see Appendix.)

[3] W. Heitler and H. W. Peng, Proc. Roy. Irish. Acad. **49 A** 101 (1943). W. Heitler, Proc. Roy. Irish Acad. **50 A** 155 (1945). (Møller and Rosenfeld mixture—Weizsäcker-Williams method—damping theory—extreme relativistic approximation.)

[4] Cécile Morette, Thèse de Doctorat, Paris (1947). (Møller and Rosenfeld mixture—damping theory—non-relativistic approximation—mesonic analog to the bremsstrahlung, see Appendix.)

[5] Cécile Morette and H. W. Peng, Nature **160**, 59 (1947) and Proc. Roy. Irish Acad. **51A** 217 (1948). (Møller and Rosenfeld mixture — damping theory — non-relativistic approximation — third-order process, see Appendix.)

[6] W. G. McMillan and E. Teller, Phys. Rev. **72**, 1 (1947). (Influence of the binding of the nucleons in a nucleus and corrections due to the electromagnetic interactions.)

[7] W. Horning and R. Weinstein, Phys. Rev. **72**, 251 (1947). (Scalar mesons. Born approximation—third-order process, see Appendix.)

[8] Lewis, Oppenheimer and Wouthuysen, Phys. Rev. **73**, 127 (1948). (Multiple meson production.)

[9] L. L. Foldy and R. E. Marshak, Phys. Rev. **75**, 1493 (1949). (Pseudoscalar mesons—pseudovector coupling. Born approximation non-relativistic region, mesonic analog to the bremsstrahlung, see Appendix.) The lists in references 1–9 are not exhaustive, and the notes between brackets are only very rough indications.

between the meson field and the nucleon field. The transitions from a state where two nucleons are present to a state where two nucleons and one meson are present can be represented by the following diagrams (Fig. 1). A full line is the world line of a nucleon; a dotted line is the world line of a meson. The q's and the k's and their combinations are the four-wave vectors of the wave function of the corresponding particle.

According to Dyson[10] and Feynman,[11] the collision operator **O**, which transforms the wave function of the system in the initial state $\Psi(-\infty)$ to the wave function of the system in the final state $\Psi(\infty)$, is obtained by the following prescription:

Replace each intermediate (or virtual) nucleon and meson line by the inverse of the operator of its equation of motion. Replace each vertex by the Hamiltonian of interaction. The cross section is given by the following relation:

$$\sigma=|\mathbf{O}|^2 1/[\varphi \text{ (initial state)}]\rho \text{ (final states)}, \quad (1)$$

where $\varphi$ is the flux of the particles in the incident beam, and $\rho$ is the density of the final states.

Natural units will be chosen so that $\hbar=c=1$ and the following notation[12] will be used:

$$\Psi(-\infty)=\Psi(p_0{}^1,p_0{}^2),$$
$$\Psi(+\infty)=\Psi(p^1,p^2,k).$$

Hence

$$q_0=p_0{}^1 \text{ or } p_0{}^2, \qquad q'_0=p_0{}^2 \text{ or } p_0{}^1,$$
$$q=p^1 \text{ or } p^2, \qquad q'=p^2 \text{ or } p^1.$$

$p_0{}^1$ is a four vector whose components are $E_0{}^1=(|\mathfrak{p}_0{}^1|^2+M^2)^{\frac{1}{2}}$ and $\mathfrak{p}_0{}^1$. $k$ is a four vector whose components are $\epsilon=(|\mathfrak{k}|^2+\mu^2)^{\frac{1}{2}}$ and $\mathfrak{k}$. The functions $\varphi$ and $\rho$ in a covariant representation[13] are

$$\varphi(p_0{}^1,p_0{}^2)=(\tfrac{1}{2}(p_0{}^1\times p_0{}^2)\cdot(p_0{}^1\times p_0{}^2))^{\frac{1}{2}} \quad (2)$$

$$\rho(p^1,p^2,k)=\frac{dp^1}{(2\pi)^4}\frac{dp^2}{(2\pi)^4}\frac{dk}{(2\pi)^4}2M2\pi\delta[(p^1)^2-M^2]$$
$$\times 2M2\pi\delta[(p^2)^2-M^2]2\pi\delta[k^2-\mu^2]$$
$$\times(2\pi)^4\delta(p_0{}^1+p_0{}^2-p^1-p^2-k). \quad (3)$$

The equations of motion of the field operators are:

$$\text{meson field } (\square^2-\mu^2)\phi=0,$$

$$\text{nucleon field } (i\boldsymbol{\nabla}-M)\psi=0,$$

$$\boldsymbol{\nabla}=\sum_\mu \gamma_\mu\frac{\partial}{\partial x_\mu} \quad \mathbf{p}=\sum_\mu \gamma_\mu p_\mu, \quad \mathbf{k}=\sum_\mu \gamma_\mu k_\mu,$$

$$a_\mu b_\mu=a_4b_4-\sum_{i=1}^{3}a_ib_i=a\cdot b,$$

$$\gamma_4=\beta=\rho_3,$$

$$\gamma_i=\beta\alpha_i=\rho_1\sigma_i,$$

where $\beta$, $\rho_1$, $\rho_3$, $\alpha_i$, and $\sigma_i$ are the usual Dirac matrices.

$$\bar{\psi}=\psi^*\gamma_4,$$

where $\psi^*$ means the complex conjugate transposed of $\psi$ and $\bar{\psi}$ the adjoint of $\psi$. The Hamiltonian of pseudoscalar interaction between the nucleon field and the pseudoscalar meson field is

$$H=if\bar{\psi}\gamma_5\tau_c\psi\phi_c.$$

The expression which is to be associated with each vertex of the diagram is simply $if\gamma_5\tau_c$.

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,$$

f is the coupling constant, and $\tau_c$ are the charge operators. We shall use the charge operators in the representation best suited when one is interested in the charge of the particles.

