where V is the atomic volume. The values of A calculated in this way for Zn and Al are given in Table III. The specific heat of the superconducting electrons calculated in this way is 87 times as great as the lattice specific heat for Al (Debye $\theta=419$[14]) and 33 times as great as that for Zn (Debye $\theta=320$[15]). The accuracy of this interpretation of $C_{s.el.}$ is, of course, strongly dependent on the accuracy with which the magnetic threshold curve follows a parabolic law. Our results for Al and Zn as well as those obtained elsewhere[2, 18] show that the magnetic threshold curves are not exactly parabolic and consequently it cannot be excluded that for Zn and Al further approximation would lead to some deviation from the T^3 function for $C_{s.el}$.

[18] A. D. Misener, Proc. Roy. Soc. A166, 43 (1938).

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Magnetic Interaction between Neutrons and Electrons*

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The phenomenological interaction operator between neutrons and electrons is usually assumed to be $-\mu\mathbf{s}\cdot\mathbf{B}$. ($\mu\mathbf{s}$ magnetic moment of the neutron, $\mathbf{B}$ magnetic field of the electrons.) Bloch has pointed out that other operators are possible. It is shown that the class of operators considered by Bloch is equivalent with the form $-\mu\mathbf{s}\cdot(\mathbf{H}+4\pi C\mathbf{M})$ where $\mathbf{M}$ is the spin momentum density of the electrons, $\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}$ and C is an indeterminate constant. Diffraction by ferromagnetic crystals and magnetic double refraction are calculated for this interaction. Present experimental evidence excludes the value $C=0$. Experimental methods for measuring C are discussed.

I. THE GENERALIZED INTERACTION OPERATOR

It is commonly assumed that the magnetic interaction between neutrons and electrons is described by the interaction Hamiltonian

$$H_{\text{int}}=-\mu\mathbf{s}\cdot\mathbf{B}, \tag{1}$$

where μ is the neutron's magnetic moment, $\mathbf{s}$ the Pauli spin operator acting on the neutron wave function and $\mathbf{B}$ the magnetic field of all electrons. It was pointed out by Bloch,[1] however, that other, more general forms of the interaction Hamiltonian are possible and that the decision should be left to experiment. The purpose of this paper is to examine the experimentally verifiable consequences of the class of Hamiltonians proposed by Bloch, to interpret the existing experimental evidence in this respect, and to suggest experiments which can give a definite decision on the interaction operator.

The interaction of neutrons with ferromagnetic crystals is most likely to supply the desired information. For the case of thermal neutrons, inelastic scattering is small, and the neutron can be described by a wave equation

$$(\nabla^2+k_0^2)\psi-(2m/\hbar^2)[V(\mathbf{r})+\langle H_{\text{int}}\rangle_{\text{Av}}]\psi=0, \tag{2}$$

where $\langle H_{\text{int}}\rangle_{\text{Av}}$ is the interaction operator, averaged over the electron wave functions of the ground state, ψ is the Pauli spinor of the neutron, $k_0^2=2mE/\hbar^2$ in the usual notation and $V(\mathbf{r})$ the nuclear "quasi-potential."

The magnetic field of the electrons $\mathbf{B}(\mathbf{r})$ is

$$\mathbf{B}(\mathbf{r})=e\sum_{i=1}^{z}\int\varphi_0^*(\mathbf{r}_1\cdots\mathbf{r}_z)\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}\times\boldsymbol{\alpha}_i\varphi_0(\mathbf{r}_1\cdots\mathbf{r}_z)d\tau_1\cdots d\tau_z. \tag{3}$$

In non-relativistic approximation, and in absence of orbital moments, the expectation value of the current $e\boldsymbol{\alpha}=\mathbf{j}$ is

$$\mathbf{j}=c\nabla\times\mathbf{M},\quad \mathbf{M}=\sum_{i=1}^{z}\left[\int\Phi_0^*\boldsymbol{\sigma}_i\Phi_0\prod_{n=1}^{z}{}'\,d\tau_n\right]_{\mathbf{r}_i=\mathbf{r}}, \tag{4}$$

where Φ_0 is the Pauli spinor and $\boldsymbol{\sigma}$ the Pauli moment operator. The symbol $\prod'$ means that the integration is carried out with respect to the space and spin coor-

* This work was sponsored by ONR.
[1] F. Bloch, Phys. Rev. 51, 994 (1937).

