Crystal Statistics. II. Partition Function Evaluated by Spinor Analysis

BRURIA KAUFMAN*
Columbia University, New York City, New York
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The partition function for a two-dimensional binary lattice is evaluated in terms of the eigenvalues of the 2^n-dimensional matrix $\mathbf{V}$ characteristic for the lattice. Use is made of the properties of the 2^n-dimensional "spin"-representation of the group of rotations in $2n$-dimensions. In consequence of these properties, it is shown that the eigenvalues of $\mathbf{V}$ are known as soon as one knows the angles of the $2n$-dimensional rotation represented by $\mathbf{V}$.

Together with the eigenvalues of $\mathbf{V}$, the matrix $\mathbf{\Psi}$ which diagonalizes $\mathbf{V}$ is obtained as a spin-representation of a known rotation. The determination of $\mathbf{\Psi}$ is needed for the calculation of the degree of order.

The approximation, in which all the eigenvalues of $\mathbf{V}$ but the largest are neglected, is discussed, and it is shown that the exact partition function does not differ much from the approximate result.

THE partition function for a 2-dimensional binary alloy has been evaluated exactly by Onsager,[1] using the approach introduced by Kramers and Wannier,[2] and Montroll.[3] According to this method, a matrix characteristic of the crystal is set up, and it is shown that the partition function is approximately equal to the largest eigenvalue of the characteristic matrix. The eigenvalue problem has been treated approximately by various authors.[4] Onsager has shown that the characteristic matrix is decomposable into a direct product of 2-dimensional matrices and so was able to find the exact eigenvalues through the solution of n-quadratic equations.

In this paper, a shorter method, though closely related, is employed to reduce the characteristic matrix, $\mathbf{V}$. It is shown that $\mathbf{V}$ is a 2^n-dimensional "spin"-representative of a $2n$-dimensional rotation (i.e., orthogonal matrix), and that, as a consequence, its eigenvalues were known as soon as the eigenvalues of the rotation were given. Further considerations of symmetry reduce the $2n$-dimensional matrix into a product of n-plane rotations, so that again the eigenvalues are found from the solution of n-quadratic equations.

As a by-product of the solution of the eigenvalue problem, we are also able to find the matrix $\mathbf{\Psi}$ of eigenvectors of $\mathbf{V}$. Since every operation in 2^n-space is mirrored in the $2n$-dimensional space, the operation of $\mathbf{\Psi}$ corresponds to the diagonalization of the $2n$-dimensional rotation. This latter operation is easy to carry out, again because of the symmetry of the $2n$-dimensional rotation.

As has been shown,[5] the knowledge of $\mathbf{\Psi}$ makes it possible to state the average probabilities of crystal configurations. In particular, it is possible to evaluate average correlations between pairs of atoms within the crystal. This will be done in III.

1. SETTING UP THE MATRIX PROBLEM

Our physical model is a rectangular lattice with m rows and n sites per row. These sites are to be occupied by two kinds of constituent atoms. We assume that interactions exist only between nearest neighbors, and that the energy of interaction is $+J$ between unlike neighbors in a row, and $+J'$ between unlike neighbors in a column; but that it is $-J$, $-J'$ between like neighbors in a row, or column, respectively. The total energy of a configuration of the lattice is found simply by counting the number of like neighbors in the lattice, and subtracting from it the number of unlike neighbors. (Since we distinguish between row and column interactions, this count has to be made separately for the J and the J' (see Fig. 1).)

The probability of finding the lattice in a given configuration, at the temperature T, is proportional to $\exp\{-E_c/kT\}$, where E_c is the total energy of the configuration. Clearly, the exponent which appears in the expression for the probability will always be of the form

$$(n_c\cdot J+n_c'\cdot J')/kT.$$

Here n_c, n_c' are integers which depend on the configuration of the lattice. It is convenient to introduce the variables $H\equiv J/kT$, $H'\equiv J'/kT$ instead of J, J'. And then the probability of a configuration is

$$(1/Z)\cdot\exp\{n_cH+n_c'H'\}.$$

Z is the "partition function" for the lattice,

$$Z=\sum_{\substack{\text{all}\\ \text{configurations}}} e^{n_cH+n_c'H'}.$$

The thermodynamic functions for the crystal can be

* Now at the Institute for Advanced Study, Princeton, New Jersey.

[1] L. Onsager, "Crystal statistics. I. A two-dimensional model with an order-disorder transition," Phys. Rev. **65**, 117 (1944). This paper, first in the present series, will be henceforth referred to as I.

[2] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252, 263 (1941).

[3] E. Montroll, J. Chem. Phys. **9**, 706 (1941).

[4] H. A. Kramers and G. H. Wannier (footnote 2); E. Montroll (footnote 3); E. N. Lassettre and J. P. Howe, J. Chem. Phys. **9**, 747 (1941); J. Ashkin and W. E. Lamb, Jr., Phys. Rev. **64**, 159 (1943).

[5] E. N. Lassettre and J. P. Howe, J. Ashkin and W. E. Lamb, Jr. (footnote 4).

found from a knowledge of Z, and, as can be seen from its definition, it is a simple matter to calculate Z for small crystals. However, since the number of terms in Z is $2^{m\cdot n}$, it is not feasible to carry out such a calculation except when m and n are of the order of unity. Nor is it necessary to proceed in such a tedious manner. It has been shown[2,3] that the terms in Z are members of the mth power of a matrix $\mathbf{V}$ which is characteristic for the lattice. Z is then seen to be the trace of $\mathbf{V}^m$.

To show this, it is convenient to introduce the fiction of spin attributed to each atom. All atoms of one kind will be given the spin $+1$, and the other kind -1. When this is done, the interaction between two neighboring atoms with spins μ, μ' is: $-\mu\mu' H$ (or $-\mu\mu' H'$) for row (or column) neighbors.

The configurations of the crystal can now be specified by stating the value of μ at each site of the crystal. However, it is more convenient to specify row-configurations. Since there are n atoms in a row, there are 2^n possible configurations, $1\leqslant\nu\leqslant 2^n$. And then the configuration of the crystal is given by the set $\{\nu_1, \nu_2, \cdots, \nu_m\}$.