This representation, in which the charge density is diagonal, is determined by the condition

$$\tau_c\psi_{\mathfrak{N}}=\text{const. }\psi_{\mathfrak{N}}.$$

The index $\mathfrak{N}$ stands for neutron or proton.

$$\tau_c=A_{c\nu}\boldsymbol{\tau}_\nu,$$

where the $\tau$'s printed in boldface are the Pauli matrices and A is

| c \ ν | 1 | 2 | 3 |
|---|---|---|---|
| + | $\frac{1}{\sqrt{2}}$ | $\frac{-i}{\sqrt{2}}$ | 0 |
| − | $\frac{1}{\sqrt{2}}$ | $\frac{i}{\sqrt{2}}$ | 0 |
| 0 | 0 | 0 | 1 |

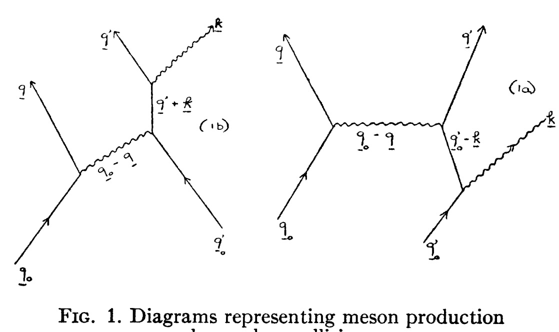


FIG. 1. Diagrams representing meson production by nuclear collision.

[10] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[11] R. P. Feynman, Phys. Rev. **76**, 769 (1949). We are indebted to Professor Feynman for his most illuminating lectures.

[12] Here is summarized the meaning of the various printings of the vectors: $p$ (italics) is a four vector, **p** (boldface) is a matrix, $\mathfrak{p}$ (German type) is a three-vector (momentum); so that $p^2=E^2-\mathfrak{p}^2$ (German type). The lower indices refer to the components of the vector; the upper indices refer to the particles. The primes indicate that the quantity is considered in the laboratory system.

[13] C. Møller, Det. Kgl. Danske Vid. Sels. Math.-Fys. Medd., Bind XXIII, Nr. 1 (1945).

c=+ or − in the charged theory, and c=+, −, or 0 in the charged symmetrical theory. By convention $\tau_+=\tau_-^*$ will be the operator for the emission of a positive meson. The following relations will be used throughout the calculation

$$\tau_{c'}^*\tau_c+\tau_c\tau_{c'}^*=2\delta_{cc'}$$

$$\left(\sum_c \tau_c^{*(1)}\tau_c^{(2)}\right)\tau_{c'}^{(1)}+\tau_{c'}^{(1)}\left(\sum_c \tau_c^{*(1)}\tau_c^{(2)}\right)=2\tau_{c'}^{(2)}. \quad (4)$$

The upper index refers to one or the other particle.

With this notation the operator 0 is:

$$0=-if^3\bar{\psi}(q_0)\gamma_5\tau_{c'}^*\psi(q)\frac{1}{(q_0-q)^2-\mu^2}\bar{\psi}(q_0')$$
$$\times\left\{\gamma_5\tau_{c'}\frac{1}{(\mathbf{q}'+\mathbf{k})-\mathrm{M}}\gamma_5\tau_c+\gamma_5\tau_c\frac{1}{(\mathbf{q}_0'-\mathbf{k})-\mathrm{M}}\gamma_5\tau_{c'}\right\}$$
$$\times\psi(q')\phi(k). \quad (5)$$

The following simplification can be made in Eq. (5):

$$\gamma_5\frac{1}{\mathbf{q}'+\mathbf{k}-\mathrm{M}}\gamma_5\psi(q')=\frac{-\mathbf{k}}{(\mathrm{q}'+\mathrm{k})^2-\mathrm{M}^2}\psi(q'),$$

$$\bar{\psi}(q_0')\gamma_5\frac{1}{\mathbf{q}_0'-\mathbf{k}-\mathrm{M}}\gamma_5=\bar{\psi}(q_0')\frac{+\mathbf{k}}{(\mathrm{q}_0'-\mathrm{k})^2-\mathrm{M}^2}.$$

Before we complete the calculation of the cross section, we shall set up some computational tools to simplify the calculation. We shall set them up in a general form so that they can be used for other calculations. In the paragraph called density of states we give the angular relations and the energy relations of these particles whose total energy and momentum is conserved.

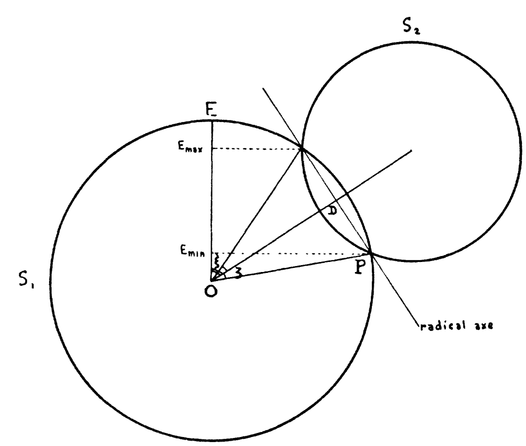



FIG. 2. This drawing is only an analog to the real configuration. The hyperquadrics $S_1$ and $S_2$ have been replaced by circles. $\mathrm{OP}=p^1$, $\mathrm{OD}=D$, OE is in the energy direction. $\mathrm{E}_{\min}\leqslant \mathrm{ch}\alpha_1\leqslant \mathrm{E}_{\max}$.

### A. DENSITY OF STATES

The integration over the density of the final states is an integration of the following type:

$$\int F(p_0^1,p_0^2,p^1,p^2,k)\rho(p^1,p^2,k). \quad (6)$$

A convenient way to proceed is the following:

(i) integration of one of the four-vectors ($p^2$ for instance) with the help of $\delta(p_0^1+p_0^2-p^1-p^2-k)$.

