be due to fluctuations in the counter supply and the coincidence circuit, and were subsequently eliminated. The standard deviation given in Table I for antimony was calculated from the deviation of the individual sets from the mean.

The measured values for the total cross section are in very good agreement (approximately 1 percent) with the predictions of theory (Table I and Figs. 2 and 3). Any attempt to obtain a closer check at the present time would have little meaning inasmuch as the energies of the gamma-rays are known only to about 1 percent. The results of this investigation indicate, therefore, that the Klein-Nishina formula is in agreement with experiment in the energy range one to three Mev for aluminum.

TABLE I. Absorption coefficients for aluminum (cm^{-1}).

Energy Mev	Theoretical Compton	Photo-electric	Pair	Total	Measured value
1.11	0.1572	0.0001	——	0.157	0.156±0.001
1.708	0.1253	——	0.0009	0.126	0.125±0.002
2.76	0.0949	——	0.0042	0.0991	0.0996±0.0008

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Theory of Complex Spectra. IV

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The calculation of the coefficients of fractional parentage and of the energy matrices for the configurations f^n is simplified very much by the use of the theory of groups. Tables of results are given.

1. INTRODUCTION

It was shown in two previous papers[1] that the calculations on complex spectra may be simplified by the introduction of tensor operators and coefficients of fractional parentage. These coefficients may be calculated by Eqs. (9) of III and (11) of III, but it appears that for the configurations f^n Eqs. (11) of III are too cumbersome for practical use.

By considering the meaning and the properties of the coefficients of fractional parentage from the standpoint of the theory of groups, we shall see that these calculations may be somewhat simplified and that a very fortunate and important simplification takes place exactly for the configurations f^n.

In Section 4 we shall classify the states of f^n as the basis of some group representations and in Section 5 we shall find some properties of the coefficients of fractional parentage which will avoid the use of Eqs. (11) of III; the results of the calculations will be given in Tables III and IV. The energy matrices will be calculated in Section 6, and also these calculations will be simplified by group-theoretical considerations.

Before treating the very argument of this paper, we shall give in Section 2 a formula which should have its natural place in Section 5 of III, but was unfortunately obtained only after the publication of that paper, and we shall prove in Section 3 a corollary of Schur's lemma, which will be very useful in the following calculations.

2. THE MATRIX OF SYMMETRIC SCALAR OPERATORS

The matrix components between two states of l^n of the scalar operator (30) of III were calculated in (33a) of III by taking only the last term of the summation and then multiplying the result by $\frac{1}{2}n(n-1)$. It appears, on the contrary, more convenient to limit the sum of (30) of III to the first $n-1$ electrons and then to multiply by $n/(n-2)$. Thus, we obtain easily

$$\begin{aligned}(l^n\alpha SL|G|l^n\alpha' SL) &= [n/(n-2)]\sum_{\alpha_1\alpha_1' S_1L_1}(l^n\alpha SL\{|l^{n-1}(\alpha_1S_1L_1)lSL)\\ &\quad\times(l^{n-1}\alpha_1S_1L_1|G|l^{n-1}\alpha_1'S_1L_1)\\ &\quad\times(l^{n-1}(\alpha_1'S_1L_1)lSL|\}l^n\alpha' SL).\quad(1)\end{aligned}$$

[1] G. Racah, Phys. Rev. **62**, 438 (1942) and **63**, 367 (1943) (which will be referred to as II and III. We refer to these papers for definitions and notations.

This formula advantageously replaces (33a) of III, since it does not need the use of (32) of III. Moreover, if it is used for operators the eigenvalues of which are already known, it gives a great number of equations between the coefficients of fractional parentage. Operators of this type are not only the operator Q defined by (34) of III and the operator R which will be defined by (23), but, first of all, the operators

$$\sum_{i<j}(\mathbf{l}_i\cdot\mathbf{l}_j)=[L(L+1)-nl(l+1)]/2 \tag{2}$$

and

$$\sum_{i<j}(\mathbf{s}_i\cdot\mathbf{s}_j)=\tfrac{1}{2}S(S+1)-3n/8. \tag{3}$$

3. A LEMMA

The irreducible representations $(a|U_A(s)|a')$ of a group $\mathfrak{g}$ are generally reducible as representations of a subgroup $\mathfrak{h}$ of $\mathfrak{g}$; i.e., a constant matrix R_A exists, so that for every element t of $\mathfrak{h}$,

$$\sum_{aa'}(\beta Bb|R_A^{-1}|a)(a|U_A(t)|a')(a'|R_A|\beta'B'b')=(b|V_B(t)|b')\delta(BB')\delta(\beta\beta'), \tag{4}$$

where the $(b|V_B(t)|b')$ are the irreducible representations of $\mathfrak{h}$. Instead of the representations $U_A(s)$, we shall always consider the equivalent representations ("reduced with respect to $\mathfrak{h}$"),

$$(\beta Bb|W_A(s)|\beta'B'b')=(\beta Bb|R_A^{-1}U_A(s)R_A|\beta'B'b'); \tag{5}$$

it follows from (4) and (5) that for every element t of $\mathfrak{h}$,

$$(\beta Bb|W_A(t)|\beta'B'b')=(b|V_B(t)|b')\delta(BB')\delta(\beta\beta'). \tag{6}$$

The external (Kronecker's) product of two irreducible representations of $\mathfrak{g}$ is generally reducible, i.e., a constant matrix S exists, so that

$$\sum_{\beta_1B_1b_1\beta_2B_2b_2\beta_1'B_1'b_1'\beta_2'B_2'b_2'}(A_1A_2\alpha A\beta Bb|S^{-1}|A_1\beta_1B_1b_1;A_2\beta_2B_2b_2)(\beta_1B_1b_1|W_{A_1}(s)|\beta_1'B_1'b_1')$$
$$\times(\beta_2B_2b_2|W_{A_2}(s)|\beta_2'B_2'b_2')(A_1\beta_1'B_1'b_1';A_2\beta_2'B_2'b_2'|S|A_1A_2\alpha'A'\beta'B'b')=(\beta Bb|W_A(s)|\beta'B'b')\delta(AA')\delta(\alpha\alpha'). \tag{7}$$

Also the external product of two irreducible representations of $\mathfrak{h}$ is generally reducible and a constant matrix T exists, so that

$$\sum_{b_1b_2b_1'b_2'}(B_1B_2\gamma Bb|T^{-1}|B_1b_1B_2b_2)(b_1|V_{B_1}(t)|b_1')\times(b_2|V_{B_2}(t)|b_2')(B_1b_1'B_2b_2'|T|B_1B_2\gamma'B'b')=(b|V_B(t)|b')\delta(BB')\delta(\gamma\gamma'). \tag{8}$$

From (8), (6), and (7) we obtain

$$\sum_{b_1b_2b''}(b|V_B(t)|b'')(B_1B_2\gamma Bb''|T^{-1}|B_1b_1B_2b_2)\times(A_1\beta_1B_1b_1;A_2\beta_2B_2b_2|S|A_1A_2\alpha A\beta B'b')$$
$$=\sum_{b_1b_2b''}(B_1B_2\gamma Bb|T^{-1}|B_1b_1B_2b_2)\times(A_1\beta_1B_1b_1;A_2\beta_2B_2b_2|S|A_1A_2\alpha A\beta B'b'')\times(b''|V_{B'}(t)|b'), \tag{9}$$

and it follows from the lemma of Schur that the constant matrix $T^{-1}S$ is diagonal with respect to B and b and is independent of b:

$$\sum_{b_1b_2}(B_1B_2\gamma Bb|T^{-1}|B_1b_1B_2b_2)\times(A_1\beta_1B_1b_1;A_2\beta_2B_2b_2|S|A_1A_2\alpha A\beta B'b')=(A_1\beta_1B_1+A_2\beta_2B_2|X_\gamma|\alpha A\beta B)\delta(BB')\delta(bb'). \tag{10}$$

Multiplying from the left by T we have, finally,

$$(A_1\beta_1B_1b_1;A_2\beta_2B_2b_2|S|A_1A_2\alpha A\beta Bb)=\sum_\gamma(B_1b_1B_2b_2|T|B_1B_2\gamma Bb)\times(A_1\beta_1B_1+A_2\beta_2B_2|X_\gamma|\alpha A\beta B); \tag{11}$$

this formula will be very useful for practical calculations, since it expresses the dependence of the matrix S on b_1, b_2, and b by means of the simpler matrices T.

4. THE GROUP-THEORETICAL CLASSIFICATION OF THE TERMS OF l^n

1. The Spin and the Orbital Momentum

The configuration l^n has $\binom{4l+2}{n}$ independent states which may be characterized by a set of quantum numbers Γ; if the $4l+2$ eigenfunctions $\phi(m_sm_l)$ of the individual electrons undergo a linear unimodular transformation,

$$\phi'(m_s'm_l')=\sum_{m_sm_l}\phi(m_sm_l)c(m_sm_l;m_s'm_l'), \tag{12}$$

the eigenfunctions $\psi(l^n,\Gamma)$ undergo the linear transformation which is induced by (12) on the ansitymmetrical tensors of degree n in the $(4l+2)$-dimensional space; i.e., the $\Psi(l^n,\Gamma)$ are the basis of the antisymmetrical representation $\{\mathfrak{c}_{4l+2}\}^n$ of the linear unimodular group $\mathfrak{c}_{4l+2}$. The rows and columns of this representation are characterized by the quantum numbers Γ.

If we limit $\mathfrak{c}_{4l+2}$ to the subgroup $\mathfrak{c}_2\times\mathfrak{c}_{2l+1}$ defined by

$$c(m_sm_l;m_s'm_l')=\gamma(m_sm_s')c(m_lm_l'), \tag{13}$$

where γ and c are two independent linear unimodular transformations, the representation $\{\mathfrak{c}_{4l+2}\}^n$ breaks up into irreducible representations of $\mathfrak{c}_2\times\mathfrak{c}_{2l+1}$, each of which is the external product of a representation $\mathfrak{D}_S$ of $\mathfrak{c}_2$ and a representation $\mathfrak{H}_{n,S}$ of $\mathfrak{c}_{2l+1}$. It is well known that the symmetry schemes of $\mathfrak{D}_S$ and $\mathfrak{H}_{n,S}$ must be

TABLE I. Reduction of $\mathfrak{B}_W$ as representation of G_2.

Values of S and v I S	I v	II S	II v	W	U					
0	0	7/2	7	(000)	(00)					
1/2	1	3	6	(100)	(10)					
1	2	5/2	5	(110)	(10)	(11)				
0	2	5/2	7	(200)	(20)					
3/2	3	2	4	(111)	(00)	(10)	(20)			
1/2	3	2	6	(210)	(11)	(20)	(21)			
1	4	3/2	5	(211)	(10)	(11)	(20)	(21)	(30)	
0	4	3/2	7	(220)	(20)	(21)	(22)			
1/2	5	1	6	(221)	(10)	(11)	(20)	(21)	(30)	(31)
0	6	1/2	7	(222)	(00)	(10)	(20)	(30)	(40)	

dual; since the scheme of $\mathfrak{D}_S$ has two lines, the lengths of which are, respectively, $(n/2)+S$ and $(n/2)-S$, the scheme of $\mathfrak{H}_{n,S}$ will have two columns of these lengths. The basis of these representations of $\mathfrak{c}_2\times\mathfrak{c}_{2l+1}$ are the functions $\Psi(l^nSM_S\Delta)$; the quantum number M_S characterizes the rows and columns of $\mathfrak{D}_S$, the quantum numbers Δ those of $\mathfrak{H}_{n,S}$.

If we limit $\mathfrak{c}_{2l+1}$ to its subgroup composed by the elements of the representation $\mathfrak{D}_l$ of order $2l+1$ of the three-dimensional rotation group $\mathfrak{d}_3$, the representation $\mathfrak{H}_{n,S}$ breaks up into representations $\mathfrak{D}_L$ of $\mathfrak{d}_3$, the basis of which are the functions $\Psi(l^n\alpha SLM_SM_L)$. The quantum number (or set of quantum numbers) α must be introduced in order to distinguish the different equivalent representations of $\mathfrak{d}_3$ which may appear in the reduction of $\mathfrak{H}_{n,S}$, i.e., the different terms of the same kind which are allowed in l^n.

