Effects of the Proton Radius on Nuclear Motion Correction for the Hyperfine Structure of Hydrogen*

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Contributions to the hyperfine structure of the ground state of hydrogen resulting from the assumption of a non-vanishing proton radius are evaluated in this paper. These effects occur because of the assumed modification of the Coulomb potential inside the proton and the consequent change in the wave function the above assumption implies. Since the proton radius and the shape of the potential function in the interior are not known, there occur uncertainties of relative order $\alpha^2 m/M$ in the hyperfine structure arising from nuclear mass motion.

I. INTRODUCTION

THE transformation[1] (I, 7.6) brings the Eqs. (I, 7.5), with neglect of the terms in $\hbar/Mc$, to the form of radial equations for a Dirac electron in a Coulomb field, and allows calculation of the effects of the mass motion terms if the assumption is made that the potential in Eq. (I, 7.5) is Coulombian. This potential is modified inside the proton if one assumes a non-vanishing proton radius,[1] however, and there result effects the evaluation of which is the object of this note. The potential function in the interior of the proton is assumed to have the constant value $J=e^2/b$, corresponding to the potential of a charged shell of radius b. This modification of the Coulomb potential is an admittedly arbitrary one. It has advantages of simplicity, however. Since it is the change in the wave functions and the approximate values of the quantities dJ/dr, ΔJ that are of primary importance, it is believed that the results of the calculations give the correct order of magnitude of the effects under consideration. The details of the way in which the potential energy curve is rounded off are believed to be immaterial provided the curve is nearly smooth and reproduces approximately the values used below. The change in the wave function produced by the change of the potential from the Coulombian is considered below. It will be seen that this effect combines with the effects of nuclear mass motion to change the hyperfine structure splitting to relative order $\alpha^2 m/M$.

II. CALCULATIONS AND CONCLUSIONS

Effects of the modification of the Coulomb potential inside the proton will now be considered. The potential function J is assumed to have the constant value e^2/b inside this region as discussed in the preceding section. Consequently the quantity dJ/dr vanishes in the nuclear interior and the integral which arises in the calculation of the hyperfine structure splitting through the evaluation of $\langle Q''\rangle$ of the preceding paper[1] has the limits $r=b$ and ∞. This effect of the exclusion of the region $0<r<b$ from the hyperfine structure integral, just referred to, has been discussed in Section V of I.

There is now a second effect to be considered. From the modification of the potential in the interior there results a change of order α^2 in the wave function in the exterior region $r>b$ and the hyperfine structure integral is changed also.

Effects of relative order α^2 and $\alpha^2 m/M$ result from both of the above causes. Only the latter type is concerned with nuclear mass motion and its evaluation will now be made.

In order to obtain the change in the wave function for $r>b$, one first considers the wave function inside the nucleus. For $r<b$, the modified potential $J=e^2/b$ is transformed by Eq. (I, 7.6) into $J'=e'^2/(AB)^{\frac{1}{2}}b$, to replace e'^2/r' in Eq. (I, 7.6′). Equations (I, 7.6′) take the form

$$\begin{aligned}(W+1+J')F'-(1/\alpha')\Omega_+'G'=0,\\ (W-1+J')G'+(1/\alpha')\Omega_-'F'=0\end{aligned} \tag{1}$$

with

$$\alpha'=e'^2/\hbar c \tag{1.1}$$

and with the other quantities defined by Eqs. (I, 7.2), (I, 7.6). Here energies are expressed in units mc^2 and lengths in e'^2/mc^2. For the ground state of hydrogen the Dirac quantum number k appearing in the Ω's, which are defined by Eq. (I, 7.2), has the value -1. On elimination of F' from Eqs. (1), one obtains

$$d^2G'/dr'^2+\kappa^2G'=0 \tag{1.2}$$

where

$$\kappa=\alpha'[(W+J')^2-1]^{\frac{1}{2}} \tag{1.3}$$

giving $G'=\sin\kappa r'$. The function F' may be obtained from the first of the two Eqs. (1). The ratio of the functions is

$$F'/G'\cong-(1/3)\alpha'(W+J'-1)r'\cong-(1/3)\alpha'J'r' \tag{2}$$

where the second approximate equality holds because $W\cong 1$ in mc^2 units. The ratio F/G has to vary continuously at the nuclear radius $r=b$ as a general property of Dirac's equation. The value of F'/G' for $r=b-0$ is the same as for $r=b+0$ and will be used below to determine the solution for $r>b$. For the special case of the spherical shell distribution the right side of Eq. (2) simplifies to

$$F'/G'\cong-(1/3)\alpha'. \tag{2.1}$$

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[1] Breit, Brown, and Arfken, Phys. Rev. **76**, 1299 (1949). This paper will be referred to as I in the text. Formulas occurring in this paper will be referred to by prefixing the Roman numeral I ahead of the number of the formula.