(ii) integration of $p^1$ say. The two $\delta$-functions $\delta[(p^1)^2-\mathrm{M}^2]$ and $\delta[(p_0^1+p_0^2-p^1-k)^2-\mathrm{M}^2]$ shows that $p^1$ is to be integrated on the intersection of the two surfaces.

$$\begin{array}{ll} \mathrm{S}_1 & (p^1)^2-(\mathrm{M}^1)^2=0, \\ \mathrm{S}_2 & (p_0^1+p_0^2-p^1-k)^2-(\mathrm{M}^2)^2=0. \end{array} \quad (7)$$

For greater generality we have set the masses of the two nucleons different. The radical plane of $\mathrm{S}_1$ and $\mathrm{S}_2$ is at a distance

$$D=\tfrac{1}{2}(p_0^1+p_0^2-k)\left(1+\frac{(\mathrm{M}^1)^2-(\mathrm{M}^2)^2}{|p_0^1+p_0^2-k|^2}\right),$$

from the origin of the vector $p^1$.

Hence, $p^1$ is to be integrated on $\mathrm{S}_1$ with the condition

$$p^1\cdot\frac{(p_0^1+p_0^2-k)}{|p_0^1+p_0^2-k|}=\mathrm{D}, \quad (8)$$

(see Fig. 2). This condition can also be derived immediately analytically from (7). $\mathrm{S}_1$ and $\mathrm{S}_2$ intersect each other only if $\mathrm{D}^2\geqslant(\mathrm{M}^1)^2$.

(iii) Integration of $k$. The condition (8) fixes the limit of integration of $k$ on the surface $k^2-\mu^2=0$.

Let us fix a system of coordinates. Let us set

$$\begin{aligned} \epsilon_0&=\mu\mathrm{ch}\theta_1 \\ \mathrm{k}_1&=\mu\mathrm{sh}\theta_1\cos\theta_2 \\ \mathrm{k}_2&=\mu\mathrm{sh}\theta_1\sin\theta_2\cos\theta_3 \\ \mathrm{k}_3&=\mu\mathrm{sh}\theta_1\sin\theta_2\sin\theta_3. \end{aligned} \quad (9)$$

The volume element is then:

$$\delta(k^2-\mu^2)\mathrm{d}k=\mu^2\mathrm{sh}^2\theta_1\mathrm{d}\theta_1\sin\theta_2\mathrm{d}\theta_2\mathrm{d}\theta_3.$$

Similarly, $p^1$ and $p^2$ will be expressed in terms of $\mathrm{M}^1$, $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\mathrm{M}^2$, $\beta_1$, $\beta_2$, $\beta_3$, respectively.

The scalar product of two vectors is:

$$k\cdot p^1=\mu\mathrm{M}[+\mathrm{ch}\theta_1\mathrm{ch}\alpha_1-\mathrm{sh}\theta_1\cos\alpha_2\mathrm{sh}\alpha_1\cos\alpha_2$$
$$-\mathrm{sh}\theta_1\sin\theta_2\mathrm{sh}\alpha_1\sin\alpha_2\cos(\theta_3-\alpha_3)].$$

Let us now fix a frame of reference: We choose the center of gravity system so that $p_0^1+p_0^2$ has only a component along the energy direction. We choose $\mathfrak{k}$ as the polar axis in the $\mathfrak{p}^1$ space. We shall now express the condition (7) for the integration of $p^1$ and determine the limits of integration of $p^1$ and $k$. Let us call $\zeta$ the angle between $p^1$ and $p_0^1+p_0^2-k$ and $\xi$ the angle between $p_0^1+p_0^2$ and $p_0^1+p_0^2-k$. The condition (7) can be expressed by the following equation:

$$\mathrm{ch}\alpha_1\mathrm{ch}\xi-\cos\alpha_2\mathrm{sh}\alpha_1\mathrm{sh}\xi=\cosh\zeta. \qquad (10)$$

It is convenient to consider $\alpha_2$ as fixed and to introduce the condition (10) into (6) through the following $\delta$-function.

$$\delta(\cos\alpha_2\mathrm{sh}\alpha_1\mathrm{sh}\xi-\mathrm{ch}\alpha_1\mathrm{ch}\xi+\cosh\zeta). \qquad (11)$$

$\alpha_1$ varies then between the following limits

$$\xi-\zeta\leqslant\alpha_1\leqslant\xi+\zeta.$$

Hence,

$$\frac{1}{4\mathrm{M}^1\mathrm{M}^2}\rho(p^1,p^2,k)=\frac{1}{(2\pi)^5}\frac{\mu\mathrm{M}^1}{2}\int_1^{\mathrm{ch}\Theta}\mathrm{dch}\theta_1\int_\pi^0\mathrm{d}\cos\theta_2$$
$$\times\int_0^{2\pi}\mathrm{d}\alpha_3\int_{\mathrm{ch}(\xi-\zeta)}^{\mathrm{ch}(\xi+\zeta)}\mathrm{dch}\alpha_1\int_\pi^0\mathrm{d}\cos\alpha_2\delta(\cos\alpha_2-A)$$
$$\times\int_0^{2\pi}\mathrm{d}\alpha_3. \qquad (12)$$

A is given by Eq. (10) and $\mathrm{ch}\Theta$ is given by Eq. (9). Translated into energy terms the density of states is:

$$\frac{1}{4\mathrm{M}^1\mathrm{M}^2}\rho(p^1,p^2,k)=\frac{1}{(2\pi)^5}\frac{1}{2}\int_\mu^{\epsilon_{\max}}\mathrm{d}\epsilon\int_\pi^0\mathrm{d}\cos\theta_2\int_0^{2\pi}\mathrm{d}\theta_3$$
$$\times\int_{\mathrm{E}^1\mathrm{min}(\epsilon)}^{\mathrm{E}^1{}_{\max}(\epsilon)}\mathrm{dE}_1\int_\pi^0\mathrm{d}\cos\alpha_2\delta(\cos\alpha_2-\mathrm{A})\int_0^{2\pi}\mathrm{d}\alpha_3, \qquad (13)$$

$$\epsilon_{\max}=\frac{1}{4\mathrm{E}}[4(\mathrm{E}_0)^2+\mu^2-(\mathrm{M}^1+\mathrm{M}^2)^2], \qquad (14)$$

$$\mathrm{E}_0=\mathrm{E}_0{}^1=\mathrm{E}_0{}^2$$
$$2\mathrm{E}_{0\min}=\mu+\mathrm{M}^1+\mathrm{M}^2. \qquad (15)$$