The energy due to interactions within a row may be denoted by $E(\nu_i)$; the energy due to interaction between two adjacent rows—by $E(\nu_i, \nu_{i+1})$. As a result, the energy of a configuration of the crystal is

$$E_c=\sum_{i=1}^{m} E(\nu_i)+\sum_{i=1}^{m} E(\nu_i, \nu_{i+1}).$$

(Here it is assumed, for purposes of symmetry, that the mth row of the crystal interacts with the first row.) If we now make the abbreviations

$$(\mathbf{V}_1)_{\nu_i\nu_{i+1}}\equiv\exp\{-E(\nu_i, \nu_{i+1})/kT\}, \tag{1}$$

$$(\mathbf{V}_2)_{\nu_i\nu_i}\equiv\exp\{-E(\nu_i)/kT\}, \tag{2}$$

we see that the probability of a configuration is proportional to

$$e^{-E_c/kT}=(\mathbf{V}_2)_{\nu_1\nu_1}(\mathbf{V}_1)_{\nu_1\nu_2}(\mathbf{V}_2)_{\nu_2\nu_2}\times(\mathbf{V}_1)_{\nu_2\nu_3}\cdots(\mathbf{V}_2)_{\nu_m\nu_m}(\mathbf{V}_1)_{\nu_m\nu_1}.$$

We therefore get for the partition function

$$Z=\sum_{\nu_1,\nu_2,\cdots\nu_m}(\mathbf{V}_2)_{\nu_1\nu_1}(\mathbf{V}_1)_{\nu_1\nu_2}\cdots(\mathbf{V}_1)_{\nu_m\nu_1}\equiv\text{trace }(\mathbf{V}_2\mathbf{V}_1)^m. \tag{3}$$

Since for each i: $1\leqslant\nu_i\leqslant 2^n$, we see that $\mathbf{V}_1$ and $\mathbf{V}_2$ are 2^n-dimensional matrices, and $\mathbf{V}_2$ is diagonal. $\mathbf{V}_1$ and $\mathbf{V}_2$ have been given explicitly in I:

$$\mathbf{V}_2=\exp\{H'\cdot\sum_{r=1}^{n}\mathbf{s}_r\mathbf{s}_{r+1}\}\equiv\exp\{H'\cdot\mathbf{A}\},$$
$$\mathbf{V}_1=(2\sinh 2H)^{n/2}\cdot\exp\{H^*\cdot\sum_{r=1}^{n}\mathbf{C}_r\}. \tag{4}$$

Here $\mathbf{s}_r$ and $\mathbf{C}_r$ are 2^n-dimensional quaternion matrices,

$$\mathbf{s}_r\equiv\mathbf{1}\times\mathbf{1}\times\cdots\times\mathbf{s}\times\mathbf{1}\times\cdots\times\mathbf{1}$$
$$\mathbf{C}_r\equiv\mathbf{1}\times\mathbf{1}\times\cdots\times\mathbf{C}\times\mathbf{1}\times\cdots\times\mathbf{1}; \tag{5}$$

there are n factors in each direct-product, with $\mathbf{s}$ or $\mathbf{C}$ appearing in the rth position. $\mathbf{s}$ and $\mathbf{C}$ are generators of the Pauli matrices:

$$\mathbf{s}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \mathbf{1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{6}$$

H^* is defined by

$$e^{-2H}\equiv\tanh H^*. \tag{7}$$

We may redefine $\mathbf{V}_1$ so as to remove from it the scalar coefficient:

$$\mathbf{V}_1\equiv\exp\{H^*\sum_{r=1}^{n}\mathbf{C}_r\}\equiv\exp\{H^*\cdot\mathbf{B}\}. \tag{8}$$

Then

$$Z=(2\sinh 2H)^{n/2}\cdot\text{trace }(\mathbf{V}_2\mathbf{V}_1)^m=(2\sinh 2H)^{n/2}\cdot\sum_{i=1}^{2^n}\lambda_i{}^m, \tag{9}$$

where λ_i are the eigenvalues of $\mathbf{V}\equiv\mathbf{V}_2\cdot\mathbf{V}_1$.

We now propose to show that matrices of the type $\exp(a\cdot\mathbf{s}_r\mathbf{s}_{r+1})$, $\exp\{b\cdot\mathbf{C}_r\}$, and their products, form a 2^n-dimensional representation of the group of rotations in $2n$-dimensions. Thus, the matrix $\mathbf{V}$ itself will be the representative of some such rotation.

We will further show that there is a very simple relationship between the eigenvalues of the rotation and the eigenvalues of its 2^n-dimensional representative. As a result, all our work will be reduced to finding eigenvalues in a $2n$-dimensional space.

2. ELEMENTS OF SPINOR ANALYSIS

The relation between the $2n$- and 2^n-spaces is best brought out by the study of sets of anticommuting matrices. The Pauli and Dirac matrices are examples of such sets. The general case was treated by Brauer and Weyl.[6] We will now derive those of their results which will be needed in what follows.[7]

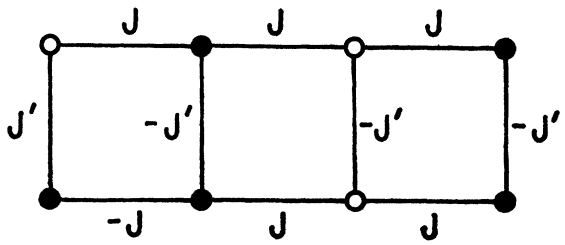

FIG. 1. Energy and probability of a given lattice configuration. The two kinds of atoms are denoted by ○ and ●.

[6] R. Brauer and H. Weyl, Am. J. Math. 57, 425 (1935). See also F. D. Murnaghan, *The Theory of Group Representations* (The Johns Hopkins Press, Baltimore, Maryland), Chapter 10.

[7] The discussion which follows is rather similar to that given by Pauli in treating the relativistic invariance of the Dirac wave-

We start out with a set of $2n$-quantities Γ_k which obey the commutation rules

$$\Gamma_k^2=\mathbf{1}, \quad \Gamma_k\Gamma_l=-\Gamma_l\Gamma_k, \quad (1\leqslant k, l, \leqslant 2n). \quad (10)$$

These quantities may be realized by matrices. For example, for the case $n=1$, a possible realization is: $\Gamma_1=\mathbf{s}$, $\Gamma_2=i\mathbf{sC}$. Γ_1 and Γ_2 are the generators of the set of Pauli spin matrices. That is to say, the complete set is formed by taking all products of the generators: $\mathbf{1}$, $\mathbf{s}$, $i\mathbf{sC}$, $\mathbf{C}=i\Gamma_1\Gamma_2$.