dinates of all but the i'th electron.[2] As usual in the classical theory, one can express $\mathbf{B}$ by

$$\mathbf{B}=4\pi\mathbf{M}+\mathbf{H}, \quad (5)$$

where $\mathbf{H}$ is irrotational;

$$\nabla\times\mathbf{H}=0, \quad (6)$$

while

$$\nabla\cdot\mathbf{B}=0. \quad (7)$$

An interaction Hamiltonian more general than (1) is

$$\langle H_{\text{int}}\rangle_{\text{Av}}=-\mu\mathbf{s}\cdot(\mathbf{H}+4\pi C\mathbf{M}), \quad (8)$$

where C is an undetermined constant. If a single magnetic ion interacts with the incident plane wave of neutrons, Born's approximation gives for the scattered amplitude

$$\psi_{\text{sc}}\sim\frac{m}{2\pi\hbar^2}\frac{\exp(ik_0r)}{r}\int\exp(i\mathbf{q}\cdot\mathbf{r})\times[V(\mathbf{r})-\mu\mathbf{s}\cdot(\mathbf{H}+4\pi C\mathbf{M})]\chi_0 d\mathbf{r} \quad (9)$$

where χ_0 is the constant spinor of the incident wave

$$\psi_0=\chi_0\exp(i\mathbf{k}_0\cdot\mathbf{r}), \quad \mathbf{q}=\mathbf{k}_0-\mathbf{k} \quad (10)$$

and $\mathbf{k}$ is the propagation vector of the scattered wave.

The Fourier integral

$$\mathbf{h}(\mathbf{q})=\int\mathbf{H}(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r} \quad (11)$$

can be expressed in terms of $\mathbf{M}$ as follows: Let

$$\int\mathbf{M}(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}=\mathbf{m}(\mathbf{q}), \quad (12)$$

and therefore,

$$\mathbf{M}(\mathbf{r})=\frac{1}{(2\pi)^3}\int\mathbf{m}(\mathbf{q})\exp(-i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}, \quad (13)$$

and

$$\mathbf{H}(\mathbf{r})=\frac{1}{(2\pi)^3}\int\mathbf{h}(\mathbf{q})\exp(-i\mathbf{q}\cdot\mathbf{r})d\mathbf{q}. \quad (14)$$

Substitution of (13) and (14) into (5), (6) and (7) yields

$$\mathbf{q}\times\mathbf{h}=0 \quad (15)$$

and

$$\mathbf{q}\cdot(\mathbf{h}+4\pi\mathbf{m})=0. \quad (16)$$

Equations (15) and (16) can be solved for $\mathbf{h}$:

$$\mathbf{h}=-4\pi\mathbf{q}(\mathbf{q}\cdot\mathbf{m})/\mathbf{q}^2. \quad (17)$$

[2] H. A. Kramers, *Grundlagen der Quantentheorie* (Akademische Verlagsgesellschaft, Leipzig, 1938), p. 408.

The scattered amplitude is now, in terms of $\mathbf{M}$:

$$\psi_{sc}\sim\frac{m\exp(ik_0r)}{2\pi\hbar^2 r}\left[\int V(\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}+4\pi\mu\mathbf{s}\cdot\left(\frac{\mathbf{q}(\mathbf{q}\cdot\mathbf{m})}{\mathbf{q}^2}-C\mathbf{m}\right)\right]\chi_0. \quad (18)$$

Equation (18) is identical with Bloch's result.* We have shown that the interaction Hamiltonian (8) is equivalent to Bloch's assumption. If, in particular, $C=1$, we obtain the result derived by Schwinger[3] from the Hamiltonian (1) whereas $C=0$ leads to the earlier formula of Bloch.[4]

II. INTERACTION WITH A FERROMAGNETIC LATTICE

The wave equation of the neutron is

$$(\nabla^2+k_0^2)\psi-\frac{2m}{\hbar^2}[V(\mathbf{r})-\mu\mathbf{s}\cdot(\mathbf{H}+4\pi C\mathbf{M})]\psi=0. \quad (19)$$

In a periodic lattice, V, $\mathbf{H}$ and $\mathbf{M}$ are periodic functions of $\mathbf{r}$ and can be represented by Fourier sums

$$V(\mathbf{r})=\sum v_n\exp(2\pi i\mathbf{A}_n\cdot\mathbf{r}), \quad (20)$$

$$\mathbf{H}(\mathbf{r})=\sum\mathbf{h}_n\exp(2\pi i\mathbf{A}_n\cdot\mathbf{r}), \quad (21)$$

$$\mathbf{M}(\mathbf{r})=\sum\mathbf{m}_n\exp(2\pi i\mathbf{A}_n\cdot\mathbf{r}), \quad (22)$$

where $\mathbf{A}_n$ are the reciprocal lattice vectors, and the summation index n is understood to mean a triplet of integers $(n_1n_2n_3)$. The coefficients $\mathbf{m}_n$ are

$$\mathbf{m}_n=\frac{1}{\tau}\int_{\text{cell}}\mathbf{M}(\mathbf{r})\exp(-2\pi i\mathbf{A}_n\cdot\mathbf{r})d\mathbf{r}, \quad (23)$$

where τ is the volume of the unit cell, which is also the domain of integration. Analogous expressions give v_n and $\mathbf{h}_n$.

