Equivalence Theorems for Meson-Nucleon Couplings

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The equivalence of pseudoscalar and pseudovector couplings of a pseudoscalar meson field with a nucleon field is examined. It is found that to the second order in the coupling constants the two couplings are equivalent except for two terms. One of these extra terms is predominantly an additional self-energy. The other occurs only for charged mesons and modifies the electromagnetic properties of a meson-nucleon system and the scattering of charged mesons. To this order (g^2) it does not contribute to nuclear forces. Similarly, it is shown that the vector coupling of scalar mesons to nucleons can be replaced (to g^2) by a term identical with the last of the above-mentioned terms. These results show that the delta-function interactions which have been found in the past are actually not present.

I. INTRODUCTION

FOR some time it has been known that there exist some relations between the results to be expected from a pseudoscalar meson theory with pseudoscalar coupling to the nucleon field and one with pseudovector coupling. Similarly, there have been indications that the vector coupling of a scalar meson field with nucleons is to a certain approximation illusory.

However, previous statements and proofs of these theorems have been either unconvincing[1] or incomplete.[2] These theorems are important labor saving devices and so an accurate statement of their limits of validity is useful. Below, two theorems are proved which show when one type of coupling may be omitted and when it may not. In order to include cases in which electromagnetic properties of nucleons are calculated by means of meson theory, an external electromagnetic field has been explicitly included. For generality, the proof is carried out using the Schwinger-Tomonaga many-time formalism.

The two theorems may be stated as follows:

Theorem I

To the second power in the nucleon-meson coupling constants, the Hamiltonian for a scalar meson field having both scalar and vector interaction with a nucleon field in the presence of an external electromagnetic field may be replaced by an equivalent one. The equivalent Hamiltonian is one without vector coupling but with an additional term. This additional term is zero for a neutral field and equal to

$$\frac{-g^2}{2\hbar c\kappa^2}\bar{\psi}\gamma_\nu\tau_3\psi\left\{\phi^*\left(\frac{\partial}{\partial x_\nu}-\frac{ie}{\hbar c}A_\nu\right)\phi-\phi\left(\frac{\partial}{\partial x_\nu}+\frac{ie}{\hbar c}A_\nu\right)\phi^*\right\}+\frac{ieg^2}{\hbar^2c^2\kappa^2}\phi^*\phi n_\rho n_\nu A_\rho\bar{\psi}\gamma_\nu\tau_3\psi \tag{1}$$

for a charged field.

Here ϕ, ϕ^* are the meson field operators, ψ, $\bar{\psi}$ the spinor operators of the nucleon field. κ and κ_0 are the reciprocal meson and nucleon Compton wave-lengths, respectively. g is the vector (pseudovector) coupling constant and f that for scalar (pseudoscalar) coupling, A_μ is the four-vector potential of the electromagnetic field, n_μ the unit normal at the point x to the surface σ on which we consider the wave function. τ_1, τ_2, τ_3 are the usual isotopic spin operators chosen so the eigenvalue, $+1$ or τ_3 corresponds to neutrons, -1 to protons. The γ_ν are the quantities ordinarily used in the Dirac equation.

Theorem II

To the same approximation as above, the Hamiltonian for a pseudoscalar meson field with both pseudoscalar and pseudovector coupling in an external field may also be replaced by an equivalent one. The new Hamiltonian is that for a system without pseudovector interaction but with an additional pseudoscalar interaction and two other additional terms. The first additional term is the same as the extra term in the scalar theory. The second new term is

$$(zg/\hbar c\kappa)(f-z\kappa_0 g/\kappa)\phi^*\phi\bar{\psi}\psi \tag{2}$$

for charged mesons and

$$(zg/\hbar c\kappa)(f-z\kappa_0 g/\kappa)\phi^2\bar{\psi}\psi \tag{3}$$

for neutral mesons.