In order to classify in a suitable way these different terms, it is convenient to perform the passage from $\mathfrak{c}_{2l+1}$ to $\mathfrak{d}_3$ by successive steps.

2. The Seniority Number

As a first step, we limit $\mathfrak{c}_{2l+1}$ to the orthogonal subgroup $\mathfrak{d}_{2l+1}$ which leaves invariant the quadratic form

$$\sum_{-l}^{l}{}_m(-1)^m\phi(m)\phi(-m), \tag{14}$$

and the representations $\mathfrak{H}_{n,S}$ then break up into irreducible representations $\mathfrak{B}_W$ of $\mathfrak{d}_{2l+1}$; since the group $\mathfrak{d}_{2l+1}$ is of rank l, each $\mathfrak{B}_W$ is characterized by a set W of l integral numbers

$$w_1\geqslant w_2\geqslant\cdots\geqslant w_l\geqslant 0, \tag{15}$$

and since in the symmetry scheme of $\mathfrak{H}_{n,S}$ no row has a length greater than 2, also the w_i will not be greater than 2 and it will be

$$w_1=\cdots=w_a=2,\quad w_{a+1}=\cdots=w_{a+b}=1,\quad w_{a+b+1}=\cdots=w_l=0. \tag{16}$$

It is known from the theory of tensors that the passage from the linear to the metric space (or from $\mathfrak{c}_r$ to $\mathfrak{d}_r$) allows the decomposition of tensors by trace operation or contraction, i.e., some linear combinations

Table II. Reduction of $\mathfrak{G}_U$ as representation of $\mathfrak{d}_3$.

U	$12g(U)$	L
(00)	0	S
(10)	6	F
(11)	12	PH
(20)	14	DGI
(21)	21	$DFGHKL$
(30)	24	$PFGHIKM$
(22)	30	$SDGHILN$
(31)	32	$PDFFGHHIIKKLMNO$
(40)	36	$SDFGGHIIKLLMNQ$

of the components of tensors of degree n transform themselves as components of tensors of degree $n-2$; the classification of the terms of l^n according to the representations of $\mathfrak{d}_{2l+1}$ will therefore introduce a correspondence between some of them and the terms of l^{n-2}. It is easy to see that this correspondence is the same which was introduced in Section 6, Subsection 2 of III i.e., that the separation of the terms with $Q\neq 0$ from those with $Q=0$ is equivalent to the decomposition of a tensor by trace operation.

If we subtract (54), III, from (37), III, and add (52), III, we get

$$q_{ij}=-\tfrac{1}{2}-2(\mathbf{s}_i\cdot\mathbf{s}_j)-2\sum_{1}^{l}{}_t(4t-1)(\mathbf{u}_i^{(2t-1)}\cdot\mathbf{u}_j^{(2t-1)}); \tag{17}$$

owing to (38) of III and to (3) we have, for Q, the expression

$$Q=\tfrac{1}{4}n(4l+4-n)-S(S+1)-\sum_{1}^{l}{}_t(4t-1)\mathbf{U}^{(2t-1)^2}, \tag{18}$$

and it may be shown that

$$\sum_{1}^{l}{}_t(4t-1)\mathbf{U}^{(2t-1)^2}=(2l-1)G(\mathfrak{d}_{2l+1}), \tag{19}$$

where $G(\mathfrak{d}_{2l+1})$ is Casimir's[2] operator G for the group $\mathfrak{d}_{2l+1}$.

It may also be shown that the numbers a and b, which characterize the representations $\mathfrak{B}_W$ according to (16), are connected to the spin and the seniority number by the relations

$$a=(v/2)-S,\quad b=\min(2S,\,2l+1-v). \tag{20}$$

The basis of the representations $\mathfrak{B}_W$ are the functions $\Psi(l^n\alpha vSLM_SM_L)$.

The seniority number could also be introduced before the spin number, by limiting $\mathfrak{c}_{4l+2}$ to its symplectic subgroup which leaves invariant the bilinear antisymmetric form

$$\sum_{-\frac{1}{2}}^{\frac{1}{2}}{}_{m_s}\sum_{-l}^{l}{}_{m_l}(-1)^{m_s+m_l-\frac{1}{2}}\phi_1(m_sm_l)\phi_2(-m_s,\,-m_l). \tag{21}$$

[2] H. Casimir, Proc. Roy. Acad. Amsterdam 34, 844 (1931).

TABLE IIIa. $(WU|W'U'+f)$ for $W'=(000)$, (100), (110), (200), (111), (210).

		W'U'										
		(000)	(100)	(110)		(200)		(111)			(210)	
W	U	(00)	(10)	(10)	(11)	(20)	(00)	(10)	(20)	(11)	(20)	(21)
(000)	(00)	0	1	0	0	0						
(100)	(10)	1	0	$(1/3)^{1/2}$	$(2/3)^{1/2}$	1						
(110)	(10)	0	1	0	0	0	$(3/35)^{1/2}$	$(2/5)^{1/2}$	$(18/35)^{1/2}$	$(2/5)^{1/2}$	$(3/5)^{1/2}$	0
	(11)	0	1	0	0	0	0	$-(1/10)^{1/2}$	$-(9/10)^{1/2}$	0	$(3/35)^{1/2}$	$(32/35)^{1/2}$
(200)	(20)	0	1	0	0	0	0	0	0	$(2/15)^{1/2}$	$(9/35)^{1/2}$	$(64/105)^{1/2}$
(111)	(00)			1	0	0	0	-1	0	0	0	0
	(10)			$(2/3)^{1/2}$	$-(1/3)^{1/2}$	0	$-(1/7)^{1/2}$	$-(3/8)^{1/2}$	$(27/56)^{1/2}$	0	0	0
	(20)			$(2/9)^{1/2}$	$-(7/9)^{1/2}$	0	0	$(1/8)^{1/2}$	$-(7/8)^{1/2}$	0	0	0
(210)	(11)			1	0	1	0	0	0	0	0	0
	(20)			$(7/9)^{1/2}$	$(2/9)^{1/2}$	1	0	0	0	0	0	0
	(21)			0	1	1	0	0	0	0	0	0
(211)	(10)						$(27/35)^{1/2}$	$-(9/40)^{1/2}$	$(1/280)^{1/2}$	$-(3/5)^{1/2}$	$(2/5)^{1/2}$	0
	(11)						0	$(9/10)^{1/2}$	$-(1/10)^{1/2}$	0	$(32/35)^{1/2}$	$-(3/35)^{1/2}$
	(20)						0	$(7/8)^{1/2}$	$(1/8)^{1/2}$	$(1/3)^{1/2}$	$(2/7)^{1/2}$	$-(8/21)^{1/2}$
	(21)						0	0	1	$(3/16)^{1/2}$	$-(25/112)^{1/2}$	$(33/56)^{1/2}$
	(30)						0	0	1	0	$(1/7)^{1/2}$	$(6/7)^{1/2}$
(220)	(20)									$(8/15)^{1/2}$	$-(16/35)^{1/2}$	$(1/105)^{1/2}$
	(21)									$-(1/8)^{1/2}$	$(27/56)^{1/2}$	$(11/28)^{1/2}$
	(22)									0	0	1

3. The Special Case of f^n

It was remarked in Section 6 of III that the seniority number suffices to distinguish the different terms of the same kind in d^n, but not for greater l; for $l\geqslant 3$ we must therefore seek for a subgroup of $\mathfrak{d}_{2l+1}$ which contains $\mathfrak{D}_l$, and it is a very fortunate chance that such a subgroup exists exactly for $l=3$: it is the subgroup of $\mathfrak{d}_7$ which leaves invariant the trilinear antisymmetric form

$$\sum_{mm'm''} V(333;mm'm'')\phi_1(m)\phi_2(m')\phi_3(m''), \tag{22}$$

where $V(abc;\alpha\beta\gamma)$ is defined by (17′) of II. This group is the first of the five simple groups which exist besides the four great classes of simple groups, and is usually denoted as G_2.

If we limit $\mathfrak{d}_7$ to its subgroup G_2, the representations $\mathfrak{B}_W$ break up into irreducible representations $\mathfrak{C}_U$ of G_2; since G_2 is of rank 2, the $\mathfrak{C}_U$ are characterized by a set $U\equiv(u_1u_2)$ of two integral numbers. If we limit G_2 to its subgroup composed by the elements of the representation $\mathfrak{D}_3$ of $\mathfrak{d}_3$, the $\mathfrak{C}_U$ also break up into representations $\mathfrak{D}_L$ of $\mathfrak{d}_3$, the basis of which are the functions $\Psi(f^n\alpha UvSLM_SM_L)$, and these functions will form our definitive system of eigenfunctions of f^n.

The law of reduction of $\mathfrak{B}_W$ as representation of G_2 is given in Table I, that of $\mathfrak{C}_U$ as representation of $\mathfrak{d}_3$ is given in Table II; we see from this last table that the quantum number α must be maintained only for $U\equiv(31)$ and $U\equiv(40)$.

Also the quantum numbers U could be introduced in a similar way as the seniority number in Section 6

TABLE IIIb. $(WU|(211)U'+f)$.

			U'				
W	U	N	(10)	(11)	(20)	(21)	(30)
(111)	(00)	1	1	0	0	0	0
	(10)	$24^{-1/2}$	-1	$8^{1/2}$	$15^{1/2}$	0	0
	(20)	$5832^{-1/2}$	1	$-56^{1/2}$	$135^{1/2}$	$2560^{1/2}$	$3080^{1/2}$
(210)	(11)	$42^{-1/2}$	$-7^{1/2}$	0	$15^{1/2}$	$20^{1/2}$	0
	(20)	$1701^{-1/2}$	$98^{1/2}$	$448^{1/2}$	$270^{1/2}$	$-500^{1/2}$	$385^{1/2}$
	(21)	$672^{-1/2}$	0	$-7^{1/2}$	$-60^{1/2}$	$220^{1/2}$	$385^{1/2}$
(211)	(10)	$72^{-1/2}$	$-5^{1/2}$	$40^{1/2}$	$-27^{1/2}$	0	0
	(11)	$126^{-1/2}$	$35^{1/2}$	0	$-27^{1/2}$	$64^{1/2}$	0
	(20)	$2520^{-1/2}$	$-245^{1/2}$	$-280^{1/2}$	$-867^{1/2}$	$-512^{1/2}$	$616^{1/2}$
	(21)	$315^{-1/2}$	0	$35^{1/2}$	$-27^{1/2}$	$-176^{1/2}$	$77^{1/2}$
	(30)	$315^{-1/2}$	0	0	$27^{1/2}$	$64^{1/2}$	$-224^{1/2}$

of III, by classifying the terms of f^n according to the eigenvalues of

$$R=\sum_{i<j} r_{ij}, \tag{23}$$

where the scalar operator r_{ij} is defined by the relation

$$(f^2LM|r_{ij}|f^2LM)=6\delta(L,3); \tag{24}$$

the equations which correspond to (17), (18), and (19) are

$$r_{ij}=\tfrac{1}{2}-2(\mathbf{s}_i\cdot\mathbf{s}_j)-2q_{ij}-18(\mathbf{u}_i^{(1)}\cdot\mathbf{u}_j^{(1)}) -66(\mathbf{u}_i^{(5)}\cdot\mathbf{u}_j^{(5)}), \tag{25}$$

$$R=\tfrac{1}{4}n(n+26)-S(S+1)-2Q-9\mathbf{U}^{(1)2}-33\mathbf{U}^{(5)2}, \tag{26}$$

and

$$9\mathbf{U}^{(1)2}+33\mathbf{U}^{(5)2}=12G(G_2); \tag{27}$$

TABLE IVa. $(UL|U'L'+f)$ for $U'=(00)$, (10), (11), (20).