Substitution of the Fourier sums into Eqs. (5), (6) and (7) gives

$$\mathbf{A}_n\cdot(\mathbf{h}_n+4\pi\mathbf{m}_n)=0 \quad (24)$$

and

$$\mathbf{A}_n\times\mathbf{h}_n=0. \quad (25)$$

Therefore,

$$\mathbf{h}_n=-\frac{4\pi\mathbf{A}_n(\mathbf{A}_n\cdot\mathbf{m}_n)}{\mathbf{A}_n^2}, \quad (n\neq0) \quad (26)$$

whereas $\mathbf{h}_0$ remains indeterminate. This corresponds to

* Bloch defines the constant C as the limit of the integral

$$C=\frac{1}{4\pi}\int_S\frac{zds_z}{r^3}$$

as the surface S is contracted to zero; ds is the surface element. The value of the integral depends on the shape of S.

[3] J. S. Schwinger, Phys. Rev. **51**, 544 (1937).

[4] F. Bloch, Phys. Rev. **50**, 259 (1936).

the physical fact that the macroscopic mean value of **H** depends on the shape of the magnet, even if it is infinitely large.

We can now proceed as in the dynamical theory of x-rays.[5] The wave function has the form

$$\psi=\exp(i\mathbf{k}\cdot\mathbf{r})\textstyle\sum_n c_n \exp(2\pi i\mathbf{A}_n\cdot\mathbf{r}), \qquad (27)$$

where the c_n are constant spinors. For convenience we put

$$\begin{aligned} w(\mathbf{r})&=\frac{2m}{\hbar^2}[V(\mathbf{r})-\mu\mathbf{s}\cdot(\mathbf{H}+4\pi C\mathbf{M})] \\ &=\sum w_n \exp(2\pi i\mathbf{A}_n\cdot\mathbf{r}), \end{aligned} \qquad (28)$$

where according to Eqs. (20), (21), (22), and (26)

$$w_n=\frac{2m}{\hbar^2}\left[v_n+4\pi\mathbf{s}\cdot\left(\frac{\mathbf{A}_n(\mathbf{A}_n\cdot\mathbf{m}_n)}{\mathbf{A}_n^2}-C\mathbf{m}_n\right)\right]. \qquad (29)$$

Substitution of the sums (27) and (28) into Eq. (19) yields:

$$[k_0^2-(\mathbf{k}+2\pi\mathbf{A}_m)^2]c_m-\textstyle\sum_n w_{m-n}c_n=0. \qquad (30)$$

When the direction **k** is far from a Laue reflection, only c_0 is large, and we have

$$(k_0^2-k^2)c_0-w_0c_0\approx 0 \qquad (31)$$

with

$$w_0=\frac{2m}{\hbar^2}[V_{\text{Av}}-\mu\mathbf{s}\cdot(\mathbf{H}_{\text{Av}}+4\pi C\mathbf{M}_{\text{Av}})], \qquad (32)$$

where $\mathbf{V}_{\text{Av}}$, $\mathbf{H}_{\text{Av}}$ and $\mathbf{M}_{\text{Av}}=\mathbf{m}_0$ are the macroscopic mean values of the nuclear potential, the field intensity and the magnetization. If the z direction is chosen parallel to $(\mathbf{H}_{\text{Av}}+4\pi C\mathbf{M}_{\text{Av}})$, the obvious solutions of Eq. (31) are the two eigen states of the z-component of **s**, i.e., $\chi_{\frac{1}{2}}$ and $\chi_{-\frac{1}{2}}$, respectively. If we call the corresponding wave numbers $k=2\pi/\lambda$, k_+ and k_-, respectively, we have from Eqs. (31) and (32):

$$k_0^2-k_\pm^2=\frac{2m}{\hbar^2}[V_{\text{Av}}\pm\mu|\mathbf{H}_{\text{Av}}+4\pi C\mathbf{M}_{\text{Av}}|]. \qquad (33)$$

This result, with the special value $C=1$, was derived by Achieser and Pomeranchuk[6] for neutron wavelengths large in comparison to the lattice spacing. Equation (33) should be valid for unrestricted wavelengths, except near-Laue interference directions.