II. PROCEDURE

The essence of the proof is the following. The vector and pseudovector coupling terms are mainly $(\partial\phi/\partial x_\mu)\Gamma_\mu$, where Γ_μ is a four-vector or pseudovector, respectively. Here μ is to be summed from

[1] E. C. Nelson, Phys. Rev. 60, 830 (1941).
[2] F. J. Dyson, Phys. Rev. 73, 929 (1948).

1 to 4. This term may be written

$$(\partial/\partial x_\mu)\phi\Gamma_\mu-\phi(\partial\Gamma_\mu/\partial x_\mu).$$

Using the Dirac equation one may show that the second term is zero or a pseudoscalar coupling. The first term is a four-dimensional divergence and may be eliminated by a contact transformation. This transformation can be shown to depend only on the space-like surface on which the wave function is considered and not on the past history of the the system. Using the commutation properties of the various fields, the extra terms arising from the transformation may all be expressed as point functions. It turns out that the transformation which eliminates the divergence also eliminates the direct δ-function interaction terms which arise on going from Lagrangian to Hamiltonian form with vector or pseudovector coupling. By an additional transformation most of the extra terms arising from the original may be combined with the additional electromagnetic terms, which occur when we have vector or pseudovector coupling, to give zero.

The equations used for meson theory in the many-time formalism are essentially those proposed by various Japanese workers.[3,4] Where notation is not defined, it is the same as that employed by Schwinger.[5]

III. CHARGED THEORY

In the interaction representation the equations of motion of the field operators are

$$(\Box^2-\kappa^2)\phi=(\Box^2-\kappa^2)\phi^*=0, \tag{4}$$

$$[\gamma_\mu(\partial/\partial x_\mu)+\kappa_0]\psi=[\gamma_\mu(\partial/\partial x_\mu)-\kappa_0]\bar{\psi}=0, \tag{5}$$

$$\kappa=\mu c/\hbar;\quad \kappa_0=Mc/\hbar, \tag{6}$$

where μ is the meson mass, M the nucleon mass.

The commutation relations are

$$\begin{aligned}[][\phi(x),\quad \phi^*(x')]&=i\hbar c\Delta(x-x'),\\ [\phi(x),\quad \phi(x')]&=[\phi^*(x),\quad \phi^*(x')]=0\end{aligned} \tag{7}$$

and

$$\begin{aligned}\{\psi_\alpha(x),\quad \bar{\psi}_\beta(x')\}&=1/iS_{\alpha\beta}(x-x'),\\ \{\psi_\alpha(x),\quad \psi_\beta(x')\}&=\{\bar{\psi}_\alpha(x),\quad \bar{\psi}_\beta(x')\}=0,\end{aligned} \tag{8}$$

where Δ and S are Schwinger's Δ and S functions for mass μ and M, respectively.

The Schrödinger equation for the wave functional is

$$i\hbar c\frac{\delta\Psi}{\delta\sigma}=\sum_{i=1}^{7}H_i(x)\Psi+\left(\Gamma_\mu\frac{\partial\phi^*}{\partial x_\mu}+\Gamma_\mu^*\frac{\partial\phi}{\partial x_\mu}\right)\Psi \tag{9}$$

with

$$\begin{aligned}H_1&=-1/cj_\mu A_\mu\\ H_2&=(ie/\hbar c)A_\mu[\phi^*(\partial\phi/\partial x_\mu)-\phi(\partial\phi^*/\partial x_\mu)]\\ H_3&=(e^2/\hbar^2c^2)\phi^*\phi(A_\mu^2+(n_\mu A_\mu)^2)\\ H_4&=R^*\phi+R\phi^*\\ H_5&=(ie/\hbar c)A_\mu(\Gamma_\mu\phi^*-\Gamma_\mu^*\phi)\\ H_6&=(ie/\hbar c)n_\mu A_\mu(n_\nu\Gamma_\nu\phi^*-n_\nu\Gamma_\nu^*\phi)\\ H_7&=\frac{n_\mu\Gamma_\mu n_\nu\Gamma_\nu^*+n_\nu\Gamma_\nu^*n_\mu\Gamma_\mu}{2},\end{aligned} \tag{10}$$