$$\mathrm{E}^1{}_{\substack{\max\\ \min}}(\epsilon)=(2\mathrm{E}_0-\epsilon)\mathrm{B}$$
$$\pm\left((\epsilon^2-\mu^2)\left[\mathrm{B}^2-\frac{(\mathrm{M}^1)^2}{4(\mathrm{E}_0)^2-4\mathrm{E}_0\epsilon+\mu^2}\right]\right)^{\frac{1}{2}},$$

with

$$\mathrm{B}=\frac{1}{2}\left[1+\frac{(\mathrm{M}^1)^2-(\mathrm{M}^2)^2}{4(\mathrm{E}_0)^2-4\mathrm{E}_0\epsilon+\mu^2}\right]$$

$$\mathrm{A}=\frac{\mathrm{B}[4(\mathrm{E}_0)^2-4\mathrm{E}_0\epsilon+\mu^2]-\mathrm{E}^1(2\mathrm{E}_0-\epsilon)}{\mathrm{M}^1\mu[(\mathrm{E}^1)^2-(\mathrm{M}^1)^2]^{\frac{1}{2}}[\epsilon^2-\mu^2]^{\frac{1}{2}}}.$$

The integration over the density of states yields:

$$\frac{1}{4\mathrm{M}^1\mathrm{M}^2}\int\rho(p^1,p^2,k)=\frac{2}{(2\pi)^3}\int_\mu^{\epsilon_{\max}}\left((\epsilon^2-\mu^2)(\epsilon_{\max}-\epsilon)E_0\right.$$
$$\left.\times\frac{4\mathrm{E}_0(\epsilon_{\max}-\epsilon)+4\mathrm{M}^1\mathrm{M}^2}{[4\mathrm{E}_0(\epsilon_{\max}-\epsilon)+(\mathrm{M}^1+\mathrm{M}^2)^2]^2}\right)^{\frac{1}{2}}\mathrm{d}\epsilon. \qquad (16)$$

The energy distribution of the nucleon of mass $\mathrm{M}^1$ (or $\mathrm{M}^2$) after the collision is obtained by interchanging $\mu$ with $\mathrm{M}^1$ (or $\mathrm{M}^2$) in the formulas (16) and (14). The dummy variable $\epsilon$ and the upper limit of integration $\epsilon_{\max}$ would then be called $\mathrm{E}^1$ and $\mathrm{E}^1{}_{\max}$ (or $\mathrm{E}^2$ and $\mathrm{E}^2{}_{\max}$).

The maximum of the energy distribution is shifted towards $\mathrm{E}_0$ for increasing values of $\mathrm{E}_0$. The shift is more rapid for a light particle than for a heavy particle.

The asymptotic values of $(1/4\mathrm{M}^1\mathrm{M}^2)\int\rho(p^1,p^2,k)$ are

$$\frac{4}{(2\pi)^3}\left(\frac{2\mathrm{E}_0\mu\mathrm{M}^1\mathrm{M}^2}{(\mathrm{M}^1+\mathrm{M}^2)^2}\right)^{\frac{1}{2}}\int_\mu^{\epsilon_{\max}}[(\epsilon-\mu)(\epsilon_{\max}-\epsilon)]^{\frac{1}{2}}\mathrm{d}\epsilon$$
$$=\frac{1}{(2\pi)^2}[(\mu+\mathrm{M}^1+\mathrm{M}^2)\mu\mathrm{M}^1\mathrm{M}^2]^{\frac{1}{2}}\left(\frac{\mathrm{E}_0-\mathrm{E}_{0\min}}{\mu+\mathrm{M}^1+\mathrm{M}^2}\right)^2$$
$$\text{for } \mathrm{E}_0-\mathrm{E}_{0\min}\ll\frac{\mathrm{M}^1+\mathrm{M}^2}{2}$$

$$\frac{2}{(2\pi)^3}\int_\mu^{E_0}\epsilon\mathrm{d}\epsilon=\frac{\mathrm{E}_0{}^2}{(2\pi)^3}\qquad\text{for } \mathrm{E}_0\gg\frac{\mathrm{M}^1+\mathrm{M}^2}{2}.$$

We shall now convert from the rest system to the laboratory system.

Set M, $\mathrm{E}_0'\mathfrak{p}_0'$ the mass, the energy, and the momentum of the incoming nucleon; the other nucleon is at rest. The angle between the meson emitted and the incoming nucleon is such that:

$$1-\mathrm{M}\frac{(\mathrm{E}_0{}^{2\prime}-\mathrm{M}^2)^{\frac{1}{2}}-(\epsilon^{2\prime}-\mu^2)^{\frac{1}{2}}}{(\mathrm{E}_0{}^{2\prime}-\mathrm{M}^2)^{\frac{1}{2}}(\epsilon^{2\prime}-\mu^2)^{\frac{1}{2}}}\leqslant\cos(\mathfrak{p}_0',\mathfrak{k}')\leqslant1$$
$$\text{for } (\epsilon^{2\prime}-\mu^2)^{\frac{1}{2}}\geqslant\frac{\mathrm{M}}{2}-\frac{\mathrm{M}^2}{4(\mathrm{E}_0{}^{2\prime}-\mathrm{M}^2)^{\frac{1}{2}}}$$

$$-1\leqslant\cos\mathfrak{p}_0'\cdot\mathfrak{k}'\leqslant1\quad\text{for } (\epsilon^{2\prime}-\mu^2)^{\frac{1}{2}}\leqslant\frac{\mathrm{M}}{2}-\frac{\mathrm{M}^2}{4(\mathrm{E}_0{}^{2\prime}-\mathrm{M}^2)^{\frac{1}{2}}}.$$

The angle between one of the nucleons after the collision and the incoming nucleon is:

$$1-\mathrm{M}\frac{\mathrm{E}_0'-\mathrm{E}^{n\prime}}{\mathrm{E}_0'\mathrm{E}^{n\prime}}\leqslant\cos\mathfrak{p}_0'\cdot\mathfrak{p}_n'\leqslant1 \text{ for } \mathrm{E}^{n\prime}>\mathrm{M}.$$