We have to justify the approximation made in Eq. (31). If all quantities but w_0 are neglected, we obtain for $c_m(m\neq 0)$

$$c_m=\frac{w_{-m}c_0}{k_0^2-(\mathbf{k}+2\pi\mathbf{A}_{-m})^2}. \qquad (34)$$

The approximation in Eq. (31) is justified if

$$\sum_{n\neq 0}\frac{w_nw_{-n}c_0}{k_0^2-(\mathbf{k}+2\pi\mathbf{A}_{-n})^2}\ll w_0c_0. \qquad (35)$$

Since we are not interested in the nuclear forces, we consider only the magnetic term in w_n.[7] Furthermore, we consider only lattices with a center of symmetry so that $\mathbf{m}_n=\mathbf{m}_{-n}$.

Because of the anticommutative properties of the Pauli matrices

$$(\mathbf{s}\cdot\mathbf{a})(\mathbf{s}\cdot\mathbf{a})=\mathbf{a}^2 \qquad (36)$$

for any vector **a**.

Equation (35) can be written

$$\left(\frac{8\pi m}{\hbar^2}\right)^2\sum_{n\neq 0}\frac{\mathbf{m}_n^2(C^2+\cos^2\varphi_n-2C\cos^2\varphi_n)}{k_0^2-(\mathbf{k}+2\pi\mathbf{A}_{-n})^2} \ll\frac{2m}{\hbar^2}|\mathbf{H}_{\text{Av}}+4\pi C\mathbf{M}_{\text{Av}}|, \qquad (37)$$

where φ_n is the angle between $\mathbf{m}_n$ and $\mathbf{A}_{-n}$. Usually, $\mathbf{H}_{\text{Av}}$ is very small as compared to $4\pi\mathbf{m}_0$. If C is of the order of one, we can neglect the first term of the second member. The validity of Eq. (37) depends then on the numerical value of the dimensionless quantity

$$\frac{8\pi m}{\hbar^2k_0^2}|\mathbf{m}_0|=\alpha, \qquad (38)$$

and on the manner in which the function

$$f(n)=\frac{|\mathbf{m}_n|}{|\mathbf{m}_0|} \qquad (39)$$

decreases with increasing n.

The Fourier coefficients $\mathbf{m}_n$ of **M** have been estimated for Fe by Halpern, Hamermesh, and Johnson[8] and could be used for the estimate. However, it is easier to use an argument due to the analogy with the dynamical theory of x-rays. The quantity α is of the order of $5\cdot 10^{-6}$ for neutrons, the same order as the corresponding quantity in x-rays. The behavior of the function $f(n)$ is determined by the microscopic distribution of $\mathbf{M}(\mathbf{r})$ which is close to the electronic density of the magnetically unsaturated electrons.[8] But in x-ray theory, the corresponding quantity is the total electronic density so that the similarity of the two cases is obvious. We can conclude from the experimental verification of the dynamical theory of x-rays that our approximation is justified.

However, the approximation method breaks down if $C\ll 1$. In this case, the term $\mathbf{m}_n^2\cos^2\varphi_n$ in the first

[5] M. von Laue, *Röntgenstrahlinterferenzen* (Akademische Verlagsgesellschaft, Leipzig, 1941).

[6] A. Achieser and J. Pomeranchuk, J. Exper. and Theor. Phys. U.S.S.R. **18**, 475 (1948).

[7] For the nuclear quasi-potential the approximation cannot be justified directly. An indirect justification has been given by M. L. Goldberger and F. Seitz, Phys. Rev. **71**, 296 (1947).

[8] Halpern, Hamermesh, and Johnson, Phys. Rev. **59**, 981 (1941).

member of Eq. (37) will predominate, and make the justification impossible. But, in this case, the entire effect of "magnetic birefringence" expressed by Eq. (33) will be immeasurably small, unless $\mathbf{H}_{Av}$ is made very large. In this latter case, i.e., when $\mathbf{H}_{Av}$ is of the order of $\mathbf{m}_0$, the above argument holds again.

The case of Laue interference in the dynamical theory will not be investigated here since actual ferromagnetic crystals are so imperfect that they can hardly be expected to approach the infinite ideal crystal.