Similarly, for the case $n=2$, a possible realization is given by the generators of the set of Dirac matrices:

$$\Gamma_1=\mathbf{s}\times\mathbf{1}, \quad \Gamma_2=i\mathbf{sC}\times\mathbf{1}, \quad \Gamma_3=\mathbf{C}\times\mathbf{s}, \quad \Gamma_4=\mathbf{C}\times i\mathbf{sC}.$$

In the general case we may choose for the Γ_k:

$$\begin{aligned}\Gamma_{2r-1}&\equiv\mathbf{C}\times\mathbf{C}\times\cdots\times\mathbf{s}\times\mathbf{1}\times\mathbf{1}\times\cdots\equiv\mathbf{P}_r,\\ \Gamma_{2r}&\equiv\mathbf{C}\times\mathbf{C}\times\cdots\times i\mathbf{sC}\times\mathbf{1}\times\mathbf{1}\times\cdots\equiv\mathbf{Q}_r,\\ &\qquad\qquad 1\leqslant r\leqslant n,\end{aligned} \quad (11)$$

n factors appear in each product; $\mathbf{s}$ or $i\mathbf{sC}$ appear in the rth place. The Γ_k are thus 2^n-dimensional matrices.

If we take all possible products of the Γ_k, we form a set of 2^{2n} matrices:

$$1, \Gamma_1, \Gamma_2, \cdots, \Gamma_{2n}, \Gamma_1\cdot\Gamma_2, \Gamma_1\cdot\Gamma_3, \cdots, \Gamma_1\cdot\Gamma_2\cdot\Gamma_3, \cdots.$$

And now, any 2^n-dimensional matrix can be written as a linear combination of these "base matrices," just as any point in a vector space can be specified in terms of the "base vectors" which span that space. For this reason, the base matrices are said to span the complete algebra of 2^n-dimensional matrices.[8]

In particular, $\mathbf{V}_1$ and $\mathbf{V}_2$ are easily expressed in terms of the base matrices. We have:

$$\begin{aligned}\mathbf{C}_r&=i\mathbf{P}_r\mathbf{Q}_r=\mathbf{1}\times\mathbf{1}\times\cdots\times\mathbf{C}\times\mathbf{1}\times\cdots,\\ \mathbf{s}_r&=\mathbf{C}_1\mathbf{C}_2\cdots\mathbf{C}_{r-1}\mathbf{P}_r=\mathbf{1}\times\mathbf{1}\times\cdots\times\mathbf{s}\times\mathbf{1}\times\cdots.\end{aligned} \quad (12)$$

Therefore

$$\mathbf{V}_1=\prod_{r=1}^{n}\exp\{iH^*\mathbf{P}_r\mathbf{Q}_r\}. \quad (13)$$

On the other hand, since

$$\mathbf{s}_{r+1}\mathbf{s}_r=-i\mathbf{P}_{r+1}\mathbf{Q}_r \quad \text{for} \quad 1\geqslant r\geqslant n-1,$$

but

$$\mathbf{s}_1\mathbf{s}_n=+i\mathbf{P}_1\mathbf{Q}_n\cdot\mathbf{C}_1\mathbf{C}_2\cdots\mathbf{C}_n\equiv i\mathbf{P}_1\mathbf{Q}_n\cdot\mathbf{U},$$

we have[9]

$$\mathbf{V}_2=\prod_{r=1}^{n-1}\exp\{-iH'\mathbf{P}_{r+1}\mathbf{Q}_r\}\cdot\exp\{iH'\mathbf{P}_1\mathbf{Q}_n\mathbf{U}\}. \quad (14)$$

The end factor differs from the others. This is a somewhat annoying feature of the boundary, which will be treated in detail below. The new notation for $\mathbf{V}_1$, $\mathbf{V}_2$ reveals a striking similarity between these two matrices. $\mathbf{V}_2$ is obtained (except for its end factor) from $\mathbf{V}_1$ by replacing $\mathbf{P}_r$ by $\mathbf{P}_{r+1}$ in all its factors. Such a transformation (in $2n$-space) is the basic operation in spinor analysis, and it will be seen to have same effect as a similarity transformation in 2^n-space.

An Alternative Realization

In (11) we have given a particular matrix realization of the set of quantities Γ_k which obey the commutation rules

$$\Gamma_l\Gamma_k+\Gamma_k\Gamma_l=2\delta_{kl}. \quad (10)$$

Clearly, the commutation rules remain invariant if all the Γ_k undergo a similarity transformation in 2^n-space. Let $\Gamma_k^*=\mathbf{S}\Gamma_k\mathbf{S}^{-1}$ $(1\leqslant k\leqslant n)$. Then

$$\Gamma_k^*\Gamma_l^*+\Gamma_l^*\Gamma_k^*=2\delta_{kl}.$$

Thus we see that the set Γ_k^* provides another matrix realization obeying (10). For example, the set

$$\begin{aligned}\Gamma_{2k-1}^*&=\mathbf{s}\times\mathbf{s}\times\cdots\times\mathbf{C}\times\mathbf{1}\times\mathbf{1}\times\cdots\equiv\mathbf{P}_k^*,\\ &\qquad\qquad (1\leqslant k\leqslant n),\\ \Gamma_{2k}^*&=\mathbf{s}\times\mathbf{s}\times\cdots\times i\mathbf{Cs}\times\mathbf{1}\times\mathbf{1}\times\cdots\equiv\mathbf{Q}_k^*,\end{aligned} \quad (15)$$

is equally as good a realization as (11). It is obtained from (11) by a transformation which interchanges $\mathbf{C}$ and $\mathbf{s}$ in all operators. This realization is often easier to handle than (11), and we will use it upon occasion.