		$U'L'$						
		(00)	(10)	(11)		(20)		
U	L	S	F	P	H	D	G	I
(00)	S	0	-1	0	0	0	0	0
(10)	F	1	1	$(3/14)^{1/2}$	$(11/14)^{1/2}$	$-(5/27)^{1/2}$	$-(1/3)^{1/2}$	$-(13/27)^{1/2}$
(11)	P	0	1	0	0	$(10/21)^{1/2}$	$-(11/21)^{1/2}$	0
	H	0	1	0	0	$(20/189)^{1/2}$	$(65/231)^{1/2}$	$-(182/297)^{1/2}$
(20)	D	0	1	$-(27/49)^{1/2}$	$-(22/49)^{1/2}$	$-4/7$	$(33/49)^{1/2}$	0
	G	0	1	$(33/98)^{1/2}$	$-(65/98)^{1/2}$	$(55/147)^{1/2}$	$-(125/539)^{1/2}$	$(13/33)^{1/2}$
	I	0	1	0	1	0	$(3/11)^{1/2}$	$(8/11)^{1/2}$
(21)	D			$-(22/49)^{1/2}$	$(27/49)^{1/2}$	$(33/49)^{1/2}$	$4/7$	0
	F			$-(11/14)^{1/2}$	$(3/14)^{1/2}$	$-(55/126)^{1/2}$	$-(8/77)^{1/2}$	$(91/198)^{1/2}$
	G			$(65/98)^{1/2}$	$(33/98)^{1/2}$	$(13/882)^{1/2}$	$(104/147)^{1/2}$	$(5/18)^{1/2}$
	H			0	1	$(13/27)^{1/2}$	$-(16/33)^{1/2}$	$-(10/297)^{1/2}$
	K			0	1	0	$(16/33)^{1/2}$	$-(17/33)^{1/2}$
	L			0	1	0	0	1
(30)	P					$(11/21)^{1/2}$	$(10/21)^{1/2}$	0
	F					$(143/378)^{1/2}$	$-(130/231)^{1/2}$	$(35/594)^{1/2}$
	G					$(11/18)^{1/2}$	$(2/33)^{1/2}$	$-(65/198)^{1/2}$
	H					$(26/63)^{1/2}$	$(18/77)^{1/2}$	$(35/99)^{1/2}$
	I					0	$(8/11)^{1/2}$	$-(3/11)^{1/2}$
	K					0	$(17/33)^{1/2}$	$(16/33)^{1/2}$
	M					0	0	1

TABLE IVb. $(UL|(21)L'+f)$.

			L'					
U	L	N	D	F	G	H	K	L
(11)	P	$1344^{-1/2}$	$220^{1/2}$	$-539^{1/2}$	$-585^{1/2}$	0	0	0
	H	$4928^{-1/2}$	$-270^{1/2}$	$147^{1/2}$	$-297^{1/2}$	$1078^{1/2}$	$1470^{1/2}$	$-1666^{1/2}$
(20)	D	$31360^{-1/2}$	$8910^{1/2}$	$8085^{1/2}$	$351^{1/2}$	$-14014^{1/2}$	0	0
	G	$4312^{-1/2}$	$330^{1/2}$	$147^{1/2}$	$1287^{1/2}$	$1078^{1/2}$	$-1470^{1/2}$	0
	I	$18304^{-1/2}$	0	$-1911^{1/2}$	$1485^{1/2}$	$220^{1/2}$	$4590^{1/2}$	$10098^{1/2}$
(21)	D	$5390^{-1/2}$	$375^{1/2}$	$1960^{1/2}$	$-1144^{1/2}$	$1911^{1/2}$	0	0
	F	$154^{-1/2}$	$-40^{1/2}$	7	$-65^{1/2}$	0	0	0
	G	$630630^{-1/2}$	$-74360^{1/2}$	455	$226941^{1/2}$	$-51744^{1/2}$	$70560^{1/2}$	0
	H	$15730^{-1/2}$	$-2535^{1/2}$	0	$1056^{1/2}$	$-3179^{1/2}$	$7260^{1/2}$	$1700^{1/2}$
	K	$2860^{-1/2}$	0	0	$-192^{1/2}$	$968^{1/2}$	$-85^{1/2}$	$1615^{1/2}$
	L	$572^{-1/2}$	0	0	0	$-40^{1/2}$	$-285^{1/2}$	$247^{1/2}$
(30)	P	$2688^{-1/2}$	34	$-245^{1/2}$	$1287^{1/2}$	0	0	0
	F	$112^{-1/2}$	$-39^{1/2}$	0	$24^{1/2}$	7	0	0
	G	$7920^{-1/2}$	$1375^{1/2}$	$2450^{1/2}$	$-858^{1/2}$	$1617^{1/2}$	$1620^{1/2}$	0
	H	$640640^{-1/2}$	$-42250^{1/2}$	-455	$3971^{1/2}$	$261954^{1/2}$	$490^{1/2}$	$124950^{1/2}$
	I	$9152^{-1/2}$	0	$1274^{1/2}$	$2750^{1/2}$	$1320^{1/2}$	$2125^{1/2}$	$-1683^{1/2}$
	K	$1040^{-1/2}$	0	0	$-204^{1/2}$	$-136^{1/2}$	$605^{1/2}$	$95^{1/2}$
	M	$64^{-1/2}$	0	0	0	0	$-15^{1/2}$	-7
(22)	S	1	0	1	0	0	0	0
	D	$640^{-1/2}$	$-130^{1/2}$	$195^{1/2}$	$297^{1/2}$	$18^{1/2}$	0	0
	G	$51480^{-1/2}$	$18590^{1/2}$	65	$-429^{1/2}$	$-23826^{1/2}$	$-4410^{1/2}$	0
	H	$6160^{-1/2}$	$980^{1/2}$	$-2450^{1/2}$	$1078^{1/2}$	$-132^{1/2}$	$245^{1/2}$	$-1275^{1/2}$
	I	$18304^{-1/2}$	0	$1105^{1/2}$	$9163^{1/2}$	$-2244^{1/2}$	$-4802^{1/2}$	$990^{1/2}$
	L	$364^{-1/2}$	0	0	0	$152^{1/2}$	$-147^{1/2}$	$-65^{1/2}$
	N	$16^{-1/2}$	0	0	0	0	$-5^{1/2}$	$11^{1/2}$

it may also be shown[3] that the eigenvalues of $G(G_2)$ are

$$g(U)=g(u_1u_2)=(u_1^2+u_1u_2+u_2^2+5u_1+4u_2)/12. \quad (28)$$

Although this method of introducing the quantum numbers U avoids the explicit use of the theory of groups, the group-theoretical definition appeared this time more convenient, since the properties of the coefficients of fractional parentage and of the energy matrices, which are connected with this classification and will be obtained in the next sections, could be demonstrated only with the use of the theory of groups.

[3] The general expression of the eigenvalues of Casimir's operator G for every semisimple group will be published elsewhere.

TABLE IVc. $(UL|(30)L'+f)$.

U	L	N	L': P	F	G	H	I	K	M
	D	$490^{-\frac{1}{2}}$	$-54^{\frac{1}{2}}$	$-91^{\frac{1}{2}}$	$189^{\frac{1}{2}}$	$-156^{\frac{1}{2}}$	0	0	0
(20)	G	$5929^{-\frac{1}{2}}$	$-330^{\frac{1}{2}}$	$910^{\frac{1}{2}}$	$126^{\frac{1}{2}}$	$-594^{\frac{1}{2}}$	$2184^{\frac{1}{2}}$	$-1785^{\frac{1}{2}}$	0
	I	$22022^{-\frac{1}{2}}$	0	$-245^{\frac{1}{2}}$	$-1755^{\frac{1}{2}}$	$-2310^{\frac{1}{2}}$	$-2106^{\frac{1}{2}}$	$-4320^{\frac{1}{2}}$	$-11286^{\frac{1}{2}}$
	D	$2695^{-\frac{1}{2}}$	$-578^{\frac{1}{2}}$	$1092^{\frac{1}{2}}$	$700^{\frac{1}{2}}$	$325^{\frac{1}{2}}$	0	0	0
	F	$1694^{-\frac{1}{2}}$	$-55^{\frac{1}{2}}$	0	$-560^{\frac{1}{2}}$	$-715^{\frac{1}{2}}$	$-364^{\frac{1}{2}}$	0	0
	G	$630630^{-\frac{1}{2}}$	$-83655^{\frac{1}{2}}$	$-87360^{\frac{1}{2}}$	$-56784^{\frac{1}{2}}$	$-3971^{\frac{1}{2}}$	$227500^{\frac{1}{2}}$	$171360^{\frac{1}{2}}$	0
(21)	H	$55055^{-\frac{1}{2}}$	0	$12740^{\frac{1}{2}}$	$-7644^{\frac{1}{2}}$	$18711^{\frac{1}{2}}$	$-7800^{\frac{1}{2}}$	$-8160^{\frac{1}{2}}$	0
	K	$165165^{-\frac{1}{2}}$	0	0	$-16848^{\frac{1}{2}}$	$77^{\frac{1}{2}}$	$-27625^{\frac{1}{2}}$	$79860^{\frac{1}{2}}$	$-40755^{\frac{1}{2}}$
	L	$17017^{-\frac{1}{2}}$	0	0	0	$-1785^{\frac{1}{2}}$	$-1989^{\frac{1}{2}}$	$-1140^{\frac{1}{2}}$	$12103^{\frac{1}{2}}$
	P	$16^{-\frac{1}{2}}$	0	$13^{\frac{1}{2}}$	$3^{\frac{1}{2}}$	0	0	0	0
	F	$1232^{-\frac{1}{2}}$	$429^{\frac{1}{2}}$	$-77^{\frac{1}{2}}$	$273^{\frac{1}{2}}$	$-33^{\frac{1}{2}}$	$-420^{\frac{1}{2}}$	0	0
	G	$55440^{-\frac{1}{2}}$	$-3465^{\frac{1}{2}}$	$-9555^{\frac{1}{2}}$	$7203^{\frac{1}{2}}$	$27797^{\frac{1}{2}}$	$1300^{\frac{1}{2}}$	$-6120^{\frac{1}{2}}$	0
(30)	H	$11440^{-\frac{1}{2}}$	0	$-195^{\frac{1}{2}}$	$-4693^{\frac{1}{2}}$	$1232^{\frac{1}{2}}$	$2600^{\frac{1}{2}}$	$2720^{\frac{1}{2}}$	0
	I	$32032^{-\frac{1}{2}}$	0	$5880^{\frac{1}{2}}$	$520^{\frac{1}{2}}$	$-6160^{\frac{1}{2}}$	$-1911^{\frac{1}{2}}$	$15680^{\frac{1}{2}}$	$-1881^{\frac{1}{2}}$
	K	$510510^{-\frac{1}{2}}$	0	0	$33813^{\frac{1}{2}}$	$89012^{\frac{1}{2}}$	$-216580^{\frac{1}{2}}$	$8085^{\frac{1}{2}}$	$163020^{\frac{1}{2}}$
	M	$3808^{-\frac{1}{2}}$	0	0	0	0	$153^{\frac{1}{2}}$	$960^{\frac{1}{2}}$	$2695^{\frac{1}{2}}$