III. EXPERIMENTAL POSSIBILITIES FOR DETERMINING BLOCH'S CONSTANT C

A. Laue Interference

For a sufficiently small ferromagnetic crystal, Born's approximation can be used to calculate the wave scattered from an incident plane wave of the form (10). It is a well-known result of the kinematic theory of x-rays that the scattered "integrated intensity" of an interference maximum characterized by

$$2\pi\mathbf{A}_n=\mathbf{k}-\mathbf{k}_0 \tag{40}$$

is proportional to the square modulus of $w_n\chi_0$, where w_n is given by (29) and χ_0 is given by the form (10). In general, this scattered intensity will depend on the spin state χ_0 of the incident wave. This results in polarization effects for the beam transmitted through a polycrystalline magnetized plate. These effects have been theoretically investigated by Halpern, Hamermesh, and others[8–10] for the usual assumption $C=1$.

The easiest method for measuring C seems to be an experiment in which the magnetic term in (29) is made to vanish for all existing Laue maxima. In this case, no single transmission effect is observable. If a monoenergetic beam of neutrons has the longest reflecting wave-length for the lattice (e.g., twice the spacing of the 110 planes in Fe), then the incident beam is reflected by 180° by those crystals which are in reflecting position, whereas all other crystals act as a transparent medium. In this case, $\mathbf{k}$, $\mathbf{k}_0$ and $\mathbf{A}_n$ have the same direction. If φ is the angle between the incident beam and the direction of magnetization $\mathbf{M}$ (which is parallel to all $\mathbf{m}_n$'s) then the magnetic term in (29) can be written

$$4\pi\mu\mathbf{m}_n{}^2\mathbf{s}\cdot\mathbf{e}(C^2+\cos^2\varphi-2C\cos^2\varphi), \tag{41}$$

where $\mathbf{e}$ is a unit vector without interest for the following. The magnetic interaction vanishes if

$$C=1,\quad \varphi=0;\quad C=0,\quad \varphi=\pi/2. \tag{42}$$

It should be possible to measure Bloch's constant C by determining that angle φ between magnetization and incident beam which leads to the smallest transmission effect. If the common assumption $C=1$ is correct, then the single transmission effect should vanish when the magnetization is parallel to the incident beam.

The observations of Hughes, Wallace, and Holtzman[11] show a large single transmission effect for the case where the magnetization is normal to the incident beam, i.e., $\varphi=\pi/2$. This seems to rule out the possibility $C=0$. It is true that in these experiments the wave-length band covered a number of reflections, but the contribution of the largest lattice spacing (110) is so large that their results would be difficult to explain if this reflection did not contribute to the magnetic interaction.

B. Total Reflection

If the effective nuclear potential V_{Av} is positive, as in most cases, the birefringence expressed in Eq. (33) can give rise to two angles of total reflection, as pointed out by Achieser and Pomeranchuk.[6] One obtains in the usual way for the two glancing angles of total reflection

$$\theta_\pm=\left(\frac{2m}{\hbar^2k_0{}^2}\right)^{\frac{1}{2}}[V_{Av}\pm\mu|\mathbf{H}_{Av}+4\pi C\mathbf{M}_{Av}|]^{\frac{1}{2}}, \tag{43}$$

corresponding to the total reflection of neutrons with spin either parallel or antiparallel to the vector $\mathbf{H}_{Av}+4\pi C\mathbf{M}_{Av}$. In particular, with the usual assumption $C=1$, we obtain the expression

$$\theta_\pm=\left(\frac{2m}{\hbar^2k_0{}^2}\right)^{\frac{1}{2}}[V_{Av}\pm\mu|\mathbf{B}_{Av}|]^{\frac{1}{2}} \tag{44}$$

in agreement with Achieser and Pomeranchuk, but for unrestricted neutron wave-length. The angles $\theta_\pm$ are independent of the direction of the induction $\mathbf{B}_{Av}$. This conclusion differs from opinions expressed by Halpern[12] and by Hughes and Burgy.[13]

Hughes and Burgy[13] reported observation of two distinct angles of total reflection, in a case where the mean magnetic field $\mathbf{H}_{Av}$ was very small. This is an additional evidence against the assumption $C=0$.

A precise measurement of $\theta_\pm$ with a field $\mathbf{H}_{Av}=0$ provides a method for measuring Bloch's constant by using Eq. (43).

[9] O. Halpern and M. H. Johnson, Phys. Rev. 55, 898 (1939).
[10] O. Halpern and T. Holstein, Phys. Rev. 59, 960 (1941).
[11] Hughes, Wallace, and Holtzman, Phys. Rev. 73, 1277 (1948).
[12] O. Halpern, Phys. Rev. 75, 343 (1949).
[13] D. J. Hughes, and M. T. Burgy, Phys. Rev. 76, 463 (1949).