The asymptotic values of $(1/4\mathrm{M}^1\mathrm{M}^2)\int\rho(p^{1\prime},p^{2\prime},k')$ are

$$\frac{[(\mu+\mathrm{M}^1+\mathrm{M}^2)\mu\mathrm{M}^1\mathrm{M}^2]^{\frac{1}{2}}}{(2\pi)^2 16}\left(\frac{\mathrm{E}_0'-\mathrm{E}'_{0\min}}{\mu+\mathrm{M}^1+\mathrm{M}^2}\right)^2$$
$$\text{for } 2\mu<\mathrm{E}_0'-\mathrm{M}\ll\mathrm{M}$$
$$\frac{1}{2}\frac{\mathrm{E}_0'\mathrm{M}}{(2\pi)^3}\quad\text{for } \mathrm{E}_0'\gg\mathrm{M}.$$

The density of states of four particles (or more) can be

TABLE I.

| | $\alpha$ | $\beta$ |
|---|---|---|
| Charged theory | 0 | 4 |
| Symmetrical theory | 6 | 9 |

studied in the same way. In Eq. (7) and following $k$ is then replaced by $k^1+k^2$.

It seems convenient to compare both experimental results and the influence of the various couplings to these invariant statistical factors.

### B. PERMUTATION OPERATOR

The number of terms in Eq. (4) can be reduced in the following way: The operation of the emission and absorption operators $\psi$ and $\bar{\psi}$ of the nucleon field on the wave function of the system in the initial and in the final state is equivalent to the anti-symmetrization of the nucleon wave function in the configuration space.

Let us set $\mathcal{P}$ a permutation operator which permutes simultaneously the spin, the charge, and the momentum of the nucleons. The anti-symmetrical wave function $\psi$ of two nucleons can be expressed in terms of the product of the wave function of each nucleon in the following manner:

$$\psi_{\alpha\beta}(p_0^1,p_0^2)=\frac{1-\mathcal{P}}{\sqrt{2}}\psi_\alpha(p_0^1)\psi_\beta(p_0^2).$$

The square of a matrix element symmetrical in the nucleons 1 and 2 is equal to

$$|\psi_{\alpha\beta}(p^1,p^2)\mathrm{A}\bar{\psi}_{\gamma\delta}(p_0^1p_0^2)|^2$$
$$=(1-\mathcal{P})|\psi_\alpha(p^1)\psi_\beta(p^2)\mathrm{A}\bar{\psi}_\gamma(p_0^1)\bar{\psi}_\delta(p_0^2)|^2.$$

Hence, in Eq. (4) we can simultaneously give an identity to the nucleons and multiply the whole expression by $(1-\mathcal{P})$. This procedure simplifies the writing of the cross section. It reduces the number of terms by a factor 4 for transitions involving the same number of particles in the initial and in the final state; it reduces the number of terms by a factor 2 for transitions involving a different number of particles in the initial and in the final states. In the actual case for instance, once a name has been given to the nucleons, two diagrams must be added which describe the emission of the meson by the other nucleon.

Moreover, this procedure separates the terms in the cross section into two groups. The terms which are not proportional to $\mathcal{P}$, are obtained when the Pauli exchange principle is neglected. The other terms are the contribution of the Pauli exchange forces. In some cases these terms can be neglected. In the actual case, these terms are small because the wave-length of the nucleon is small as compared to the range of the nuclear forces. ($\mu c/\mathrm{Mv}<1$, v is the velocity of the nucleon.)

The operator which permutes the charge of the nucleons can be expressed algebraically by the expression:

$$\frac{1+\sum_{\nu=1}^{3}\boldsymbol{\tau}_\nu^{(1)}\cdot\boldsymbol{\tau}_\nu^{(2)}}{2}=\frac{1+\sum_{c}\tau_c^{(1)}\tau_c^{(2)}}{2}.$$

The covariant algebraic form of the operator which permutes the spin of the nucleons is derived as follows:

A four component Dirac wave function can be expressed as a wave function with two two-valued indices.

$$\psi_{\alpha^1}(1)\psi_{\alpha^2}(2)=\psi_{\sigma^1\rho^1}(1)\psi_{\sigma^2\rho^2}(2).$$

The indices $\rho^1\sigma^1$ and $\rho^2\sigma^2$ can be permuted simultaneously by the operator.

$$\Gamma=\left[\frac{1+\sum_{i=1}^{3}\sigma_i^{(1)}\cdot\sigma_i^{(2)}}{2}\right]\left[\frac{1+\sum_{i=1}^{3}\rho_i^{(1)}\cdot\rho_i^{(2)}}{2}\right]$$
$$=\frac{1}{4}\left\{1+\sum_{\mu=1}^{5}\left[\gamma_\mu^{(1)}\gamma_\mu^{(2)}+\frac{\delta_{\mu\nu}-1}{2}\gamma_\mu^{(1)}\gamma_\mu^{(2)}\gamma_\nu^{(1)}\gamma_\nu^{(2)}\right]\right\}.$$

One checks that

$$\Gamma\gamma_\mu^{(1)}=\gamma_\mu^{(2)}\Gamma.$$

For the case of Einstein Bose particles, $1-\mathcal{T}$ is replaced by $1+\mathcal{T}$.

### C. PROJECTION OPERATOR

The covariant operator $\Lambda(p)=(\mathbf{p}+\mathrm{M})/2\mathrm{M}$ operating on $\gamma_4\psi(p)$ is such that

$$\Lambda(p)\gamma_4\psi_\alpha(p)=\begin{cases}\psi_\alpha(p) & \text{if } \alpha \text{ designates a state whose energy has the same sign as the fourth component of } p.\\ 0 & \text{otherwise}\end{cases}$$

$(\Lambda)^2=\Lambda.$

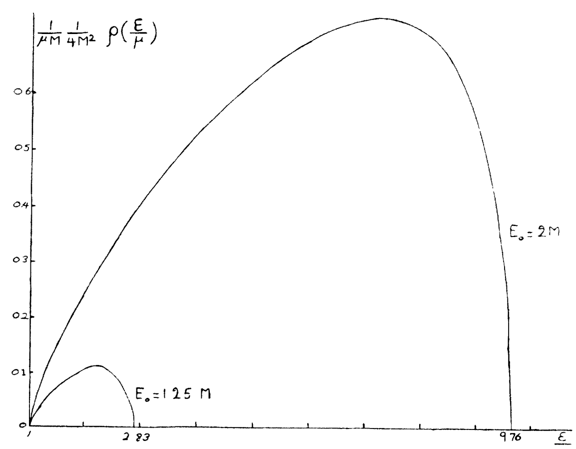



FIG. 3. Statistical energy distribution of the meson emitted (in the rest system).