The transformation from Γ_k to the Γ_k^* given in (15) is effected by

$$\mathbf{g}=2^{n/2}(\mathbf{C}+\mathbf{s})\times(\mathbf{C}+\mathbf{s})\times\cdots\times(\mathbf{C}+\mathbf{s})=\mathbf{g}^{-1}$$

$$\mathbf{C}+\mathbf{s}=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \quad (16)$$

(To see this, note that $\frac{1}{2}(\mathbf{C}+\mathbf{s})^2=\mathbf{1}$ and $\frac{1}{2}(\mathbf{C}+\mathbf{s})\cdot\mathbf{s}\cdot(\mathbf{C}+\mathbf{s})=\mathbf{C}$.) In this realization we have, of course, $i\mathbf{P}_r^*\mathbf{Q}_r^*=\mathbf{s}_r$, and thus

$$\mathbf{g}\cdot\mathbf{V}_1\cdot\mathbf{g}=\prod_{r=1}^{n}\exp\{iH^*\mathbf{P}_r^*\mathbf{Q}_r^*\}$$

is diagonal in this coordinate system. On the other hand, $-i\mathbf{P}_{r+1}^*\mathbf{Q}_r^*=\mathbf{C}_{r+1}\mathbf{C}_r$ and therefore $\mathbf{V}_2\rightarrow\mathbf{g}\mathbf{V}_2\mathbf{g}$, which is no longer diagonal.

The Spin-Representation of the Orthogonal Group

We have seen that if the set Γ_k is a matrix realization of the commutation rules (10), then all the sets $\mathbf{S}\Gamma_k\mathbf{S}^{-1}$ are also realizations of (10).

The converse of this statement is a very important theorem for our purpose: *If two sets of matrices, Γ_k and Γ_k^*, both obey the commutation rules (10), then a transformation $\mathbf{S}$ can be found such that $\Gamma_k^*=\mathbf{S}\Gamma_k\mathbf{S}^{-1}$.* If we agree to consider two sets of matrices which are related set-wise by a similarity transformation as being the same set, we can say simply: For a given n there is only

equation [*Handbuch der Physik* (Verlag. Julius Springer, Berlin), Vol. 24/1, second edition, pp. 219–224].

[8] The proof of the completeness of the algebra is given by Brauer and Weyl (footnote 6). See also I, p. 122.

[9] The operator $\mathbf{U}$ was denoted by $\mathbf{C}$ in I. See Eq. (32) in I.

one realization of the commutation rules. This theorem is not trivial. Its proof[6,10] rests upon the fact that the only possible automorphism of a complete matrix algebra is given by a similarity transformation.

In what follows, we will assume that we have already found one realization Γ_k (e.g., as in (11)). We will then show that certain linear combinations of the Γ_k will also obey the commutation rules. Therefore, according to the theorem above, these linear combinations (Γ_k^*) must be related to the Γ_k by a similarity transformation. This gives us two relations between the Γ_k and the Γ_k^*. One of these relations will be seen to be a "rotation" in $2n$-space, and the other will be referred to as the "spin-representation" in 2^n-space of the rotation in $2n$-space. Let the linear combinations be written as

$$\Gamma_k^*=\sum_{j=1}^{2n} o_{jk}\Gamma_j, \quad 1\leqslant k\leqslant 2n, \tag{17}$$

where o_{jk} are as yet arbitrary numbers, in general complex. Then:

$$(\Gamma_k^*)^2=(\textstyle\sum_j o_{jk}\Gamma_j)^2=\sum_i o_{ik}o_{jk}\Gamma_i\Gamma_j.$$

In this sum, terms will cancel in pairs. For $i\neq j$, $o_{jk}o_{ik}\Gamma_j\Gamma_i+o_{ik}o_{jk}\Gamma_i\Gamma_j=0$ because of (10). Only the terms $i=j$ have no partners and do not cancel. But by (10), $\Gamma_j\Gamma_j=\mathbf{1}$. Therefore:

$$(\Gamma_k^*)^2=\sum_{j=1}^{2n} o_{jk}^2.$$

If we now demand of the constants o_{jk} that they satisfy

$$\sum_{j=1}^{2n} o_{jk}^2=1, \quad (1\leqslant k\leqslant 2n),$$

we find that $(\Gamma_k^*)^2=\mathbf{1}$. Similarly, if we demand that

$$\sum_{j=1}^{2n} o_{jk}o_{jl}=0, \quad k\neq l, \quad (1\leqslant k,l\leqslant 2n),$$

we find $\Gamma_k^*\Gamma_l^*=-\Gamma_l^*\Gamma_k^*$. Thus the set Γ_k^* will form a realization of (10) if the o_{jk} fulfill:

$$\sum_{j=1}^{2n} o_{jk}o_{jl}=\delta_{kl}, \quad (1\leqslant k,l\leqslant 2n). \tag{18}$$

We may view the o_{jk} as members of a $2n$-dimensional matrix. From the restrictions on its members, this matrix is seen to be orthogonal. (If its members are real, it describes a rotation in $2n$-dimensional Euclidean space.) We then say that the rotation $\mathbf{o}$ operates on the $2n$ quantities Γ_k and sends them into Γ_k^*.

$$\mathbf{o}:\Gamma_k\rightarrow\Gamma_k^*=\sum_{j=1}^{2n} o_{jk}\Gamma_j, \quad (1\leqslant k\leqslant 2n). \tag{19}$$

[10] An elegant group-theoretical proof was also given by Jordan and Wigner, Zeits. f. Physik **47**, 631 (1928).

However, by the theorem, Γ_k^* is also related to Γ_k through a similarity transformation:[11]

$$\Gamma_k^*=\mathbf{S}(\mathbf{o})\cdot\Gamma_k\cdot\mathbf{S}(\mathbf{o})^{-1}, \quad 1\leqslant k\leqslant 2n. \tag{20}$$

We write $\mathbf{S}(\mathbf{o})$, to indicate that the similarity transformation depends on the orthogonal matrix $\mathbf{o}$, and proceed to discuss the nature of this dependence.