where

$$\begin{aligned}j_\mu&=iec\bar{\psi}\gamma_\mu(1-\tau_3)/(2)\psi\\ R&=if\bar{\psi}\gamma_5'\tau_+\psi\\ \Gamma_\mu&=(ig/\kappa)\bar{\psi}\gamma_\mu\gamma_5'\tau_+\psi.\end{aligned} \tag{11}$$

γ_5' is unity for the scalar meson theory and the conventional γ_5 for pseudoscalar mesons.

$$\tau_+=(\tau_1+i\tau_2)/2.$$

The unfamiliar terms containing n_μ result from generalizing the conventional theory so as to obtain manifest covariance and integrability of the Schrödinger equation. Pure scalar (or pseudoscalar) coupling is obtained by putting g equal to zero in (10). Vector (pseudovector) coupling is seen to give rise, in addition to the last term in (9), to

H_7—direct δ function interaction,
H_5+H_6—additional electromagnetic interaction.

Now:

$$\Gamma_\mu\frac{\partial\phi^*}{\partial x_\mu}+\Gamma_\mu^*\frac{\partial\phi}{\partial x_\mu}=\frac{\partial}{\partial x_\mu}(\Gamma_\mu\phi^*+\Gamma_\mu^*\phi)-\phi^*\frac{\partial\Gamma_\mu}{\partial x_\mu}-\phi\frac{\partial\Gamma_\mu^*}{\partial x_\mu}. \tag{12}$$

Using Eq. (5) gives

$$\phi^*\frac{\partial\Gamma_\mu}{\partial x_\mu}+\phi\frac{\partial\Gamma_\mu^*}{\partial x_\mu}=\begin{cases}0 & \text{—scalar theory}\\ \dfrac{-2\kappa_0 g}{\kappa f}(R\phi^*+R^*\phi) & \text{—pseudoscalar theory.}\end{cases} \tag{13}$$

Thus, the last term in (12) contributes nothing (scalar theory) or else gives an additional pseudoscalar coupling.

The divergence is now to be eliminated by a contact transformation. Let $\Psi=\exp(-iG_1)\Psi'$, (9) becomes

$$\begin{aligned}&i\hbar c(\delta\Psi'/\delta\sigma)+i\hbar c\exp(iG_1)(\delta\exp(-iG_1)/\delta\sigma)\Psi'\\ &=\exp(iG_1)\sum_{i=1}^{7}H_i\exp(-iG_1)\Psi'\\ &\quad+\exp(iG_1)\partial/\partial x_\mu(\Gamma_\mu\phi^*+\Gamma_\mu^*\phi)\\ &\quad\times\exp(-iG_1)\Psi'+\exp(iG_1)[-\phi^*(\partial\Gamma_\mu/\partial x_\mu)\\ &\qquad-\phi(\partial\Gamma_\mu^*/\partial x_\mu)]\exp(-iG_1)\Psi'.\end{aligned} \tag{14}$$

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[3] S. Kanasawa and S. Tomonaga, Prog. Theor. Phys. **3**, 101 (1948).
[4] Y. Miyamoto, Prog. Theory. Phys. **3**, 124 (1948).
[5] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

Taking G_1 to be of order g, and expanding to terms in g^2 gives

$$i\hbar c \exp(iG_1)\frac{\delta \exp(-iG_1)}{\delta\sigma} = \hbar c\frac{\delta G_1}{\delta\sigma}+\frac{i\hbar c}{2}\left[G_1, \frac{\delta G_1}{\delta\sigma}\right]. \quad (15)$$