$\Lambda(p)$ is a "self-selecting" operator: it selects the energy states of the same sign as the fourth component of $p$. Consequently:

$$\sum|\bar{\psi}(p_0)\mathbf{A}\psi(p)|^2=\mathrm{Spur}\mathbf{A}\Lambda(p_0)\bar{\mathbf{A}}\Lambda(p) \quad (17)$$

with

$$\bar{\mathbf{A}}=\gamma_4\mathbf{A}^*\gamma_4.$$

The sum is performed over those energy states which have the same sign as the fourth component of $p_0$ and $p$.

We notice that we can include the incoming and the outgoing lines in the prescription which give the collision operator and forget at the same time the projection operators: the world line of a nucleon in the final or in the initial state is translated in the collision operator by

$$1/(\mathbf{p}-\mathrm{M})=2\mathrm{M}\Lambda(p)/[(p)^2-\mathrm{M}^2]$$
$$=2\mathrm{M}\Lambda(p)2\pi\delta[(p)^2-\mathrm{M}^2]$$

and the world line of a meson is translated by

$$1/(\mathrm{k}^2-\mu^2)=\delta(k^2-\mu^2)$$

To avoid the mathematical difficulty raised by such expressions as $\{\delta[(p)^2-\mathrm{M}^2]\}^2$, one can also say that the cross section for a process represented by a certain diagram is obtained in the following way:

The diagram and its adjoint (i.e., the same diagram with the time direction reversed) are linked together. The prescription for the collision operator is followed on that "double-diagram" to obtain the probability of the transition.

In any case, the propagation of a particle is described similarly whether the particle is in an observable state or in a virtual state. The propagation of a particle is characterized by its four-wave vector. There is a conservation law for the four-wave vectors of the particles. When the particle is in an observable state the components of the four-wave vector are the energy and the momentum of the particle. When the particle is in a virtual state the 4th component of the wave vector is not the energy of the particle; the three other components are the momentum of the particle.

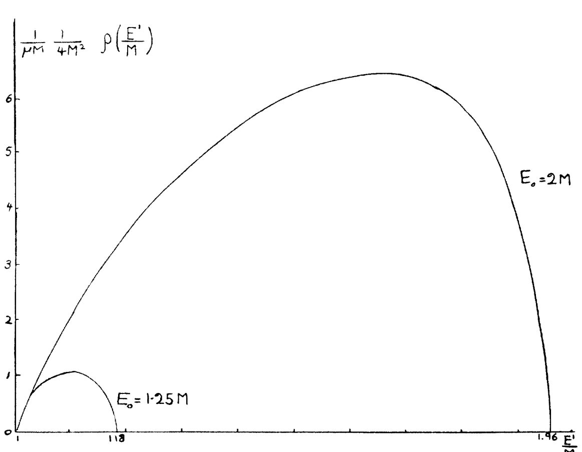



FIG. 4. Statistical energy distribution of one of the nucleons after the collision (in the rest system).

We shall now complete the computation of the cross section.

$$\sigma=\frac{1}{\varphi(p_0^1,p_0^2)}\rho(p^1,p^2,k)\Bigg\{\phi(k)\psi(p^1)\psi(p^2)\times\Bigg[-if^3\frac{\gamma_5^{(1)}\mathbf{k}^{(2)}}{(p_0^1-p^1)^2-\mu^2}\Bigg[\frac{\tau_c^{(2)}\sum_{c'}\tau_{c'}^{(1)}\tau_{c'}^{(2)}}{(p_0^2-k)^2-\mathrm{M}^2}-\frac{\sum_{c'}\tau_{c'}^{(1)}\tau_{c'}^{(2)}\tau_c^{(2)}}{(p^2+k)^2-\mathrm{M}^2}\Bigg]+1\leftrightarrow2\Bigg]\bar{\psi}(p_0^1)\bar{\psi}(p_0^2)\Bigg\}^2. \quad (18)$$

The term explicitly written corresponds to the emission of the meson by the particle 2. The other one "1↔2" corresponds to the emission of the meson by the particle 1. In agreement with the discussion of the paragraph B, the contribution of the Pauli exchange forces has been neglected.

Because of Eqs. (4) and (17) and by use of the following spurs:

$$\frac{\mathrm{M}^2}{4}\mathrm{Spur}(-\Lambda(p_0^1)\gamma_5\Lambda(p^1)\gamma_5)=(p_0^1\cdot p^1)-\mathrm{M}^2=-\tfrac{1}{2}(p_0^1-p^1)^2,$$

$$\frac{\mathrm{M}^2}{4}\mathrm{Spur}\Lambda(p_0^2)\mathbf{k}\Lambda(p^2)\mathbf{k}=2[2(p_0^2\cdot k)(p^2\cdot k)-k^2(p_0^2\cdot p^2)+k^2\mathrm{M}^2]$$
$$=-\{[-\mathrm{M}^2+(p^2+k)^2][-\mathrm{M}^2+(p_0^2-k^2)]+\mathrm{k}^2(p_0^1-p^1)^2\},$$

$$\mathrm{Spur}\Lambda(p_0^1)\gamma_5\Lambda(p^1)\mathbf{k}=0.$$

We obtain:

$$\sigma=\frac{1}{\varphi(p_0^1,p_0^2)}\rho(p^1,p^2,k)\left(\frac{\mathrm{f}^2}{4\pi}\right)^3[\alpha\mathrm{A}+\beta\mathrm{B}]$$

where the numerical values of $\alpha$ and $\beta$ are given by the following table.