5. THE CALCULATION OF THE FRACTIONAL PARENTAGES OF f^n

1. General Properties

The eigenfunctions of l^n, which are the basis of $\{\mathfrak{c}_{4l+2}\}^n$, may be obtained by reduction of the representation $\{\mathfrak{c}_{4l+2}\}^{n-1}\times(\mathfrak{c}_{4l+2})$:

$$\begin{aligned}\Psi(l^n\alpha vSLM_SM_L) &= \sum \Psi(l^{n-1}\alpha'v'S'L'M_S'M_L')\phi(m_sm_l)\\ &\quad\times(l^{n-1}\alpha'v'S'L'M_S'M_L', lm_sm_l|l^n\alpha vSLM_SM_L);\end{aligned} \tag{29}$$

owing to the particular choice of the scheme which was made in the preceding section, it follows from the lemma (11) that the coefficients of this transformation break up in a product of different factors, each of which depends only from a smaller number of variables:

$$\begin{aligned}&(l^{n-1}\alpha'v'S'L'M_S'M_L', lm_sm_l|l^n\alpha vSLM_SM_L)\\ &=(S'\tfrac{1}{2}M_S'm_s|S'\tfrac{1}{2}SM_S)(L'lM_L'm_l|L'lLM_L)\\ &\quad\times(W'\alpha'L'+l|W\alpha L)(l^{n-1}v'S'+l|\}l^nvS).\end{aligned} \tag{30}$$

Confronting this expression with (10) of III, we see that the coefficients of fractional parentage are the product of two factors:

$$\begin{aligned}&(l^{n-1}(\alpha'v'S'L')lSL|\}l^n\alpha vSL)\\ &=(W'\alpha'L'+l|W\alpha L)(l^{n-1}v'S'+l|\}l^nvS);\end{aligned} \tag{31}$$

the relations (58) of III are particular cases of this result. Owing to the unitary of all our transformations, the factors of (31) satisfy the orthogonality relations

$$\sum_{v'S'}(l^nvS\{|l^{n-1}v'S'+l)(l^{n-1}v'S'+l|\}l^nvS)=1 \tag{32}$$

and

$$\sum_{\alpha'L'}(W\alpha L|W'\alpha'L'+l)(W'\alpha'L'+l|W''\alpha''L)=\delta(WW'')\delta(\alpha\alpha''). \tag{33}$$

For the particular case $l=3$, owing to the existence of the intermediate group G_2, the coefficients of fractional parentage are the product of three factors:

$$\begin{aligned}&(f^{n-1}(\alpha'U'v'S'L')fSL|\}f^n\alpha UvSL)\\ &=(U'\alpha'L'+f|U\alpha L)(W'U'+f|WU)\\ &\quad\times(f^{n-1}v'S'+f|\}f^nvS),\end{aligned} \tag{34}$$

and the orthogonality relations (33) break up into

$$\sum_{U'}(WU|W'U'+f)(W'U'+f|W''U)=\delta(WW'') \tag{35}$$

and

$$\sum_{\alpha'L'}(U\alpha L|U'\alpha'L'+f)(U'\alpha'L'+f|U''\alpha''L)=\delta(UU'')\delta(\alpha\alpha''). \tag{36}$$

In order to find also relations of the type (61) of III, we consider now the identical representations of $\mathfrak{d}_{2l+1}$ which appears in the reduction of $\mathfrak{B}_{W_1}\times\mathfrak{B}_{W_2}$. Such representations may only appear if $W_1=W_2$, and since the tensors of odd degree are diagonal with respect to W (see (70) of III), we obtain in the same way as in Section 6 of II that

$$\begin{aligned}&(W_1\alpha_1L_1+W_2\alpha_2L_2|(00\cdots0)0)\\ &=[(2L_1+1)/g_{W_1}]^{\frac{1}{2}}\delta(W_1W_2)\delta(\alpha_1\alpha_2)\delta(L_1L_2),\end{aligned} \tag{37}$$

where g_{W_1} is the order of the representation $\mathfrak{B}_{W_1}$. Owing to the value (16') of II of $(LLMM'|LL00)$, we get also

$$\begin{aligned}&(W_1\alpha_1L_1M_1, W_2\alpha_2L_2M_2|W_1W_2, (00\cdots0)00)\\ &=g_{W_1}^{-\frac{1}{2}}(-1)^{L_1-M_1}\delta(W_1W_2)\delta(\alpha_1\alpha_2)\\ &\quad\times\delta(L_1L_2)\delta(M_1,-M_2).\end{aligned} \tag{38}$$

TABLE V. $c(UU'(40))$.

	(00)	(10)	(11)	(20)	(21)	(30)	(22)	(31)	(40)
(00)	0	0	0	0	0	0	0	0	1
(10)	0	0	0	0	0	1	0	1	1
(11)	0	0	0	0	1	1	0	1	1
(20)	0	0	0	1	1	1	1	2	2
(21)	0	0	1	1	2	2	2	3	2
(30)	0	1	1	1	2	2	1	3	2
(22)	0	0	0	1	2	1	1	2	2
(31)	0	1	1	2	3	3	2	5	3
(40)	1	1	1	2	2	2	2	3	3

If, in the general formula,[4]

$$\begin{aligned}\int &(W_1\alpha_1L_1M_1|R|W_1\alpha_1'L_1'M_1')\\ &\times(W_2\alpha_2L_2M_2|R|W_2\alpha_2'L_2'M_2')\\ &\times(W\alpha LM|R|W\alpha'L'M')^*dR\\ &=g_W^{-1}(W_1\alpha_1L_1M_1,\, W_2\alpha_2L_2M_2|W_1W_2,\, W\alpha LM)\\ &\quad\times(W_1W_2,\, W\alpha'L'M'|W_1\alpha_1'L_1'M_1',\, W_2\alpha_2'L_2'M_2')\\ &\quad\times\int dR, \qquad (39)\end{aligned}$$

we consider the special case $W\equiv(00\cdots0)$ and introduce (38), we get

$$\begin{aligned}\int &(W_1\alpha_1L_1M_1|R|W_1\alpha_1'L_1'M_1')\\ &\times(W_2\alpha_2L_2M_2|R|W_2\alpha_2'L_2'M_2')dR\\ &=g_W^{-1}(-1)^{L_1+L_1'-M_1-M_1'}\delta(W_1W_2)\\ &\quad\times\delta(\alpha_1\alpha_2)\delta(L_1L_2)\delta(M_1,-M_2)\\ &\quad\times\delta(\alpha_1'\alpha_2')\delta(L_1'L_2')\delta(M_1',-M_2')\int dR, \qquad (40)\end{aligned}$$

and confronting this result with the orthogonality relation[5]

$$\begin{aligned}\int &(W_1\alpha_1L_1M_1|R|W_1\alpha_1'L_1'M_1')^*\\ &\times(W_2\alpha_2L_2M_2|R|W_2\alpha_2'L_2'M_2')dR\\ &=g_{W_1}^{-1}\delta(W_1W_2)\delta(\alpha_1\alpha_2)\delta(L_1L_2)\delta(M_1M_2)\\ &\quad\times\delta(\alpha_1'\alpha_2')\delta(L_1'L_2')\delta(M_1'M_2')\int dR, \qquad (41)\end{aligned}$$

we obtain from the well-known corollary of Schwarz's inequality that

$$\begin{aligned}&(W\alpha LM|R|W\alpha'L'M')^*\\ &\quad=(-1)^{L+L'-M-M'}(W\alpha L-M|R|W\alpha'L'-M'). \qquad (42)\end{aligned}$$

Applying this result to the first and third factor in the left side of (39), we have

$$\begin{aligned}&(W_1\alpha_1L_1M_1,\, W_2\alpha_2L_2M_2|W_1W_2,\, W\alpha LM(W_1W_2,\, W\alpha'L'M'|W_1\alpha_1'L_1'M_1',\, W_2\alpha_2'L_2'M_2')\\ &\quad=(-1)^{L+L'-M-M'-L_1-L_1'+M_1+M_1'}(g_W/g_{W_1})(W\alpha L-M,\, W_2\alpha_2L_2M_2|WW_2,\, W_1\alpha_1L_1-M_1)\\ &\qquad\times(WW_2,\, W_1\alpha_1'L_1'-M_1'|W\alpha'L'-M',\, W_2\alpha_2'L_2'M_2'), \qquad (43)\end{aligned}$$

and since, owing to (16′) of II and (19a) of II,

$$\begin{aligned}&(L_1L_2M_1M_2|L_1L_2LM)\\ &\quad=(-1)^{L_2+M-M_1}[(2L+1)/(2L_1+1)]^{\frac{1}{2}}\\ &\qquad\times(LL_2-MM_2|LL_2L_1-M_1), \qquad (44)\end{aligned}$$

we have also

$$\begin{aligned}&(W_1\alpha_1L_1+W_2\alpha_2L_2|W\alpha L)(W\alpha'L'|W_1\alpha_1'L_1'+W_2\alpha_2'L_2')\\ &=(-1)^{L+L_2-L_1+L'+L_2'-L_1'}\\ &\quad[(2L_1+1)(2L_1'+1)/(2L+1)(2L'+1)]^{\frac{1}{2}}(g_W/g_{W_1})\\ &\quad\times(W\alpha L+W_2\alpha_2L_2|W_1\alpha_1L_1)\\ &\qquad\times(W_1\alpha_1'L_1'|W\alpha'L'+W_2\alpha_2'L_2'). \qquad (45)\end{aligned}$$

This equation may be satisfied only if

$$\begin{aligned}&(W\alpha L+W_2\alpha_2L_2|W_1\alpha_1L_1)\\ &\quad=(-1)^{L_1-L_2-L+x}[(2L+1)g_{W_1}/(2L_1+1)g_W]^{\frac{1}{2}}\\ &\qquad\times(W_1\alpha_1L_1+W_2\alpha_2L_2|W\alpha L), \qquad (46)\end{aligned}$$

where x is independent of the L and depends only on the W. The value of x is to a some extent arbitrary, since it depends from our choice of phases. For the particular case $W_2\equiv(10\cdots0)\equiv l$, which is important for us, we put $x=l=L_2$, and therefore,

$$\begin{aligned}&(W\alpha L+l|W'\alpha'L')=(-1)^{L-L'}\\ &\quad\times[(2L+1)g_{W'}/(2L'+1)g_W]^{\frac{1}{2}}(W'\alpha'L'+l|W\alpha L); \qquad (47)\end{aligned}$$

the relation (61) of III is a particular case of this result.

It is easy to see that for $l=3$ the relation (47) breaks up into

$$(WU+f|W'U')=(g_Ug_{W'}/g_{U'}g_W)^{\frac{1}{2}}(W'U'+f|WU) \qquad (48)$$

and

$$\begin{aligned}&(U\alpha L+f|U'\alpha'L')=(-1)^{L-L'}\\ &\quad\times[(2L+1)g_{U'}/(2L'+1)g_U]^{\frac{1}{2}}(U'\alpha'L'+f|U\alpha L). \qquad (49)\end{aligned}$$

2. The Calculation of $(l^{n-1}v'S'+l|\}l^nvS)$

Applying (1) to (3) and owing to (31) and (33), we have

[4] See E. Wigner, *Gruppentheorie* (Friedrich Vieweg and Sohn, Braunschweig, 1931), p. 204, Eq. (22).

[5] Reference 4, p. 110, Eq. (11).

TABLE VIa. $(U|\chi(L)|U')$ for $U, U'\neq(31), (40)$.

$(U\|\chi\|U')$	S	P	D	F	G	H	I	K	L	M	N
$(20\|\chi\|20)$	0	0	143	0	-130	0	35	0	0	0	0
$(11\|\chi\|21)$	0	0	0	0	0	1	0	0	0	0	0
$(20\|\chi\|21)$	0	0	$-39\sqrt{2}$	0	$4(65)^{1/2}$	0	0	0	0	0	0
$(21\|\chi_1\|21)$	0	0	377	455	-561	49	0	-315	245	0	0
$(21\|\chi_2\|21)$	0	0	13	-65	55	-75	0	133	-75	0	0
$(10\|\chi\|30)$	0	0	0	1	0	0	0	0	0	0	0
$(11\|\chi\|30)$	0	$-13(11)^{1/2}$	0	0	0	$(39)^{1/2}$	0	0	0	0	0
$(20\|\chi\|30)$	0	0	0	0	$-13(5)^{1/2}$	0	30	0	0	0	0
$(21\|\chi\|30)$	0	0	0	$12(195)^{1/2}$	$8(143)^{1/2}$	$11(42)^{1/2}$	0	$-4(17)^{1/2}$	0	0	0
$(30\|\chi\|30)$	0	-52	0	38	-52	88	25	-94	0	25	0
$(20\|\chi\|22)$	0	0	$3(429)^{1/2}$	0	$-38(65)^{1/2}$	0	$21(85)^{1/2}$	0	0	0	0
$(21\|\chi\|22)$	0	0	$45(78)^{1/2}$	0	$12(11)^{1/2}$	$-12(546)^{1/2}$	0	0	$-8(665)^{1/2}$	0	0
$(22\|\chi\|22)$	260	0	-25	0	94	104	-181	0	-36	0	40

TABLE VIb. $(U|\chi(L)|31)$.