Choosing

$$G_1=(1/\hbar c)(\partial/\partial x_\mu)(\Gamma_\mu\phi^*+\Gamma_\mu{}^*\phi) \quad (16)$$

gives*

$$G_1=(1/\hbar c)\int_{-\infty}^{\sigma}(\partial/\partial x_\mu)(\Gamma_\mu\phi^*+\Gamma_\mu{}^*\phi)d\omega = (1/\hbar c)\int_\sigma(\Gamma_\mu\phi^*+\Gamma_\mu{}^*\phi)d\sigma_\mu \quad (17)$$

(on applying Gauss' theorem). With the choice of G_1, (14) becomes, on inserting (15) and dropping primes,

$$i\hbar c\frac{\delta\Psi}{\delta\sigma}=\exp(iG_1)\sum_{i=1}^{7}H_i\exp(-iG_1)\Psi +\exp(iG_1)\left(-\phi^*\frac{\partial\Gamma_\mu}{\partial x_\mu}-\phi\frac{\partial\Gamma_\mu{}^*}{\partial x_\mu}\right)\exp(-iG_1)\Psi +\frac{i}{2}\left[G_1, \frac{\partial}{\partial x_\mu}(\Gamma_\mu\phi^*+\Gamma_\mu{}^*\phi)\right]\Psi. \quad (18)$$

Using the commutation rules and the properties of Δ and S on space-like surfaces gives

$$i/z[G_1, (\partial/\partial x_\mu)(\Gamma_\mu\phi^*+\Gamma_\mu{}^*\phi)] = -H_7+(\partial F_\mu/\partial x_\mu)(x)+\mathrm{He}, \quad (19)$$

where

$$F_\mu(x)=(i/2\hbar c)\int_\sigma([\Gamma_\nu(x'), \Gamma_\mu{}^*(x)]\phi^*(x')\phi(x) +[\Gamma_\nu{}^*(x'), \Gamma_\mu(x)]\phi(x')\phi^*(x))d\sigma_\nu'.$$

$$H_e(x)=\frac{g^2}{2\hbar c\kappa^2}\bar\psi\gamma_\nu\tau_3\psi\left(\phi\frac{\partial\phi^*}{\partial x_\nu}-\phi^*\frac{\partial\phi}{\partial x_\nu}\right).$$

But to order g^2,

$$\exp(iG_1)H_7\exp(-iG_1)=H_7. \quad (20)$$

Hence, the direct interaction is removed by the transformation. Equation (18) becomes

$$i\hbar c\frac{\delta\Psi}{\delta\sigma}=\exp(iG_1)\sum_{i=1}^{6}H_i\exp(-iG_1)\Psi +\exp(iG_1)\left(-\phi^*\frac{\partial\Gamma_\mu}{\partial x_\mu}-\phi\frac{\partial\Gamma_\mu{}^*}{\partial x_\mu}\right) \times\exp(-iG_1)\Psi+\frac{\partial F_\mu}{\partial x_\mu}\Psi+\mathrm{He}\Psi. \quad (21)$$

* This is just Dyson's transformation expressed in the many-time formalism.

The last term in (21) is again a divergence and is eliminated by the contact transformation.

$$\Psi=\exp(-iG_2)\Psi' \quad (22)$$

with

$$G_2=(1/\hbar c)\int_\sigma F_\mu(x'')d\sigma_\mu''. \quad (23)$$

G_2 is of order g^2 and so, on keeping only those terms which have contributions of order g^2 or lower,

$$i\hbar c\frac{\delta\Psi'}{\delta\sigma}=\exp(iG_2)\exp(iG_1) \times\sum_{i=1}^{6}H_i\exp(-iG_1)\exp(-iG_2)\Psi' +\exp(iG_1)[-\phi^*(\partial\Gamma_\mu/\partial x_\mu) -\phi(\partial\Gamma_\mu{}^*/\partial x_\mu)]\exp(-iG_1)\Psi' +\exp(iG_1)H_e\exp(-iG_1)\Psi^1. \quad (24)$$