$$\mathrm{A}=\frac{1}{(p_0^1-p^1)^2-\mu^2}-\frac{\mu^2}{[(p_0^2-k)^2-\mathrm{M}^2][(p^2+k)^2-\mathrm{M}^2]}$$

$$\mathrm{B}=\frac{1}{(p_0^1-p^1)^2-\mu^2}\left[\frac{(p_0^2-k)^2-\mathrm{M}^2}{(p^2+k)^2-\mathrm{M}^2}+\frac{(p^2+k)^2-\mathrm{M}^2}{(p_0^2-k)^2-\mathrm{M}^2}\right]-\frac{\mu^2}{[(p_0^2-k)^2-\mathrm{M}^2]^2}-\frac{\mu^2}{[(p^2+k)^2-\mathrm{M}^2]^2}.$$

In this expression $\mu^2$ has been neglected as compared to $\mathrm{M}^2$. Here we have averaged over the charges of the nucleons in the initial state and summed over the charges of the nucleons and the meson in the final state. The cross sections for the different cases according to the charge of the meson and the nucleons are not equal in the charged theory.

The structure of the cross section is given essentially by the term $1/[(p_0^1-p^1)^2-\mu^2]$. The terms proportional to $\mu^2$ are small as compared to the others. The two terms which are functions of $k$ balance each other to give a term nearly equal to two.

$$\int \frac{1}{(p_0^1-p^1)^2-\mu^2}\rho(p^1,p^2,k)$$
$$=\frac{(2M)^2}{(2\pi)^3}\int\frac{1}{2[((E_0^1)^2-M^2)^{\frac{1}{2}}((E^1)^2-M^2)^{\frac{1}{2}}]}$$
$$\times\log\frac{2[E_0^1E^1+((E_0^1)^2-M^2)^{\frac{1}{2}}((E^1)^2-M^2)^{\frac{1}{2}}]-2M^2+\mu^2}{2[E_0^1E^1-((E_0^1)^2-M^2)^{\frac{1}{2}}((E^1)^2-M^2)^{\frac{1}{2}}]-2M^2+\mu^2}$$
$$\times\left[((E^1)^2-M^2)\left(\frac{4(E_0^1)^2-4E_0^1E^1+\mu^2}{4(E_0^1)^2-4E_0^1E^1+M^2}\right)^2-\frac{4\mu^2}{4E_0^2-4E_0^1E^1+M^2}\right]^{\frac{1}{2}}dE^1. \quad (19)$$

The square root is merely a statistical factor; it corresponds to the integration over all the states of the meson available when the energy of one of the nucleons is equal to $E_1$. The logarithm divided by $2[(E_0^1)^2-M^2]^{\frac{1}{2}}(E_1^2-M^2)^{\frac{1}{2}}$ is essentially a function of $E_0^1$ alone. Its asymptotic values are

$$\frac{1+M}{(E_0^1-M)} \quad \text{for} \quad \frac{\mu}{2}\leqslant E_0^1-M\ll M,$$
$$\frac{4M^2}{(E_0^1)^2}\log(E_0^1) \quad \text{for} \quad E_0^1\gg M.$$

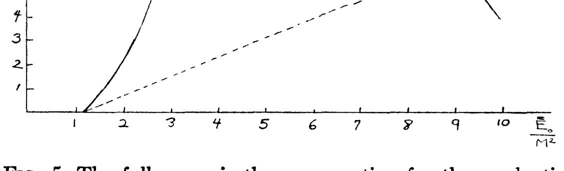



FIG. 5. The full curve is the cross section for the production of a pseudoscalar meson with pseudoscalar coupling in the charge-symmetrical theory (in the laboratory system). The dotted curve is the cross section as obtained from the invariant statistical factors (in the laboratory system). It increases up to a value of 13 which is reached in the neighborhood of $\bar{E}_0/M=20$.

Hence, the energy distribution of the meson and the nucleons is nearly the same as the energy distribution obtained from pure statistical considerations (see paragraph A). In Figs. 3 and 4 there is plotted the statistical energy distribution of the meson and the nucleon in the rest system.

The asymptotic expressions for the cross section in the laboratory system are:

$$\sigma=a\pi\left(\frac{f^2}{4\pi}\right)^3\frac{8}{M[M(E_0'-M)]^{\frac{1}{2}}}\left[E_0'-M-\frac{\mu^2-2\mu M}{M}\right]^2 \quad \text{for } 2\mu\leqslant E_0'-M\ll M$$
$$\sigma=a\left(\frac{f^2}{4\pi}\right)^3\left(\frac{8}{ME_0'}\right)\log\frac{ME_0'}{2}. \quad \text{for } E_0'\gg M$$

a=1 in the charged theory. a=3 in the symmetrical theory. $E_0'$ is the energy of the incoming nucleon, the other nucleon is at rest. The cross section in the charge symmetrical theory is about three times bigger than the cross section in the charged theory. Besides a larger number of mesons transmitted or emitted in the charge symmetrical theory, some transitions are possible only through an exchange of neutral mesons.

The total cross section is plotted on Fig. 4 in the laboratory system. On Fig. 5 there is also plotted the cross section obtained from the statistical factors: density of final states and inverse of the flux of the incoming particles (i.e., the cross section obtained from equation (2) with $\mathbf{O}=1$.

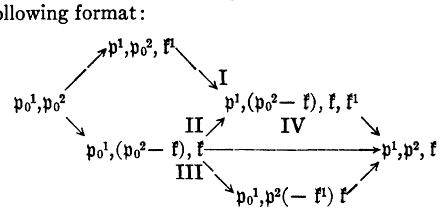
This result differs appreciably from the results obtained with a scalar meson field[1] and with the pseudovector coupling of the pseudoscalar meson field.[14] For the scalar and pseudoscalar interactions which do not depend on the energy of the meson, the energy distribution of the particles is nearly the same as the statistical energy distribution. There is no term in the cross section which favors a large or a small momentum transfer from the nucleon to the meson: The strong dependence of the cross section on large momentum transfers, at high energies, which could be expected in a pseudoscalar theory, is eliminated for two reasons: The terms corresponding to the emission of the meson before and after the scattering of the nucleons balance each other. The contribution of high energy particles limited to a small angle scattering is of the same order of magnitude as the contribution of slow energy particles with a large angle scattering; the energy denominators which look like "resonance denominators" are smeared out by the integration over the angles.