By a method discussed in III,[2] the difference between this ratio and the ratio of solutions F_0'/G_0' obtained using $J'=e'^2/r$, which follows from a Coulomb potential in Eq. (I, 7.5), can be used to calculate changes in the wave function for $r>b$ produced by the cut-off of the Coulomb potential. Employing a notation similar to that of III one obtains

$$y'=F'/G', \quad y_0'=F_0'/G_0', \quad \delta y'=y'-y_0'. \quad (2.2)$$

The quantity y_0' is easily found since equations for F_0' and G_0' take the form of Dirac equations for an electron in a Coulomb field. The known solutions of these yield $y_0'=-\alpha'/2$. The cut-off, therefore, produces a change in y which when evaluated at the proton radius is

$$\delta y'=y'-y_0'\cong\alpha'/6. \quad (2.3)$$

For $r>b$ the same equations hold for F', G' as for F_0', G_0' and one can find $\delta y'$ in this region by employing Eq. (III, 3.1). Equation (III, 3.1) becomes

$$(d/dr')(G_0'^2\delta y')+(F_0'^2+G_0'^2)\delta W/\alpha'=0 \quad (2.4)$$

with energy in units mc^2 and lengths in $a_H'=\hbar^2/me'^2$. The primes in this formula as in Eqs. (1) and following denote quantities transformed by Eqs. (I, 7.6). On integration from the proton radius b to infinity one finds δW, the change in energy produced by the cut-off, to be of order $\alpha^4 Ry$ and hence negligible. Employing Eq. (2.4) and the known value of δy at $r=b$, one finds

$$\delta y'\cong(\alpha'/6)b'^2\alpha'^4(1/r'^2+2/r'+2). \quad (3)$$

Here b' is the value of r' at the proton radius[3] in units e'^2/mc^2. The integral of interest for the hyperfine structure is

$$-I'=\int f'g'dr'\cong\int y_0'g_0'^2(1+\delta y'/y_0')dr' + \int 2g_0'y_0'\delta g'dr' \quad (3.1)$$

where f' and g' are given by $r'f'=F'$, $r'g'=G'$. The first integral on the right of the approximate equality sign is

$$[1-(8/3)b'\alpha'^2]I_0=[1-(8/3)b'\alpha^2(1-m/M)]I_0$$

where

$$-I_0=\int_0^\infty f_0'g_0'dr',$$

the hyperfine structure integral without cut-off. The second integral is of order $\alpha^4 I_0$ and will be neglected.

The term in α^2 in the above factor multiplying I_0 includes the effect of the exclusion of the region $0<r<b$ from the integral and the effect of the cut-off on the wave functions exterior to the proton. The term in $\alpha^2 m/M$ arises from the combination of both of these effects with mass motion effects. Because the proton radius and the shape of the interior potential are not known, uncertainties in the mass correction to the hyperfine structure of this latter order $\alpha^2 m/M$ occur.

The terms in ΔJ in Eqs. (I, 7.5) must also be considered, although it will be shown that their effect is negligible for the particular J assumed here. Equations (I, 7.5) relating unprimed quantities are employed to describe the behavior of the wave functions through the charged shell of radius b, which is taken to be the limit of a shell of finite thickness. Only terms in ΔJ, dG/dr and dF/dr are large in this transition region. Consequently, the first of Eqs. (I, 7.5) becomes

$$[1+r(dJ/dr)/2Mc^2]dG/dr+(rG\Delta J)/4Mc^2=0 \quad (4)$$

with the same equation holding for F. This yields with neglect of terms in $(m/M)^2$,

$$G_{(b-0)}\cong G_{(b+0)}\exp\left\{(1/4Mc^2)\int_{(b-0)}^{(b+0)} r\Delta J dr\right\} \cong G_{(b+0)}(1-m/4Mb'') \quad (4.1)$$

where b'' is the value of b in units e^2/mc^2 as in I. The functions for $r>b$ can be taken to be the same as before the introduction of the charged shell, since the change in normalization caused by the decrease in the functions in the interior is found to be negligible. Consequently, the only effect of the transition region is to diminish F and G inside the proton. This region is excluded from the hyperfine structure integral I for that special case of a constant J as discussed above and therefore the transition region does not have an appreciable effect on the hyperfine structure.

If one assumes a potential which is not constant in the region $0<r<b$, however, the quantity dJ/dr is not zero there and the hyperfine structure integral will include this region. In this case, the transition region contributes effects of order $\alpha^2 m/M$ since the wave functions are decreased in order m/M over a region of order α^2 of the important range of integration.

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[2] G. Breit and G. E. Brown, Phys. Rev. **76**, 1307 (1949). This paper will be referred to as III in the text. Formulas occurring in this paper will be referred to by prefixing the Roman numeral III ahead of the number of the formula.

[3] The quantity b' does not correspond to b of the preceding paper in a manner strictly adhering to the relations of the other primed quantities in this note to the unprimed ones in I. In I, b is expressed in units a_H whereas b'' is its value in e^2/mc^2 and corresponds to the b' of the present paper.