Using the theorem

$$\exp(iA)B\exp(-iA)=\sum_{l=0}^{\infty}(i^l/l!)[A, B]_l, \quad (25)$$

where $[A, B]_l$ is the l'th repeated commutator, (24) becomes on keeping terms at most second order in the coupling constants and again dropping primes,

$$i\hbar c(\delta\Psi/\delta\sigma)=(H_a+H_b+H_c+H_d+H_e)\Psi, \quad (26)$$

where

$$\begin{aligned} H_a&=\sum_{i=1}^{4}H_i-[\phi^*(\partial\Gamma_\mu/\partial x_\mu)+\phi(\partial\Gamma_\mu{}^*/\partial x_\mu)]\\ H_b&=i[G_1, H_4]-i[G_1, \phi^*(\partial\Gamma_\mu/\partial x_\mu) +\phi(\partial\Gamma_\mu{}^*/\partial x_\mu)]\\ H_c&=H_5+i[G_1, H_1]+i[G_1, H_5] -\tfrac{1}{2}[G_1, [G_1, H_1]]+i[G_2, H_1]\\ H_d&=H_6+i[G_1, H_2]+i[G_1, H_6] -\tfrac{1}{2}[G_1, [G_1, H_2]]+i[G_2, H_2]. \end{aligned} \quad (27)$$

In (26) the commutativity of H_3 with G_1 and G_2 has been used.

By (7) and (8) and the properties of S and Δ, one finds

$$i[G_1, H_1]=-H_5, \quad (28)$$

and then

$$H_c=i/z[G_1, H_5]+i[G_2, H_1]. \quad (29)$$

These two terms may be combined by means of Jacobi's identity and then evaluated giving

$$H_c=(+ieg^2/\hbar^2c^2\kappa^2)A_\nu\phi^*\phi\bar\psi\gamma_\nu\tau_3\psi. \quad (30)$$

Similarly,

$$i[G_1, H_2]=-H_6, \quad (31)$$

and so

$$H_d=i/z[G_1, H_6]+i[G_2, H_2] \quad (32)$$

which is

$$H_d=(+ieg^2/\hbar^2c^2\kappa^2)n_\rho A_\rho\phi^*\phi\bar\psi\tau_3 n_\nu\gamma_\nu\psi. \quad (33)$$

Equations (30) and (33) hold for both the scalar and pseudoscalar theories.

Straightforward evaluation of the commutator yields for H_b

$$H_b=0 \quad \text{(scalar theory)} \tag{34}$$

$$H_b=(zg/\hbar c\kappa)(f-z\kappa_0 g/\kappa)\phi^*\phi\bar{\psi}\psi \quad \text{(pseudoscalar theory)}. \tag{34'}$$

Substituting the results obtained in (26) gives as the Schrödinger equation for Ψ:

Scalar theory

$$i\hbar c\frac{\delta\Psi}{\delta\sigma}=\sum_{i=1}^{4}H_i\Psi - \frac{g^2}{z\hbar c\kappa^2}\bar{\psi}\gamma_\nu\tau_3\psi\left\{\phi^*\left(\frac{\partial}{\partial x_\nu}-\frac{ie}{\hbar c}A_\nu\right)\phi - \phi\left(\frac{\partial}{\partial x_\nu}+\frac{ie}{\hbar c}A_\nu\right)\phi^*\right\}\Psi + \frac{ieg^2}{\hbar^2c^2\kappa^2}\phi^*\phi n_\rho n_\nu A_\rho\bar{\psi}\gamma_\nu\tau_3\psi\Psi. \tag{35}$$