[14] The cross section for production of a pseudoscalar meson with pseudovector coupling increases like the square of the energy of the incoming particle $E_0'$. The cross section reaches the geometrical cross section for a total energy of the particle of the order of twice the rest mass. If the cross section is calculated with the damping theory, the damping terms become effective in that energy region; they reduce the cross section for proton-proton collision to a constant and the cross section for proton-neutron collision to a decreasing function of $E_0'$.

It is a great pleasure to acknowledge the hospitality of the Institute for Advanced Study and the guidance of Professor J. R. Oppenheimer. We wish to thank also Dr. K. M. Case for very helpful discussions.

## APPENDIX

The meson production is considered here from the point of view of the quantum theory of fields (case a). The meson production has been also considered from a phenomenological point of view[2,4,9] as the mesonic analog to the bremsstrahlung (case b). The process is then described as the scattering of a nucleon by another nucleon followed or preceded by the emission of a meson. The knowledge of the scattering can be obtained from experiment; thus this semi-empirical method does not ask the explanation of the momentum transfer between the nucleons from the theory. The difference between these two conceptions is more easily explained in the old form of the perturbation theory. A diagram such as Fig. 1a condenses three different processes which can be unfolded in the following format:




(20)

$\mathfrak{f}^1=\mathfrak{p}_0{}^1-\mathfrak{p}^1$ is the momentum of the virtual meson exchanged between the nucleons.

A circuit will be referred to by the corresponding number on the arrow. Circuit IV describes a direct interaction[15] between the nucleons which may or may not be needed to cancel unwanted singularities in the theoretical derivation of the nuclear potential. The cross sections obtained in the case a and b would be equivalent[5] only if the two following conditions were fulfilled:

(1°) If the matrix element for the circuit II+III+IV could be replaced as a whole by the corresponding matrix element of the nuclear potential. The diagonal elements of these two matrices are equal (elements corresponding to the transitions between two states of the same total energy); but the off diagonal elements are not equal when circuit IV is included or when there is a virtual pair creation in the intermediate state. When there is not equivalence between the circuit II+III+IV and the nuclear potential, it is difficult to speak of a nuclear potential for processes of higher order such as radiative scattering.

(2°) If circuit I were negligible as compared to the circuits II+III+IV. This condition would be necessary because the phenomenological description cannot account for circuit I where the meson is emitted during the exchange of the virtual meson. The ratio of the contribution of circuit I to the contribution of circuit II or III is of the order:

$$\mu/(\mathfrak{p}_0{}^1-\mathfrak{p}^1)<1$$

near the threshold. The importance of the contribution of the circuit I at the threshold is due to the finiteness of the meson mass. Two circumstances may affect circuit II and III in such a way that circuit I becomes of major importance. When there is no spin or charge operator, contribution from terms in which scattering precedes meson emission is equal and opposite in sign to that in which scattering follows meson emission.[7] [The contribution of circuit II (and III) cancels with that of the corresponding circuit of the mirror image of (20).] When there is a circuit IV, the contributions of circuit II and III are essentially reduced by that of circuit IV, that of circuit I is unaffected.[6]

We have pointed out the differences of the photon and meson case due to the finiteness of the meson mass. By looking at the meson production in a slightly different manner, we can bring forth another difference between these two fields. Whereas in the bremsstrahlung, the Coulomb potential is accounted for by the longitudinal part of the field, and the photon emission by the transverse part of the field; in the meson production, there is no such distinction between the field responsible for the nuclear potential and the field responsible for the meson emission. Unlike the problem of the bremsstrahlung one cannot take the difference between the total field and the part of the field associated with the nuclear potential to account for the meson emission. In the bremsstrahlung there is no difference between the case a and b.

In conclusion, the phenomenological description avoids the difficulties of the theoretical calculation of the nuclear scattering. It keeps just a part which gives meson production. It overlooks some of the differences between the meson field and the photon field.

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[15] In the first-order approximation the matrix element for the nuclear interaction $V_{A'A}$ due to the exchange of mesons between two nucleons is

$$V_{A'A}=\Sigma_i[H_{A'i}H_{iA}/(E_A-E_i)]. \quad (1)$$

Contrary to previous opinions on the static nuclear potential, the pseudoscalar mesons do not contribute to the first-order term (they contribute to the second-order term). The vector mesons give a matrix element $V_{A'A}$ which representation in coordinate space is:

$$G^2(e^{-\mu r}/r)+g^2\delta(r). \quad (2)$$

g is the coupling constant for the tensor interaction. $G^2$ is a linear combination of the square of the coupling constants of the vector and the tensor interactions. The infinite terms occurring from the tensor interaction cannot be cancelled by charge and mass renormalization alone (K. M. Case, Phys. Rev. 75, 1440 (1949)). A treatment which would remove all infinite terms may remove also the term $g^2\delta(r)$. There may not be any $\delta$-interaction in the static potential derived from $V_{A'A}$. If there is one, $V_{A'A}$ is replaced by the following expression:

$$V_{A'A}=\Sigma_i[H_{A'i}H_{iA}'/(E_A-E_i)]+H'_{A'A}. \quad (3)$$

H′ is the operator for direct interaction which is chosen so that

$$V(r)=G^2(e^{-\mu r}/r). \quad (4)$$

In Eq. (4) as in Eq. (2) the term in $G^2$ is obtained only when the static approximation is made, that is to say, when the recoil energy is neglected in Eq. (3) or in Eq. (1). This approximation is justified in Eq. (1); it is not obviously justified in Eq. (3) because $H'_{A'A}$ cancels the main contribution of $\Sigma_i[H_{A'i}H_{iA}/(E_A-E_i)]$ and the remainder is of the same order of magnitude as the first term neglected by the static approximation. It can be shown that the static approximation is justified in Eq. (3) only if the total energy of the states A and A′ is the same. Consequently, the representation of $V_{A'A}$ in coordinate space is not $G^2(e^{-\mu r}/r)$.