Consider the full group of orthogonal matrices $\mathbf{o}$ in $2n$-space. We show that the collection of matrices $\mathbf{S}(\mathbf{o})$ forms a 2^n-dimensional representation of the orthogonal group. That is to say: the correspondence between the $\mathbf{o}$ and $\mathbf{S}(\mathbf{o})$ matrices is such that $\mathbf{S}(\mathbf{o}'')$ which corresponds to the product $\mathbf{o}''=\mathbf{o}'\cdot\mathbf{o}$ is the product of the matrices corresponding to $\mathbf{o}$ and $\mathbf{o}'$:

$$\mathbf{S}(\mathbf{o}'')=\mathbf{S}(\mathbf{o}'\cdot\mathbf{o})=\mathbf{S}(\mathbf{o}')\cdot\mathbf{S}(\mathbf{o}). \tag{21}$$

To show this, let

$$\mathbf{o}:\Gamma_k\rightarrow\sum_{j=1}^{2n} o_{jk}\cdot\Gamma_j=\mathbf{S}(\mathbf{o})\cdot\Gamma_k\cdot\mathbf{S}(\mathbf{o})^{-1},$$

$$\mathbf{o}':\Gamma_k\rightarrow\sum_{l=1}^{2n} o_{lk}'\cdot\Gamma_l=\mathbf{S}(\mathbf{o}')\cdot\Gamma_k\cdot\mathbf{S}(\mathbf{o}')^{-1}.$$

Combining the two operations we have

$$\mathbf{o}''=\mathbf{o}'\cdot\mathbf{o}:\Gamma_k\rightarrow\sum_{l=1}^{2n}\sum_{j=1}^{2n} o_{jk}o_{lj}'\Gamma_j$$

$$=\sum_{l=1}^{2n} o_{lk}''\Gamma_l=\mathbf{S}(\mathbf{o}'')\cdot\Gamma_k\cdot\mathbf{S}(\mathbf{o}'')^{-1}.$$

But on the other hand

$$\mathbf{o}'\cdot\mathbf{o}: \quad \Gamma_k\rightarrow\mathbf{S}(\mathbf{o}')\cdot\mathbf{S}(\mathbf{o})\cdot\Gamma_k\cdot\mathbf{S}(\mathbf{o})^{-1}\cdot\mathbf{S}(\mathbf{o})^{-1}.$$

As a result

$$\mathbf{S}(\mathbf{o}'\cdot\mathbf{o})=\mathbf{S}(\mathbf{o}'')=\mathbf{S}(\mathbf{o}')\cdot\mathbf{S}(\mathbf{o}).$$

(It is important to notice that the Γ_k transform according to the transpose—or, what is the same here, the reciprocal—of $\mathbf{o}$, and *not* according to $\mathbf{o}$ itself.)

Every rotation in $2n$-space thus has a counterpart transformation, $\mathbf{S}(\mathbf{o})$, in 2^n-dimensional "spin" space. For example, the rotation referred to above, which sends $\mathbf{P}_r\rightarrow\mathbf{P}_{r+1}$, $\mathbf{Q}_r\rightarrow\mathbf{Q}_r$ and therefore sends $\mathbf{V}_1\rightarrow\mathbf{V}_2$, has a representative in spin space, and as a result we have $\mathbf{V}_2=\mathbf{S}\mathbf{V}_1\mathbf{S}^{-1}$ (if we disregard the difficulty with the end factor in $\mathbf{V}_2$).

Eigenvalues of the Spin-Representatives

While in principle it is possible to find $\mathbf{S}(\mathbf{o})$ explicitly for any rotation $\mathbf{o}$, this is quite a complicated job in general. Fortunately we shall need to know $\mathbf{S}(\mathbf{o})$ only for the very simple case where $\mathbf{o}$ is the product of commuting plane rotations. Consider first one plane rotation. It operates on a pair of Γ's, say Γ_k and Γ_l, and

[11] In the case of the Dirac wave equation, the matrix $\mathbf{o}$ is the Lorentz-transformation in $2n=4$ dimensional coordinate-space. The Lorentz-transformation induces a transformation in $2^n=4$-dimensional "spin space," so that $\psi\rightarrow\psi'=\mathbf{S}(\mathbf{o})\cdot\psi$, where $\mathbf{S}(\mathbf{o})$ is the spin-representative of $\mathbf{o}$, and ψ is the 4-component Dirac wave function. See Pauli, reference 7.

leaves the others unchanged. Thus

$$\mathbf{K}:\quad \begin{aligned}&\Gamma_k\to\cos\theta\cdot\Gamma_k-\sin\theta\cdot\Gamma_l\equiv\Gamma_k^*,\quad \Gamma_i\to\Gamma_i,\\ &\Gamma_l\to\sin\theta\cdot\Gamma_k+\cos\theta\cdot\Gamma_l\equiv\Gamma_l^*,\quad i\neq k,\ l.\end{aligned}\tag{22}$$

We will verify that

$$\mathbf{S}(\mathbf{K})=\exp\{\theta/2\cdot\Gamma_k\Gamma_l\},\tag{23}$$

or, using the series expansion of the right-hand term:

$$\mathbf{S}(\mathbf{K})=\cos(\theta/2)+\sin(\theta/2)\cdot\Gamma_k\Gamma_l,\tag{24}$$

(Γ_k, Γ_l are known 2^n-dimensional matrices; $\cos(\theta/2)$ stands for the 2^n-dimensional unit matrix multiplied by the scalar $\cos(\theta/2)$. Thus $\mathbf{S}(\mathbf{K})$ is given explicitly by (24)); and indeed,

$$\begin{aligned}&\mathbf{S}(\mathbf{K})\cdot\Gamma_k\cdot\mathbf{S}(\mathbf{K})^{-1}\\ &=\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\Gamma_k\Gamma_l\right)\Gamma_k\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\Gamma_k\Gamma_l\right)\\ &=\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)\Gamma_k-\left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)\cdot\Gamma_l\\ &=\cos\theta\cdot\Gamma_k-\sin\theta\cdot\Gamma_l=\Gamma_k^*.\end{aligned}$$

Similarly $\mathbf{S}(\mathbf{K})\Gamma_l\mathbf{S}(\mathbf{K})^{-1}=\sin\theta\cdot\Gamma_k+\cos\theta\cdot\Gamma_l=\Gamma_l^*$. This proves the assertion about the explicit form of $\mathbf{S}(\mathbf{K})$.

The half-angle which appears in the explicit form of $\mathbf{S}(\mathbf{K})$ is a characteristic feature of the spin-representation. It shows that the group of matrices $\mathbf{S}(\mathbf{o})$ provides a double-valued representation for the orthogonal group. Clearly, if we add 2π to the angle θ in (22), the rotation $\mathbf{K}$ is unchanged, but $\mathbf{S}(\mathbf{K})$ is multiplied by -1 (see (24)).