L	$(10\|\chi\|31)$	$(11\|\chi\|31)$	$(20\|\chi\|31)$	$(21\|\chi\|31)$	$(30\|\chi\|31)$	$(31\|\chi\|31)$
P	0	$11(330)^{1/2}$	0	0	$76(143)^{1/2}$	-6644
D	0	0	$-8(78)^{1/2}$	$-60(39/7)^{1/2}$	0	4792
F	0	0	0	$-312(5)^{1/2}$	$-48(39)^{1/2}$	$\begin{Vmatrix} 4420 & 336(143)^{1/2} \\ 336(143)^{1/2} & -902 \end{Vmatrix}$
F'	1	0	0	$12(715)^{1/2}$	$-98(33)^{1/2}$	
G	0	0	$5(65)^{1/2}$	$2024/(7)^{1/2}$	$20(1001)^{1/2}$	-2684
H	0	$11(85)^{1/2}$	0	$31(1309/3)^{1/2}$	$-20(374)^{1/2}$	$\begin{Vmatrix} -2024 & -48(6545)^{1/2} \\ -48(6545)^{1/2} & 2680 \end{Vmatrix}$
H'	0	$-25(77)^{1/2}$	0	$103(5/3)^{1/2}$	$-44(70)^{1/2}$	
I	0	0	$10(21)^{1/2}$	0	$-57(33)^{1/2}$	$\begin{Vmatrix} -12661/5 & -3366(34)^{1/2}/5 \\ -3366(34)^{1/2}/5 & 17336/5 \end{Vmatrix}$
I'	0	0	0	0	$18(1122)^{1/2}$	
K	0	0	0	$-52(323/23)^{1/2}$	$-494(19/23)^{1/2}$	$\begin{Vmatrix} 123506/23 & 144(21318)^{1/2}/23 \\ 144(21318)^{1/2}/23 & -85096/23 \end{Vmatrix}$
K'	0	0	0	$-336(66/23)^{1/2}$	$73(1122/23)^{1/2}$	
L	0	0	0	$-24(190)^{1/2}$	0	-4712
M	0	0	0	0	$-21(385)^{1/2}$	-473
N	0	0	0	0	0	1672
O	0	0	0	0	0	220

TABLE VIc. $(U|\chi(L)|40)$.

L	$(00\|\chi\|40)$	$(10\|\chi\|40)$	$(20\|\chi\|40)$	$(30\|\chi\|40)$	$(40\|\chi\|40)$
S	1	0	0	0	-1408
D	0	0	$-88(13)^{1/2}$	0	-44
F	0	1	0	$90(11)^{1/2}$	1078
G	0	0	$53(715/27)^{1/2}$	$-16(1001)^{1/2}$	$\begin{Vmatrix} -16720/9 & -34(2618)^{1/2}/9 \\ -34(2618)^{1/2}/9 & 10942/9 \end{Vmatrix}$
G'	0	0	$7(15470/27)^{1/2}$	$64(442)^{1/2}$	
H	0	0	0	$-72(462)^{1/2}$	-704
I	0	0	$34(1045/31)^{1/2}$	$-9(21945/31)^{1/2}$	$\begin{Vmatrix} -2453/31 & 60(74613)^{1/2}/31 \\ 60(74613)^{1/2}/31 & 36088/31 \end{Vmatrix}$
I'	0	0	$-12(1785/31)^{1/2}$	$756(85/31)^{1/2}$	
K	0	0	0	$-84(33)^{1/2}$	-132
L	0	0	0	0	$\begin{Vmatrix} -4268/31 & 924(1995)^{1/2}/31 \\ 924(1995)^{1/2}/31 & 11770/31 \end{Vmatrix}$
L'	0	0	0	0	
M	0	0	0	$-99(15)^{1/2}$	-1067
N	0	0	0	0	528
Q	0	0	0	0	22

$$S(S+1)-3n/4 = [n/(n-2)]\sum_{v'S'}[S'(S'+1)-3(n-1)/4] \times(l^{n-1}v'S'+l|\}l^nvS)^2; \quad (50)$$

since S' may have only the two values $S-\frac{1}{2}$ and $S+\frac{1}{2}$, we obtain from (32) and (50) that

$$\begin{aligned} \textstyle\sum_{v'}(l^{n-1}v'S-\tfrac{1}{2}\,+l|\}l^nvS)^2 &= (n+2S+2)S/n(2S+1), \\ \textstyle\sum_{v'}(l^{n-1}v'S+\tfrac{1}{2}\,+l|\}l^nvS)^2 &= (n-2S)(S+1)/n(2S+1). \end{aligned} \quad (51)$$

Since, for $n=v$, v' may have only the value $v-1$, we get

$$(l^{v-1}v-1\;S-\tfrac{1}{2}\,+l|\}l^vvS)^2=(v+2S+2)S/v(2S+1),$$
$$(l^{v-1}v-1\;S+\tfrac{1}{2}\,+l|\}l^vvS)^2=(v-2S)(S+1)/v(2S+1),$$

and owing to (58) of III,

$$\begin{aligned} (l^{n-1}v-1\;S-\tfrac{1}{2}\,+l|\}l^nvS)^2 &= (4l+4-n-v)(v+2S+2)S/ \\ &\qquad 2n(2l+2-v)(2S+1), \\ (l^{n-1}v-1\;S+\tfrac{1}{2}\,+l|\}l^nvS)^2 &= (4l+4-n-v)(v-2S)(S+1)/ \\ &\qquad 2n(2l+2-v)(2S+1); \end{aligned} \quad (52a)$$

TABLE VII. $x((210), UU')$.

	(11)	(20)	(21)
(11)	0	0	$12(455)^{\frac{1}{2}}$
(20)	0	$-6/7$	$6(66)^{\frac{1}{2}}/7$
(21)	$12(455)^{\frac{1}{2}}$	$6(66)^{\frac{1}{2}}/7$	3/7, 0

TABLE VIII. $x((211), UU')$.

	(10)	(11)	(20)	(21)	(30)
(10)	0	0	0	0	$-20(143)^{\frac{1}{2}}$
(11)	0	0	0	$10(182)^{\frac{1}{2}}$	10
(20)	0	0	$-8/7$	$4(33)^{\frac{1}{2}}/7$	$4\sqrt{3}$
(21)	0	$10(182)^{\frac{1}{2}}$	$4(33)^{\frac{1}{2}}/7$	4/7, 3	2
(30)	$-20(143)^{\frac{1}{2}}$	10	$4\sqrt{3}$	2	2

subtracting (52a) from (51) we also get

$$\begin{aligned}&(l^{n-1}v+1\ S-\tfrac{1}{2}\ +l|\}l^nvS)^2\\&=(n-v)(4l+6-v+2S)S/2n(2l+2-v)(2S+1),\\&(l^{n-1}v+1\ S+\tfrac{1}{2}\ +l|\}l^nvS)^2\\&=(n-v)(4l+4-v-2S)(S+1)/\\&\qquad 2n(2l+2-v)(2S+1).\end{aligned}\tag{52b}$$

The phases of $(l^{n-1}v'S'+l|\}l^nvS)$ are independent of n and will be denoted by $\epsilon(v'S'|\}vS)$; they are arbitrary as long as the phases of $(W'\alpha'L'+l|W\alpha L)$ are not fixed. The latter are partially fixed by (47), and, comparing it with (61) of III, we have

$$\epsilon(v+1S'|\}vS)=(-1)^{l+S-S'+\frac{1}{2}}\epsilon(vS|\}v+1S'),$$

or, in a more general form,

$$\epsilon(v'S'|\}vS)=(-1)^{l+S-S'+(v'-v)/2}\epsilon(vS|\}v'S'). \tag{53}$$

Another partial limitation in the choice of $\epsilon(v'S'|\}vS)$ is given by the fact that, according to (20), every value of W corresponds to two couples of v and S, which are related by the equations,

$$v_1+2S_2=v_2+2S_1=2l+1; \tag{54}$$

it may be shown that from this fact follows the relation

$$\frac{\epsilon(v_1-2\ S_1|\}v_1-1\ S_1-\frac{1}{2})\epsilon(v_1-1\ S_1-\frac{1}{2}|\}v_1S_1)}{\epsilon(v_1-2\ S_1|\}v_1-1\ S_1+\frac{1}{2})\epsilon(v_1-1\ S_1+\frac{1}{2}|\}v_1S_1)}=\frac{\epsilon(v_2S_2+1|\}v_2+1\ S_2+\frac{1}{2})\epsilon(v_2+1\ S_2+\frac{1}{2}|\}v_2S_2)}{\epsilon(v_2S_2+1|\}v_2-1\ S_2+\frac{1}{2})\epsilon(v_2-1\ S_2+\frac{1}{2}|\}v_2S_2)}, \tag{55}$$

when v_1, S_1, v_2, and S_2 satify (54).

In order to satisfy (53) and (55), the following choice of phases was made for $l=3$:

$$\begin{aligned}&\epsilon(v'S'|\}vS)=(-1)^{S'} &&\text{for } v \text{ odd},\\&\epsilon(v'S'|\}vS)=(-1)^{S'+(v'-v)/2} &&\text{for } v \text{ even}.\end{aligned}\tag{56}$$

3. The Calculation of $(W'U'+f|WU)$ and $(U'L'+f|UL)$

The coefficients of fractional parentage of f and f^2 are equal to unity; those of f^3 were calculated from (9) of III, and, in the cases where two doublets with the same L were allowed, $(f^2(^3F)f\ L|\}f^3\alpha\ L)$ was put equal to 0 in one of them, since for doublets of f^3 with $U\equiv(21)$ the eigenvalue of R vanishes according to (26), (27), and (28).

From the coefficients of fractional parentage of f, f^2, and f^3, several elements of $(W'U'+f|WU)$ and $(U'L'+f|UL)$ were obtained and are given in Tables III and IV. These tables were then extended by using (48), (49), (35), (36), and also (1) as it was pointed out in Section 2. In the very few cases where these equations were not sufficient for the determination of some elements, additional equations were obtained from (23) of III by requiring that $(f^nUvSL\|U^{(5)}\|f^nU'v'SL')$ should vanish unless $v=v'$ and $U=U'$, since the tensor $\mathbf{U}^{(5)}$ commutes with Q and R.

Owing to the present status of the experimental classification of the spectra of the rare earths, the terms of lower multiplicity are not yet interesting; we limited therefore Tables III and IV to those elements which are of use in the calculation of the coefficients of fractional parentage for f^4 and for the two highest multiplicities of f^5, f^6, and f^7.

6. THE SPECTRA OF f^n

1. The Choice of the Parameters

In Section 4 of II we considered the coefficients of Slater's integrals F^k as scalar products of tensors in the three-dimensional space; we shall now show that they may also be considered as particular components of tensors in the $(2l+1)$-dimensional space.

In full analogy to Section 3 of II it is possible to define as an irreducible tensor of the "type" W in the $(2l+1)$-dimensional space each operator whose components transform by a $(2l+1)$-dimensional rotation as the elements of the basis of the representation $\mathfrak{B}_W$ of $\mathfrak{d}_{2l+1}$.