Pseudoscalar theory

$$i\hbar c\frac{\delta\Psi}{\delta\sigma}=\sum_{i=1}^{4}H_i\Psi-\frac{z\kappa_0 g}{\kappa f}H_4\Psi - \frac{g^2}{z\hbar c\kappa^2}\bar{\psi}\gamma_\nu\tau_3\psi\left\{\phi^*\left(\frac{\partial}{\partial x_\nu}-\frac{ie}{\hbar c}A_\nu\right)\phi - \phi\left(\frac{\partial}{\partial x_\nu}+\frac{ie}{\hbar c}A_\nu\right)\phi^*\right\}\Psi + \frac{ieg^2}{\hbar^2c^2\kappa^2}\phi^*\phi n_\rho n_\nu A_\rho\bar{\psi}\gamma_\nu\tau_3\psi\Psi + \frac{2g}{\hbar c\kappa}\left(f-\frac{z\kappa_0 g}{\kappa}\right)\phi^*\phi\bar{\psi}\psi\Psi. \tag{35'}$$

IV. NEUTRAL THEORY

The equations for the neutral meson theory in the many-time formalism may be obtained from those of the charged theory given above by identifying ϕ and ϕ^* and omitting the isotopic spin operator. In the Schrödinger equation the terms H_2, H_3, H_5, and H_6 are to be omitted. By transformations identical with G_1 and G_2, except for the absence of starred operators and τ vectors, the equations may be put in the form given by (26) and (27).

H_d vanishes since all the H_ν's occurring there are zero.

H_c becomes

$$H_c=i[G_1, H_1]-\tfrac{1}{2}[G_1, [G_1, H_1]]+i[G_2, H_1]. \tag{36}$$

It can readily be shown that G_1 and G_2 now commute with H_1.

$$\therefore \quad H_c=0. \tag{37}$$

H_b on evaluation gives the same result as in (34) and (34′) with the replacement

$$\phi^*\phi\rightarrow\phi^2(x). \tag{38}$$

Similarly He = 0.

V. DISCUSSION

It is to be noted that the Schrödinger equation for the scalar (pseudoscalar) theory with pure scalar (pseudoscalar) coupling is given by

$$i\hbar c(\delta\Psi/\delta\sigma)=\sum_{i=1}^{4}H_i\Psi.$$

Thus, with charged mesons the effect of vector (pseudovector) coupling is to add three terms to the Schrödinger equation. The first of these terms is zero for the scalar theory and a multiple of the pseudoscalar coupling for pseudoscalar mesons. The second addition is the same for both types of field and is

$$-\frac{g^2}{z\hbar c\kappa^2}\bar{\psi}\gamma_\nu\tau_3\psi\left\{\phi^*\left(\frac{\partial}{\partial x_\nu}-\frac{ie}{\hbar c}A_\nu\right)\phi - \phi\left(\frac{\partial}{\partial x_\nu}+\frac{ie}{\hbar c}A_\nu\right)\phi^*\right\} + \frac{ieg^2}{\hbar^2c^2\kappa^2}\phi^*\phi n_\rho n_\nu A_\rho\bar{\psi}\gamma_\nu\tau_3\psi. \tag{40}$$

This term gives no magnetic moment to order eg^2. The term does not occur for neutral mesons. The last term introduced by the vector coupling is zero for scalar mesons. For the pseudoscalar case it may be written

$$2g/\hbar c\kappa(f-2\kappa_0 g/\kappa)\{\langle\phi^*\phi\rangle_{\text{vac}}\bar{\psi}\psi+\phi^*\phi\langle\bar{\psi}\psi\rangle_{\text{vac}} + (\phi^*\phi-\langle\phi^*\phi\rangle_{\text{vac}})(\bar{\psi}\psi-\langle\bar{\psi}\psi\rangle_{\text{vac}})+\text{const.}\}. \tag{41}$$

In this form it is apparent that the one particle effects due to this term are expressible as additional mass terms. This shows that the statement that the pseudovector coupling is equivalent to pseudoscalar coupling is true for many of the simpler problems. It is not true for the special case of self-energy calculations. It is to be noted that to g^2 the pseudoscalar and pseudovector couplings are equivalent for the calculation of nuclear forces and that there are no direct δ function interactions to this order.

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