Thus we find that $\mathbf{K}$ is represented by two inequivalent matrices, $\mathbf{S}(\mathbf{K})$ and $-\mathbf{S}(\mathbf{K})$. On the other hand, to any $\mathbf{S}(\mathbf{K})$ there corresponds only one rotation $\mathbf{K}$.

The angles of rotation in $\mathbf{K}$ (and in a general rotation $\mathbf{o}$) are not restricted to be real. They may be pure imaginary, or even complex numbers. In the next section it will be seen that the matrix $\mathbf{V}$ represents a rotation with pure imaginary angles. We may then write $\theta=i\gamma$ and obtain $\cosh\gamma$, $-i\sinh\gamma$ instead of $\cos\theta$, $\sin\theta$.

$\mathbf{S}(\mathbf{K})$ has a particularly simple form in the case that $\Gamma_k=\mathbf{P}_r^*$, $\Gamma_l=\mathbf{Q}_r^*$ (with the notation of (15)). We have then:

$$\begin{aligned}&\mathbf{S}(\mathbf{K})=\exp\{\theta/2\cdot\mathbf{P}_r^*\mathbf{Q}_r^*\}\\ &=\exp\{-i\theta/2\cdot\mathbf{s}_r\}=\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\cdot\mathbf{s}_r\\ &=1\times1\times\cdots\times\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\cdot\mathbf{s}\right)\\ &\qquad\times1\times1\times\cdots\times1\\ &=1\times1\times\cdots\times\begin{pmatrix}e^{-i\theta/2}&0\\0&e^{i\theta/2}\end{pmatrix}\\ &\qquad\times1\times\cdots\times1.\end{aligned}\tag{25}$$

From this we see that the eigenvalues of $\mathbf{S}(\mathbf{K})$ are $e^{i\theta/2}$, $e^{-i\theta/2}$ with a 2^{n-1} fold degeneracy for each value. The rotation $\mathbf{K}$ itself has the eigenvalues: $e^{i\theta}$, $e^{-i\theta}$, and $+1$, the latter being $2(n-1)$-fold degenerate.

Consider now a product of n commuting plane rotations. The $2n$ Γ's are grouped into pairs, and each pair $(\Gamma_{r1}, \Gamma_{r2})$ is rotated by an angle θ_r, as in (22). We have

$$\mathbf{K}=\prod_{r=1}^{n}\mathbf{K}_r,\tag{26}$$

and the eigenvalues of $\mathbf{K}$ are

$$e^{\pm i\theta_1},\ e^{\pm i\theta_2},\ \cdots,\ e^{\pm i\theta_n}.\tag{27}$$

Since we know the representative of each factor in $\mathbf{K}$, we also know the representative of the product

$$\mathbf{S}(\mathbf{K})=\prod_{r=1}^{n}\mathbf{S}(\mathbf{K}_r)=\prod_{r=1}^{n}\exp\{\theta_r/2\cdot\Gamma_{r1}\Gamma_{r2}\}.\tag{28}$$

In the special case where $\Gamma_{r1}=\mathbf{P}_r^*$, $\Gamma_{r2}=\mathbf{Q}_r^*$, we have

$$\begin{aligned}&\mathbf{S}(\mathbf{K})=\prod_{r=1}^{n}\exp(\tfrac{1}{2}\theta_r\mathbf{P}_r^*\mathbf{Q}_r^*)\\ &=\begin{pmatrix}\exp[-(i/2)\theta_1]&0\\0&\exp[(i/2)\theta_1]\end{pmatrix}\\ &\times\begin{pmatrix}\exp[-(i/2)\theta_2]&0\\0&\exp[(i/2)\theta_2]\end{pmatrix}\times\cdots\\ &\qquad\times\begin{pmatrix}\exp[-(i/2)\theta_n]&0\\0&\exp[(i/2)\theta_1]\end{pmatrix},\end{aligned}\tag{29}$$

with the eigenvalues

$$\lambda=\exp[i/2(\pm\theta_1\pm\theta_2\pm\cdots\pm\theta_n)].\tag{30}$$

All sign combinations are to be taken, giving the expected 2^n eigenvalues for $\mathbf{S}(\mathbf{K})$. Even for a more general choice of Γ_{r1}, Γ_{r2} we find the same eigenvalues for $\mathbf{S}(\mathbf{o})$. This is so because we can pass from any matrix base $\cdots$, Γ_{r1}, Γ_{r2}, $\cdots$ to the base $\cdots$, $\mathbf{P}_r^*$, $\mathbf{Q}_r^*$, $\cdots$ via a transformation in 2^n-space, according to the theorem on p. 1234. But this transformation does not change the eigenvalues of $\mathbf{S}(\mathbf{o})$.

To summarize, if the rotation $\mathbf{o}$ has the eigenvalues

$$e^{i\theta_1},\ e^{-i\theta_1},\ e^{i\theta_2},\ e^{-i\theta_2},\ \cdots,\ e^{i\theta_n},\ e^{-i\theta_n},$$

then its spin-representative $\mathbf{S}(\mathbf{o})$ has the eigenvalues

$$\lambda=\exp[i/2(\pm\theta_1\pm\theta_2\pm\cdots\pm\theta_n)],$$

for any choice of axes in 2^n spin space.

Other Representations of the Orthogonal Group

Besides the spin-representation, the orthogonal group also has representations of dimension $\binom{2n}{f}$ $1\leqslant f\leqslant 2n$. These are easily found by considering the transformation of a product of f Γ's under the rotation $\mathbf{o}$. For

example (for $k\neq l$)

$$\mathbf{o}:\Gamma_k\Gamma_l\rightarrow(\sum_{i=1}^{2n} o_{ik}\Gamma_i)(\sum_{j=1}^{2n} o_{jl}\Gamma_j)=\sum_{i,j=1}^{2n} o_{ik}o_{jl}\Gamma_i\Gamma_j$$

$$=\sum_{1\leqslant i\leqslant j\leqslant 2n}(o_{ik}o_{jl}-o_{jk}o_{il})\cdot\Gamma_i\Gamma_j+\sum_{i=j=1}^{2n} o_{ik}o_{il}.$$

But the second sum vanishes, by (18), and so

$$\mathbf{o}:\Gamma_k\Gamma_l\rightarrow\sum_{i<j}\begin{vmatrix} o_{ik} & o_{jk}\\ o_{il} & o_{jl}\end{vmatrix}\Gamma_i\Gamma_j=\mathbf{S}(\mathbf{o})\cdot\Gamma_k\Gamma_l\cdot\mathbf{S}(\mathbf{o})^{-1}. \quad (31)$$

In general, the product $\Gamma_{r1}\cdot\Gamma_{r2}\cdots\Gamma_{rf}$ will be transformed into a linear combination of all products of the same rank. The coefficients in this linear combination will be the corresponding minors in the determinant of $\mathbf{o}$.