In the three-dimensional space $\mathbf{u}^{(k)}$ was a tensor, and its components $u_q^{(k)}$ transformed as the spherical harmonics $Y(kq)$; in the $(2l+1)$-dimensional space $\mathbf{u}^{(k)}$ alone is no longer a tensor, but it may be shown that the quantities $(2k+1)^{\frac{1}{2}}u_q^{(k)}$ transform as the functions $\psi((20\cdots0)kq)$ if k is even, and as the functions $\psi((110\cdots0)kq)$ if k is odd, i.e., all the quantities $(4t+1)^{\frac{1}{2}}u_q^{(2t)}$ or $(4t-1)^{\frac{1}{2}}u_q^{(2t-1)}$ for $1\leqslant t\leqslant l$ are *together* the components of a sole tensor.

TABLE IX. $x((220), UU')$.

	(20)	(21)	(22)
(20)	3/14	$3(55)^{\frac{1}{2}}/7$	$-3(5/28)^{\frac{1}{2}}$
(21)	$3(55)^{\frac{1}{2}}/7$	$-6/7, -3$	$3/(7)^{\frac{1}{2}}$
(22)	$-3(5/28)^{\frac{1}{2}}$	$3/(7)^{\frac{1}{2}}$	3/2

TABLE X. $x((221), UU')$.

	(10)	(11)	(20)	(21)	(30)	(31)
(10)	0	0	0	0	$5(143)^{\frac{1}{2}}$	$-15(429)^{\frac{1}{2}}$
(11)	0	0	0	$14(910/11)^{\frac{1}{2}}$	$2(10)^{\frac{1}{2}}$	$2(39)^{\frac{1}{2}}/11$
(20)	0	0	$2/7$	$-10(6)^{\frac{1}{2}}/7$	$\sqrt{3}$	$9(3/7)^{\frac{1}{2}}$
(21)	0	$14(910/11)^{\frac{1}{2}}$	$-10(6)^{\frac{1}{2}}/7$	$-1/7,\ 12/11$	$5(2/11)^{\frac{1}{2}}$	$3\sqrt{2}/11$
(30)	$5(143)^{\frac{1}{2}}$	$2(10)^{\frac{1}{2}}$	$\sqrt{3}$	$5(2/11)^{\frac{1}{2}}$	$-1/2$	$3/2(11)^{\frac{1}{2}}$
(31)	$-15(429)^{\frac{1}{2}}$	$2(39)^{\frac{1}{2}}/11$	$9(3/7)^{\frac{1}{2}}$	$3\sqrt{2}/11$	$3/2(11)^{\frac{1}{2}}$	$1/22$

TABLE XI. $x((222), UU')$.

	(00)	(10)	(20)	(30)	(40)
(00)	0	0	0	0	$-30(143)^{\frac{1}{2}}$
(10)	0	0	0	$-3(1430)^{\frac{1}{2}}$	$9(1430)^{\frac{1}{2}}$
(20)	0	0	$6/11$	$-3(42/11)^{\frac{1}{2}}$	$9\sqrt{2}/11$
(30)	0	$-3(1430)^{\frac{1}{2}}$	$-3(42/11)^{\frac{1}{2}}$	-3	$1/(11)^{\frac{1}{2}}$
(40)	$-30(143)^{\frac{1}{2}}$	$9(1430)^{\frac{1}{2}}$	$9\sqrt{2}/11$	$1/(11)^{\frac{1}{2}}$	$3/11$

In the seven-dimensional space the quantities

$$\sum_{kqk'q'}[(2k+1)(2k'+1)]^{\frac{1}{2}}u_1{}^{(k)}{}_q u_2{}^{(k')}{}_{q'} \times((200)(20)kq,\ (200)(20)k'q'|WUKQ) \quad (57)$$

will transform as $\psi(WUKQ)$, and, in particular, the quantities

$$\sum_{kq}(2k+1)u_1{}^{(k)}{}_q u_2{}^{(k)}{}_{-q}(kkq-q|kk00) \times((200)(20)k+(200)(20)k|WU0)$$
$$=\sum_k(2k+1)^{\frac{1}{2}}(\mathbf{u}_1{}^{(k)}\cdot\mathbf{u}_2{}^{(k)}) \times((200)(20)k+(200)(20)k|WU0) \quad (58)$$

will have the tensorial properties of $\psi(WUS)$.

Since, according to (45) of II,

$$f_k(f^2L)=(f^2LM|(\mathbf{C}_1{}^{(k)}\cdot\mathbf{C}_2{}^{(k)})|f^2LM),$$

or also

$$f_k(f^2L)=(3\|C^{(k)}\|3)^2(f^2LM|(\mathbf{u}_1{}^{(k)}\cdot\mathbf{u}_2{}^{(k)})|f^2LM), \quad (59)$$

we shall substitute to the f_k the linear combinations

$$N_i\sum_k(2k+1)^{\frac{1}{2}}(3\|C^{(k)}\|3)^{-2}f_k \times((200)(20)k+(200)(20)k|W_iU_iS), \quad (60)$$

where the N_i are convenient normalization factors and

$$\begin{aligned}W_1U_1&\equiv(000)(00),\\ W_2U_2&\equiv(400)(40),\\ W_3U_3&\equiv(220)(22).\end{aligned} \quad (61)$$

Since the only parents of the functions $\psi((000)(00)S)$, $\psi((400)(40)S)$, and $\psi((220)(22)S)$ are, respectively, the functions

$$\psi((100)(10)F),\quad \psi((300)(30)F),\quad \text{and}\quad \psi((210)(21)F),$$

the coefficients

$$((200)(20)k+(200)(20)k|W_iU_iS)$$

will be proportional to $((20)k+f|(10)F)$, $((20)k+f|(30)F)$ and $((20)k+f|(21)F)$, which are given in Table IV. Taking the values of $(3\|C^{(k)}\|3)$ from (51) of II and remembering that

$$f^k=D_kf_k, \quad (62)$$

where the D_k are the denominators of Table II[6] of TAS,* we define for the configurations f^n,

$$\begin{aligned}e_0&=f^0=n(n-1)/2,\\ e_1&=9f^0/7+f^2/42+f^4/77+f^6/462,\\ e_2&=143f^2/42-130f^4/77+35f^6/462,\\ e_3&=11f^2/42+4f^4/77-7f^2/462;\end{aligned} \quad (63)$$

the term $9f^0/7$ was added for convenience in e_1 without changing its tensorial properties, since both f^0 and e_1 are scalars in the seven-dimensional space.

The general expression of the energy matrices of f^n will be

$$e_0E^0+e_1E^1+e_2E^2+e_3E^3 \quad (64)$$

instead of

$$f^0F_0+f^2F_2+f^4F_4+f^6F_6; \quad (65)$$

the E^i are linear combinations of Slater's parameters, which are, however, different from those adopted empirically in (96) of II:

$$\begin{aligned}E^0&=F_0-10F_2-33F_4-286F_6,\\ E^1&=(70F_2+231F_4+2002F_6)/9,\\ E^2&=(F_2-3F_4+7F_6)/9,\\ E^3&=(5F_2+6F_4-91F_6)/3;\end{aligned} \quad (66)$$

TABLE XII. $c(WW'(220))$.

	(000)	(100)	(110)	(200)	(111)	(210)	(211)	(220)	(221)	(222)
(000)	0	0	0	0	0	0	0	1	0	0
(100)	0	0	0	0	0	1	0	0	1	0
(110)	0	0	1	0	0	0	1	1	1	0
(200)	0	0	0	1	0	0	1	1	0	1
(111)	0	0	0	0	1	1	1	1	1	0
(210)	0	1	0	0	1	2	1	0	2	1
(211)	0	0	1	1	1	1	3	1	2	1
(220)	1	0	1	1	1	0	1	2	1	1
(221)	0	1	1	0	1	2	2	1	3	1
(222)	0	0	0	1	0	1	1	1	1	1

* E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, London, 1935).

TABLE XIII. $c(UU'(22))$.

	(00)	(10)	(11)	(20)	(21)	(30)	(22)	(31)	(40)
(00)	0	0	0	0	0	0	1	0	0
(10)	0	0	0	0	1	0	0	1	0
(11)	0	0	1	0	0	1	1	1	0
(20)	0	0	0	1	1	1	1	1	1
(21)	0	1	0	1	2	1	0	2	2
(30)	0	0	1	1	1	2	1	2	1
(22)	1	0	1	1	0	1	2	1	1
(31)	0	1	1	1	2	2	1	3	2
(40)	0	0	0	1	2	1	1	2	2

the formulas for f^2 assume now the form

$$
\begin{aligned}
&{}_0{}^1S = E^0+9E^1 \\
&\qquad\qquad\qquad {}_2{}^3P = E^0+33E^3 \\
&{}_2{}^1D = E^0+2E^1+286E^2-11E^3 \\
&\qquad\qquad\qquad {}_2{}^3F = E^0 \\
&{}_2{}^1G = E^0+2E^1-260E^2-4E^3 \\
&\qquad\qquad\qquad {}_2{}^3H = E^0-9E^3 \\
&{}_2{}^1I = E^0+2E^1+70E^2+7E^3
\end{aligned}
\tag{67}
$$

For $n>2$, the e_i are matrices whose order equals the number of allowed states for a given SL; the elements of these matrices may be calculated by means of (1), but most of the calculations may be avoided by considering the tensorial properties of the e_i.

e_1 is a scalar also in the seven-dimensional space; it is therefore diagonal in the $vUSL$ scheme, and its eigenvalues are independent of L and U. We have from (67) that

$$e_1(f^2SL)=q_{12}+\tfrac{1}{2}-2(\mathbf{s}_1\cdot\mathbf{s}_2), \tag{68}$$

and owing to (50) of III and (3) we obtain that, in general, the eigenvalues of e_1 are

$$e_1(f^n vUSL)=9(n-v)/2+v(v+2)/4-S(S+1). \tag{69}$$

The matrices e_2 and e_3 are particular components of tensors in the seven-dimensional space. The dependence of their elements on U and L will be analogous to (28) of II, but the result is somewhat more complicated, since, in the decomposition of the external product of two irreducible representations of $\mathfrak{d}_7$, some representation may appear more than once. We now have

$$(f^n vWUSL|e_i|f^n v'W'U'SL) = \textstyle\sum_\alpha A_\alpha(W'U'L+W_iU_iS|\alpha WUL); \tag{70}$$

or, owing to an obvious extension of (46),

$$(f^n vWUSL|e_i|f^n v'W'U'SL) = \textstyle\sum_\beta B_\beta(WUL+W'U'L|\beta W_iU_iS)/(2L+1)^{\frac{1}{2}}. \tag{71}$$

The number of values which may be assumed by α and β equals the number of times that the representation $\mathfrak{B}_{W_i}$ appears in the decomposition of $\mathfrak{B}_W\times\mathfrak{B}_{W'}$ and will be denoted by $c(WW'W_i)$. A method for calculating these numbers is given by Weyl.[6]

TABLE XIVa. $(U|\varphi(L)|U')$ for $U, U'\neq(31), (40)$.

$(U\ \ U')$	S	P	D	F	G	H	I	K	L	M	N
$(11\|\varphi\|11)$	0	-11	0	0	0	3	0	0	0	0	0
$(20\|\varphi\|20)$	0	0	-11	0	-4	0	7	0	0	0	0
$(10\|\varphi\|21)$	0	0	0	1	0	0	0	0	0	0	0
$(20\|\varphi\|21)$	0	0	$6\sqrt{2}$	0	$(65)^{\frac{1}{2}}$	0	0	0	0	0	0
$(21\|\varphi\|21)$	0	0	-57	63	55	-105	0	-14	42	0	0
$(11\|\varphi\|30)$	0	$(11)^{\frac{1}{2}}$	0	0	0	$(39)^{\frac{1}{2}}$	0	0	0	0	0
$(20\|\varphi\|30)$	0	0	0	0	$2(5)^{\frac{1}{2}}$	0	3	0	0	0	0
$(21\|\varphi\|30)$	0	0	0	$(195)^{\frac{1}{2}}$	$-(143)^{\frac{1}{2}}$	$-2(42)^{\frac{1}{2}}$	0	$-4(17)^{\frac{1}{2}}$	0	0	0
$(30\|\varphi\|30)$	0	83	0	-72	20	-15	42	-28	0	6	0
$(00\|\varphi\|22)$	1	0	0	0	0	0	0	0	0	0	0
$(20\|\varphi\|22)$	0	0	$3(429)^{\frac{1}{2}}$	0	$4(65)^{\frac{1}{2}}$	0	$3(85)^{\frac{1}{2}}$	0	0	0	0
$(22\|\varphi\|22)$	144	0	69	0	-148	72	39	0	-96	0	56

TABLE XIVb. $(U|\varphi(L)|31)$.