Particularly important is the product

$$\mathbf{U}\equiv\Gamma_1\cdot\Gamma_2\cdot\Gamma_3\cdots\Gamma_{2n}. \quad (32)$$

There is only one product of this rank. When the rotation $\mathbf{o}$ is applied to the Γ's, $\mathbf{U}$ is merely multiplied by the determinant of $\mathbf{o}$. Now, $\mathbf{o}$ is orthogonal, so that its determinant is either $+1$ (a proper rotation) or -1 (an improper rotation). Thus

$$\mathbf{o}:\quad \mathbf{U}\rightarrow\pm\mathbf{U}=\mathbf{S}(\mathbf{o})\cdot\mathbf{U}\cdot\mathbf{S}(\mathbf{o})^{-1}. \quad (33)$$

However, transformation by a matrix which is not a spin representative of any rotation will, in general, not leave $\mathbf{U}$ invariant. For example

$$\mathbf{g}\cdot\mathbf{U}\cdot\mathbf{g}\neq\pm\mathbf{U}.$$

3. THE EIGENVALUES AND EIGENVECTORS OF V

Dual Transformation

As was mentioned above, $\mathbf{V}_2$ is a transform of $\mathbf{V}_1$ through the rotation which sends $\mathbf{P}_r\rightarrow\mathbf{P}_{r+1}$, $\mathbf{Q}_r\rightarrow\mathbf{Q}_r$. This similarity between $\mathbf{V}_1$ and $\mathbf{V}_2$ was recognized by Onsager, who made use of a somewhat different transformation,[1] $\mathbf{D}$: $\mathbf{P}_r\rightarrow\mathbf{Q}_r$, $\mathbf{Q}_r\rightarrow\mathbf{P}_{r+1}$. As a result,

$$\mathbf{D}:\quad i\mathbf{P}_r\mathbf{Q}_r\rightarrow i\mathbf{Q}_r\mathbf{P}_{r+1}=-i\mathbf{P}_{r+1}\mathbf{Q}_r,$$
$$-i\mathbf{P}_{r+1}\mathbf{Q}_r\rightarrow-i\mathbf{Q}_{r+1}\mathbf{P}_{r+1}=i\mathbf{P}_{r+1}\mathbf{Q}_{r+1},$$

or, in the notation of I:

$$\mathbf{C}_r\rightarrow\mathbf{s}_r\mathbf{s}_{r+1},\quad \mathbf{s}_r\mathbf{s}_{r+1}\rightarrow\mathbf{C}_{r+1}.$$

Thus transformation by $\mathbf{D}$ interchanges $\mathbf{V}_1$ and $\mathbf{V}_2$. This provides the algebraic reason for the "dual transformation" found by Kramers and Wannier:[2] if $\mathbf{V}_1=\mathbf{V}_1(H^*)$ and $\mathbf{V}_2=\mathbf{V}_2(H')$, one has

$$\mathbf{D}:\quad \mathbf{V}_1(H^*)\cdot\mathbf{V}_2(H')\rightarrow\mathbf{V}_2(H^*)\cdot\mathbf{V}_1(H').$$

To each lattice there corresponds a "dual" lattice, in which the roles of H^* and H' are interchanged, and the partition function for the lattice described by $\mathbf{V}(H, H')$ is the same as the partition function of the dual lattice, with $\mathbf{V}(H'^*, H^*)$—except for a factor $(\sinh 2H\cdot\sinh 2H')^{-mn/2}$.

This fact enabled Kramers and Wannier to locate the critical temperature. In their model $H'\equiv H$, and the critical point occurs at the temperature where the equality $H=H^*$ is satisfied. In our model the critical point is at $H=H'^*$, or, in other words, at the temperature where

$$\sinh 2H\cdot\sinh 2H'=1.$$

Procedure for Diagonalizing V

The similarity between $\mathbf{V}_1$ and $\mathbf{V}_2$ may be utilized in order to decompose $\mathbf{V}$ into a direct product of simple components. It is possible to show that the transformation ($\mathbf{P}_r\rightarrow\mathbf{P}_{r+1}$, $\mathbf{Q}_r\rightarrow\mathbf{Q}_r$), which sends $\mathbf{V}_1$ into $\mathbf{V}_2$, can be written, in 2^n-space, as a direct product of two-dimensional matrices. And since $\mathbf{V}_1$ is also such a direct product (although $\mathbf{V}_2$ is not), the eigenvalue problem is reduced to the solution of n quadratic equations.

However, a much more direct approach is based on the fact that $\mathbf{V}_1$ and $\mathbf{V}_2$ are themselves spin representatives of certain rotations in $2n$-space, say $\mathbf{R}_1$ and $\mathbf{R}_2$. $\mathbf{V}$ is then a representative of $\mathbf{R}_1\cdot\mathbf{R}_2$, and its eigenvalues are known as soon as we know the eigenvalues of the $2n$-dimensional matrix $\mathbf{R}_1\cdot\mathbf{R}_2$. Now, the similarity between $\mathbf{V}_1$ and $\mathbf{V}_2$ implies that there is a similarity between $\mathbf{R}_1$ and $\mathbf{R}_2$; and, indeed, we will see that $\mathbf{R}_2$ is obtained from $\mathbf{R}_1$ by a permutation of rows and columns. As a result, it is quite an easy matter to find the eigenvalues of $\mathbf{R}_1\cdot\mathbf{R}_2$.