L	$(10\|\varphi\|31)$	$(11\|\varphi\|31)$	$(21\|\varphi\|31)$	$(30\|\varphi\|31)$	$(31\|\varphi\|31)$
P	0	$(330)^{\frac{1}{2}}$	0	$17(143)^{\frac{1}{2}}$	209
D	0	0	$12(273)^{\frac{1}{2}}$	0	-200
F	1	0	$-36(5)^{\frac{1}{2}}$	$-16(39)^{\frac{1}{2}}$	$\Vert\ 624 \qquad -80(143)^{\frac{1}{2}}\ \Vert$
F'	0	0	$-3(715)^{\frac{1}{2}}$	$24(33)^{\frac{1}{2}}$	$\Vert -80(143)^{\frac{1}{2}} \qquad -616\ \Vert$
G	0	0	$11(7)^{\frac{1}{2}}$	$4(1001)^{\frac{1}{2}}$	836
H	0	$(85)^{\frac{1}{2}}$	$-2(1309/3)^{\frac{1}{2}}$	$(187/2)^{\frac{1}{2}}$	$\Vert -1353/2 \qquad -5(6545)^{\frac{1}{2}}/2 \Vert$
H'	0	$(77)^{\frac{1}{2}}$	$-74(5/3)^{\frac{1}{2}}$	$31(35/2)^{\frac{1}{2}}$	$\Vert -5(6545)^{\frac{1}{2}}/2 \qquad 703/2\ \Vert$
I	0	0	0	$30(33)^{\frac{1}{2}}$	$\Vert -2662/5 \qquad 528(34)^{\frac{1}{2}}/5 \Vert$
I'	0	0	0	0	$\Vert 528(34)^{\frac{1}{2}}/5 \qquad -88/5\ \Vert$
K	0	0	$-28(323/23)^{\frac{1}{2}}$	$4(437)^{\frac{1}{2}}$	$\Vert\ 6652/23 \qquad 96(21318)^{\frac{1}{2}}/23 \Vert$
K'	0	0	$42(66/23)^{\frac{1}{2}}$	0	$\Vert 96(21318)^{\frac{1}{2}}/23 \qquad -5456/23\ \Vert$
L	0	0	$-6(190)^{\frac{1}{2}}$	0	-464
M	0	0	0	$-6(385)^{\frac{1}{2}}$	814
N	0	0	0	0	-616
O	0	0	0	0	352

[6] H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1939), p. 229.

TABLE XIVc. $(U|\varphi(L)|40)$.

L	$(20\|\varphi\|40)$	$(21\|\varphi\|40)$	$(22\|\varphi\|40)$
S	0	0	$2(2145)^{\frac{1}{2}}$
D	$11(13)^{\frac{1}{2}}$	$-6(26)^{\frac{1}{2}}$	$9(33)^{\frac{1}{2}}$
F	0	$3(455)^{\frac{1}{2}}$	0
G	$-4(715/27)^{\frac{1}{2}}$	$-131(11/27)^{\frac{1}{2}}$	$-4(11/27)^{\frac{1}{2}}$
G'	$(15470/27)^{\frac{1}{2}}$	$17(238/27)^{\frac{1}{2}}$	$-17(238/27)^{\frac{1}{2}}$
H	0	$-12(21)^{\frac{1}{2}}$	$3(286)^{\frac{1}{2}}$
I	$7(1045/31)^{\frac{1}{2}}$	0	$3(3553/31)^{\frac{1}{2}}$
I'	$3(1785/31)^{\frac{1}{2}}$	0	$75(21/31)^{\frac{1}{2}}$
K	0	$-2(119)^{\frac{1}{2}}$	0
L	0	$22(105/31)^{\frac{1}{2}}$	$4(627/31)^{\frac{1}{2}}$
L'	0	$-84(19/31)^{\frac{1}{2}}$	$12(385/31)^{\frac{1}{2}}$
N	0	0	$-(2530)^{\frac{1}{2}}$

TABLE XV. $y(f^3, {}_v{}^2U, {}_3{}^2U')$.

	${}_3{}^2(11)$	${}_3{}^2(20)$	${}_3{}^2(21)$
${}_1{}^2(10)$	0	0	$-6(22)^{\frac{1}{2}}$
${}_3{}^2(11)$	2	0	0
${}_3{}^2(20)$	0	10/7	$2(66)^{\frac{1}{2}}/7$
${}_3{}^2(21)$	0	$2(66)^{\frac{1}{2}}/7$	2/7

2. The Calculation of e_2

For the values of W and W' which satisfy (16), $c(WW'(400))$ equals unity if $W=W'$ and $w_1=2$, and vanishes in any other case; it follows that e_2 is diagonal with respect to v and vanishes for $v=2S$, and also that for $v>2S$

$$(f^n vWUSL|e_2|f^n vWU'SL)\\ =b(nvS)(WUL+WU'L|(400)(40)S)/(2L+1)^{\frac{1}{2}}. \quad (72)$$

By considerations which are very similar to the method used in Section 7 of II for calculating the energy matrices of d^n (and in particular for the proof that the relative positions of the quartets and sextets of d^5 are exactly opposed to those of the terms of d^2 with the same L), it may be shown that $b(nvS)$ is independent of n and that for two values of v and S which correspond to the same value of W the $b(nvS)$ differ only in the sign. We can therefore write

$$(f^n vUSL|e_2|f^n vU'SL)=\pm(WUL|e_2|WU'L), \quad (73)$$

where the upper sign holds for the values of v and S which appear in the first column of Table I, and the lower sign for the second column.

The actual calculation of $(WUL|e_2|WU'L)$ is simplified by the lemma (11): introducing it in (72) we have that

$$(WUL|e_2|WU'L)=\sum_\gamma x_\gamma(W, UU')(U|\chi_\gamma(L)|U'), \quad (74)$$

where x_γ is independent of L and $\chi_\gamma(L)$ is independent of W; the maximal number of independent $(U|\chi_\gamma(L)|U')$ is $c(UU'(40))$ and is given in Table V.

Not only $(WUL|e_2|WU'L)$, but also

$$\sum_{L_2}(UL|U_2L_2+f)(U_2|\chi_{\gamma 2}(L_2)|U_2')\\ \times(U_2'L_2+f|U'L) \quad (75)$$

TABLE XVI. $y(f^4, {}_v{}^3U, {}_4{}^3U')$.

	${}_4{}^3(10)$	${}_4{}^3(11)$	${}_4{}^3(20)$	${}_4{}^3(21)$	${}_4{}^3(30)$
${}_2{}^3(10)$	0	0	0	$-12(33/5)^{\frac{1}{2}}$	0
${}_2{}^3(11)$	0	6/5	0	0	6
${}_4{}^3(10)$	0	0	0	$8(11/15)^{\frac{1}{2}}$	0
${}_4{}^3(11)$	0	29/15	0	0	$-1/3$
${}_4{}^3(20)$	0	0	6/7	$-8(11/147)^{\frac{1}{2}}$	$4/\sqrt{3}$
${}_4{}^3(21)$	$8(11/15)^{\frac{1}{2}}$	0	$-8(11/147)^{\frac{1}{2}}$	$-2/21$	$-4/3$
${}_4{}^3(30)$	0	$-1/3$	$4/\sqrt{3}$	$-4/3$	1/3

TABLE XVII. $y(f^4, {}_v{}^1U, {}_4{}^1U')$.

	${}_4{}^1(20)$	${}_4{}^1(21)$	${}_4{}^1(22)$
${}_0{}^1(00)$	0	0	$-12(22)^{\frac{1}{2}}$
${}_2{}^1(20)$	$3(3/175)^{\frac{1}{2}}$	$-4(33/35)^{\frac{1}{2}}$	$-(3/5)^{\frac{1}{2}}$
${}_4{}^1(20)$	221/140	$8(11/245)^{\frac{1}{2}}$	$-(7/80)^{\frac{1}{2}}$
${}_4{}^1(21)$	$8(11/245)^{\frac{1}{2}}$	2/7	0
${}_4{}^1(22)$	$-(7/80)^{\frac{1}{2}}$	0	1/4

TABLE XVIII. $y(f^5, {}_v{}^4U, {}_5{}^4U')$.

	${}_5{}^4(10)$	${}_5{}^4(11)$	${}_5{}^4(20)$	${}_5{}^4(21)$	${}_5{}^4(30)$
${}_3{}^4(00)$	0	0	0	0	0
${}_3{}^4(10)$	0	0	0	$9(11)^{\frac{1}{2}}$	0
${}_3{}^4(20)$	0	0	$3/(7)^{\frac{1}{2}}$	$(33/7)^{\frac{1}{2}}$	$-2(21)^{\frac{1}{2}}$
${}_5{}^4(10)$	0	0	0	$-(55/3)^{\frac{1}{2}}$	0
${}_5{}^4(11)$	0	$-1/3$	0	0	$-5/3$
${}_5{}^4(20)$	0	0	5/7	$5(11/147)^{\frac{1}{2}}$	$2/\sqrt{3}$
${}_5{}^4(21)$	$-(55/3)^{\frac{1}{2}}$	0	$5(11/147)^{\frac{1}{2}}$	$-4/21$	$-2/3$
${}_5{}^4(30)$	0	$-5/3$	$2/\sqrt{3}$	$-2/3$	$-1/3$

is expressible as linear combination of the $(U|\chi_\gamma(L)|U')$; it is therefore convenient to calculate at first the expressions (75), and then to assemble the results in the summation (1), where the coefficients of fractional parentage have the form (34). It is also possible to avoid at all the summations (75) for most of the values of L, after the different $(U|\chi_\gamma(L)|U')$ allowed by Table V are obtained from few simple $\chi_{\gamma 2}(L_2)$.

Although almost all the allowed χ_γ appear in the expressions (75), the linear combinations (74) are generally proportional to each other, and it is therefore possible to express the results by means of one $\chi(L)$ for every couple UU', with the sole exception of $U=U'\equiv(21)$, where both $\chi_\gamma(L)$ allowed by Table V are necessary for expressing the different $e_2(W)$. The functions $(U|\chi(L)|U')$ are tabulated in Tables VI, the values of $x(W, UU')$ in Tables VII–XI.

3. The Calculation of e_3

Together with e_3 it is useful to consider the operator

$$\Omega=-462^{\frac{1}{2}}\sum_k(2k+1)^{\frac{1}{2}}\mathbf{U}^{(k)^2}\\ \times((110)(11)k+(110)(11)k|(220)(22)S)\\ =33(\mathbf{U}^{(1)^2}-\mathbf{U}^{(5)^2}), \quad (76)$$

TABLE XIX. $y(f^5, {}_v{}^2U, {}_5{}^2U')$.

	${}_5{}^2(10)$	${}_5{}^2(11)$	${}_5{}^2(20)$	${}_5{}^2(21)$	${}_5{}^2(30)$	${}_5{}^2(31)$
${}_1{}^2(10)$	0	0	0	$36/(5)^{1/2}$	0	$-36\sqrt{2}$
${}_3{}^2(11)$	0	$3/\sqrt{2}$	0	0	$3(5)^{1/2}/2$	$-(39/8)^{1/2}$
${}_3{}^2(20)$	0	0	$3/7$	$-11(6)^{1/2}/7$	$-4\sqrt{3}$	0
${}_3{}^2(21)$	$3(33/10)^{1/2}$	0	$-3(33/98)^{1/2}$	$3/7(11)^{1/2}$	$-3/2\sqrt{2}$	$3/2(22)^{1/2}$
${}_5{}^2(10)$	0	0	0	$43/(30)^{1/2}$	0	$4\sqrt{3}$
${}_5{}^2(11)$	0	$-5/6$	0	0	$-5(5/72)^{1/2}$	$-(13/48)^{1/2}$
${}_5{}^2(20)$	0	0	$11/7$	$-11/7(6)^{1/2}$	$4/\sqrt{3}$	0
${}_5{}^2(21)$	$43/(30)^{1/2}$	0	$-11/7(6)^{1/2}$	$25/231$	$29/6(22)^{1/2}$	$1/22\sqrt{2}$
${}_5{}^2(30)$	0	$-5(5/72)^{1/2}$	$4/\sqrt{3}$	$29/6(22)^{1/2}$	$-1/12$	$1/4(11)^{1/2}$
${}_5{}^2(31)$	$4\sqrt{3}$	$-(13/48)^{1/2}$	0	$1/22\sqrt{2}$	$1/4(11)^{1/2}$	$1/44$

TABLE XX. $y(f^6, {}_4{}^5U, {}_6{}^5U')$.

	${}_6{}^5(11)$	${}_6{}^5(20)$	${}_6{}^5(21)$
${}_4{}^5(00)$	0	0	0
${}_4{}^5(10)$	0	0	$-6(11)^{1/2}$
${}_4{}^5(20)$	0	$-2(2/7)^{1/2}$	$2(33/7)^{1/2}$

which has the same tensorial properties as e_3; from (24) of III and (27) we have

$$\Omega = \tfrac{1}{2}L(L+1) - 12G(G_2), \tag{77}$$

and therefore its matrix is diagonal in the UL scheme and has the eigenvalues

$$\omega(U, L) = \tfrac{1}{2}L(L+1) - 12g(U). \tag{78}$$

We have from (67) that

$$e_3(f^2\,{}^3L) = -3\omega, \tag{79}$$

and since for every n

$$\Omega = 66\sum_{i<j}((\mathbf{u}_i{}^{(1)}\cdot\mathbf{u}_j{}^{(1)}) - (\mathbf{u}_i{}^{(5)}\cdot\mathbf{u}_j{}^{(5)})), \tag{80}$$

we obtain that for every term of f^n with maximal spin

$$e_3(f^n\,{}^{n+1}L) = -3\omega(U, L). \tag{81}$$

The values of $c(WW'(220))$ are given in Table XII' but the results are much simpler than could be expected from that table. The calculations show that

$$(f^n vUSL|e_3+\Omega|f^n vU'SL) = a(n, v)(f^v vUSL|e_3+\Omega|f^v vU'SL), \tag{82}$$

$$(f^6\,{}_6L|e_3+\Omega|f^6\,{}_6L) = (f^7\,{}_7L|e_3+\Omega|f^7\,{}_7L) = 0; \tag{83}$$

for $n \leqslant 7$ it is

$$\begin{aligned} a(v+2, v) &= (1-v)/(7-v), \\ a(v+4, v) &= -4/(7-v), \end{aligned} \tag{84}$$

and it may be noted that these equations satisfy the relation

$$\sum_v^{14-v} n(f^n vUSL|e_3+\Omega|f^n vU'SL) = 0. \tag{85}$$

The fact that $a(n, v)$ depends on v but not on S suggests that Eqs. (82) to (85) are connected with the properties of the symplectic group which leaves invariant the form (21), but the investigation of these properties is beyond the scope of this paper.

For $v \neq v'$ we found also

$$\begin{aligned} (f^5\,{}_1{}^2L|e_3|f^5\,{}_3{}^2L) &= (2/5)^{1/2}(f^3\,{}_1{}^2L|e_3|f^3\,{}_3{}^2L), \\ (f^6\,{}_0{}^1L|e_3|f^6\,{}_4{}^1L) &= (9/5)^{1/2}(f^4\,{}_0{}^1L|e_3|f^4\,{}_4{}^1L), \\ (f^6\,{}_2L|e_3|f^6\,{}_4L) &= (1/6)^{1/2}(f^4\,{}_2L|e_3|f^4\,{}_4L), \\ (f^7\,{}_1{}^2L|e_3|f^7\,{}_5{}^2L) &= (3/2)^{1/2}(f^5\,{}_1{}^2L|e_3|f^5\,{}_5{}^2L). \end{aligned} \tag{86}$$

The values of $c(UU'(22))$ are given in Table XIII, but the calculations show that also when $c(UU'(22)) > 1$ we can write without any exception

$$(f^n vUSL|e_3+\Omega|f^n v'U'SL) = y(f^n, vSU, v'SU')(U|\varphi(L)|U'). \tag{87}$$

The functions $(U|\varphi(L)|U')$ are tabulated in Tables XIV, the values of $y(f^n, vSU, v'SU')$ which do not follow from (81), (82), (83), or (86) are given in Tables XV–XXIV.

TABLE XXI. $y(f^6, {}_v{}^3U, {}_6{}^3U')$.

	${}_6{}^3(10)$	${}_6{}^3(11)$	${}_6{}^3(20)$	${}_6{}^3(21)$	${}_6{}^3(30)$	${}_6{}^3(31)$
${}_2{}^3(10)$	0	0	0	$-48(2/5)^{1/2}$	0	-36
${}_2{}^3(11)$	0	$(6/5)^{1/2}$	0	0	$\sqrt{3}$	$3(13/10)^{1/2}$
${}_4{}^3(10)$	0	0	0	$46/(15)^{1/2}$	0	$-8(6)^{1/2}$
${}_4{}^3(11)$	0	$11/3(5)^{1/2}$	0	0	$-19/3\sqrt{2}$	$(13/60)^{1/2}$
${}_4{}^3(20)$	0	0	$-6\sqrt{2}/7$	$-22/7\sqrt{3}$	$8(2/3)^{1/2}$	0
${}_4{}^3(21)$	$-(110/3)^{1/2}$	0	$(22/147)^{1/2}$	$-16/21(11)^{1/2}$	$5/3\sqrt{2}$	$1/(22)^{1/2}$
${}_4{}^3(30)$	0	$-(5)^{1/2}/3$	$4(2/3)^{1/2}$	$4/3(11)^{1/2}$	$1/3\sqrt{2}$	$-1/(22)^{1/2}$

TABLE XXII. $y(f^6, {}_v^1U, {}_6^1U')$.

	${}_6^1(00)$	${}_6^1(10)$	${}_6^1(20)$	${}_6^1(30)$	${}_6^1(40)$
${}_2^1(20)^{\frac{1}{2}}$	0	0	$6/(55)^{\frac{1}{2}}$	$2(42/5)^{\frac{1}{2}}$	$6(2/55)^{\frac{1}{2}}$
${}_4^1(20)$	0	0	$-61/(770)^{\frac{1}{2}}$	$8(3/5)^{\frac{1}{2}}$	$-6/(385)^{\frac{1}{2}}$
${}_4^1(21)$	0	$3(22)^{\frac{1}{2}}$	$(2/7)^{\frac{1}{2}}$	$-\sqrt{3}$	$1/(7)^{\frac{1}{2}}$
${}_4^1(22)$	$-4(33/5)^{\frac{1}{2}}$	0	$-1/(22)^{\frac{1}{2}}$	0	$2/(11)^{\frac{1}{2}}$

TABLE XXIII. $y(f^7, {}_3^4U, {}_7^4U')$.

	${}_7^4(20)$	${}_7^4(21)$	${}_7^4(22)$
${}_3^4(00)$	0	0	$-12(11)^{\frac{1}{2}}$
${}_3^4(10)$	0	$6(33)^{\frac{1}{2}}$	0
${}_3^4(20)$	$-(5/7)^{\frac{1}{2}}$	$2(11/7)^{\frac{1}{2}}$	-1

TABLE XXIV. $y(f^7, {}_3^2U, {}_7^2U')$.

	${}_7^2(00)$	${}_7^2(10)$	${}_7^2(20)$	${}_7^2(30)$	${}_7^2(40)$
${}_3^2(11)$	0	0	0	$2(10)^{\frac{1}{2}}$	0
${}_3^2(20)$	0	0	$-16/(77)^{\frac{1}{2}}$	$-2(6)^{\frac{1}{2}}$	$6(2/77)^{\frac{1}{2}}$
${}_3^2(21)$	0	$-(66)^{\frac{1}{2}}$	$(6/7)^{\frac{1}{2}}$	1	$(3/7)^{\frac{1}{2}}$

From (71) and from the orthogonality between the functions $\psi((00)S)$, $\psi((40)S)$, and $\psi((22)S)$ follow the relations

$$\sum_L(2L+1)(U|\chi(L)|U) = \sum_L(2L+1)(U|\varphi(L)|U)=0 \quad (88)$$

and

$$\sum_L(2L+1)(U|\chi(L)|U')(U'|\varphi(L)|U)=0, \quad (89)$$

which were useful for checking Tables VI and XIV.

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The Moment of Inertia and Electric Dipole Moment of CsF from Radiofrequency Spectra*

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The electric resonance method of molecular beam spectroscopy was used to obtain spectra resulting from induced changes in the space-quantization of the rotational state $J=1$ of CsF in a homogeneous electric field. An analysis of the spectra for several values of the field intensity and for two different vibrational states gave the following molecular constants, $I_e=(151\pm6)10^{-40}$ g cm^2, $\mu_e=(7.88\pm0.17)10^{-18}$ e.s.u., $B_e=(185\pm7)10^{-3}$ cm^{-1}, $\alpha_e=(1.85\pm0.19)10^{-3}$ cm^{-1}, $r_e=(2.34\pm0.05)10^{-8}$ cm, $\omega_e=270\pm30$ cm^{-1}. I_e is the moment of inertia, μ_e is the electric dipole moment, B_e and α_e are rotational constants, r_e is the internuclear distance and ω_e is the vibrational constant.

THE molecular beam electric resonance method[1] yields spectra in the radiofrequency region resulting from changes in the space-quantization of a single rotational state of the molecule when the molecule passes through a homogeneous electric field, upon which is superposed a weak, transverse, oscillating field. In previous experiments with CsF determinations were made of the moment of inertia and electric dipole moment,[1] and of several nuclear-molecular interaction constants.[2] In this paper the results of further experiments with CsF under the high resolution conditions described in reference 2 are presented. The moment of inertia and electric dipole moment are redetermined with considerably greater accuracy and additional constants are obtained from a study of the vibrational effects.

The apparatus has been described in detail elsewhere.[1,2] The spectra were observed by fixing the frequency of the oscillating electric field and varying the magnitude of the steady, homogeneous field, a valid method of observation if the electric field intensity is sufficiently strong,[2] as was the case in the present experiments. Frequency was measured to 1 part in 10,000 with a General Radio Type 624A heterodyne frequency meter, which had been checked against standard frequencies broadcast by WWV. The electric field intensity in the homogeneous field was calculated from the potential drop across the field and the distance between the parallel plates forming the field boundaries.[3] "B" batteries were used as a voltage supply. A Type K potentiometer, connected to a calibrated volt box, was used to measure voltage to 1 part in 5000. The standard cell of the potentiometer was checked with another cell recently calibrated at the Bureau of Standards. No effects due to thermal e.m.f.'s were observed.

All voltage readings were corrected for the contact potential difference between the plates of the homogeneous field. This quantity was measured by making a run with the applied field intensity in one direction,

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[1] H. K. Hughes, Phys. Rev. 72, 614 (1947).
[2] J. W. Trischka, Phys. Rev. 74, 718 (1948).

[3] There is a typographical error in the value of the field gap reported in reference 2. The distance between the plates was 0.4931 ± 0.0004 cm.