One preparatory step has to be taken: From (28) it follows that

$$\mathbf{V}_1=\sum_{r=1}^{n}\exp(iH^*\cdot\mathbf{P}_r\mathbf{Q}_r)$$

is the representative of a rotation with $\frac{1}{2}\theta_r=iH^*$, $1\leqslant r\leqslant n$. Similarly in $\mathbf{V}_2$, all factors except the last are representatives of plane rotations: $\exp(-iH'\cdot\mathbf{P}_{r+1}\mathbf{Q}_r)$. In order to bring this last "boundary" factor into line with the others, we note that

$$\begin{aligned}(1+\mathbf{U})\cdot i\mathbf{U}\cdot\mathbf{P}_1\mathbf{Q}_n&=(1+\mathbf{U})\cdot i\mathbf{P}_1\mathbf{Q}_n,\\ (1-\mathbf{U})\cdot i\mathbf{U}\cdot\mathbf{P}_1\mathbf{Q}_n&=-(1-\mathbf{U})\cdot i\mathbf{P}_1\mathbf{Q}_n.\end{aligned} \quad (34)$$

We can therefore write[12]

$$\mathbf{V}=\tfrac{1}{2}(1+\mathbf{U})\cdot\mathbf{V}+\tfrac{1}{2}(1-\mathbf{U})\cdot\mathbf{V}$$

$$=\tfrac{1}{2}(1+\mathbf{U})\cdot\{\prod_{r=1}^{n}\exp(iH^*\mathbf{P}_r\mathbf{Q}_r)$$

$$\cdot\prod_{r=1}^{n-1}\exp(-iH'P_{r+1}\mathbf{Q}_r)\cdot\exp(iH'\mathbf{P}_1\mathbf{Q}_n)\}$$

$$+\tfrac{1}{2}(1-\mathbf{U})\cdot\{\prod_{r=1}^{n}\exp(iH^*\mathbf{P}_r\mathbf{Q}_r)$$

$$\cdot\prod_{r=1}^{n}\exp(-iH'\mathbf{P}_{r+1}\mathbf{Q}_r)\}$$

$$\equiv\tfrac{1}{2}(1+\mathbf{U})\cdot\mathbf{V}^+ +\tfrac{1}{2}(1-\mathbf{U})\cdot\mathbf{V}^-. \quad (35)$$

[12] As noted in footnote 9, $\mathbf{U}$ is identical with the operator $\mathbf{C}$ in I, and fulfills the same function of splitting the space into two disjoint subspaces. Kramers and Wannier operate only in the subspace of the projection $\frac{1}{2}(1+\mathbf{U})$.

It is seen that not $\mathbf{V}$ itself, but $\mathbf{V}^+$ and $\mathbf{V}^-$ are representatives of rotations in $2n$-space.

Selection of the Eigenvalues in the Two Subspaces

To indicate that $\mathbf{V}^+$ and $\mathbf{V}^-$ are spin-representatives, we write $\mathbf{V}^+=\mathbf{S}(\mathbf{R}^+)$ and $\mathbf{V}^-=\mathbf{S}(\mathbf{R}^-)$. We will proceed to find the eigenvalues of each of these separately, and will take into account the effect of the factors $\frac{1}{2}(1+\mathbf{U})$, $\frac{1}{2}(1-\mathbf{U})$ by selecting half of the eigenvalues of $\mathbf{V}^+$ and half of those of $\mathbf{V}^-$. The two half-sets together will then constitute the full set of eigenvalues of $\mathbf{V}$. The justification for this procedure may perhaps be best seen in a different coordinate system from the one in which we have been working. Suppose that we change from our original coordinate system by transforming with the matrix

$$\mathbf{g}=2^{n/2}\cdot(\mathbf{C}+\mathbf{s})\times(\mathbf{C}+\mathbf{s})\times\cdots\times(\mathbf{C}+\mathbf{s}). \tag{16}$$

This interchanges $\mathbf{s}$ and $\mathbf{C}$ in all operators. Then (in the notation of (4))

$$\mathbf{B}\rightarrow\sum_1^n \mathbf{s}_r={}^\dagger\mathbf{B},\quad \mathbf{A}\rightarrow\sum_1^n \mathbf{C}_r\mathbf{C}_{r+1}={}^\dagger\mathbf{A},\qquad \mathbf{U}\rightarrow\mathbf{s}\times\mathbf{s}\times\cdots\times\mathbf{s}={}^\dagger\mathbf{U}. \tag{36}$$

${}^\dagger\mathbf{U}$ is a diagonal matrix in which only $+1$ and -1 appear along the diagonal. Imagine that we rearrange the order of the base vectors of our new coordinate system in such a way that first come all the $+1$ members of ${}^\dagger\mathbf{U}$ and then all the -1's. Then ${}^\dagger\mathbf{U}$ is merely a unit matrix in a subspace (a half) of the 2^n-dimensional space, and it is the negative of a unit matrix in the other half. ${}^\dagger\mathbf{U}$ commutes with ${}^\dagger\mathbf{A}$:

$$(\mathbf{s}\times\mathbf{s}\times\cdots\times\mathbf{s})\cdot(\sum_1^n \mathbf{C}_r\mathbf{C}_{r+1})=(\sum_1^n \mathbf{C}_r\mathbf{C}_{r+1})\cdot(\mathbf{s}\times\mathbf{s}\times\cdots\times\mathbf{s}).$$

And, of course, it commutes with ${}^\dagger\mathbf{B}$; it therefore commutes with ${}^\dagger\mathbf{V}$. (In general, ${}^\dagger\mathbf{U}$ commutes with any operator which is a product of an even number of $\mathbf{\Gamma}_k$; it anticommutes with products of an odd number of $\mathbf{\Gamma}_k$.) However, if any matrix $\mathbf{X}$ commutes with ${}^\dagger\mathbf{U}$, it must be of the form

$$\mathbf{X}=\begin{pmatrix}\mathbf{a} & \mathbf{0}\\ \mathbf{0} & \mathbf{b}\end{pmatrix}.$$

Thus ${}^\dagger\mathbf{V}$ is of this form too. On the other hand, if $\mathbf{Y}$ anticommutes with ${}^\dagger\mathbf{U}$, it has the form

$$\mathbf{Y}=\begin{pmatrix}\mathbf{0} & \mathbf{a}\\ \mathbf{b} & \mathbf{0}\end{pmatrix}.$$

$\frac{1}{2}(1+{}^\dagger\mathbf{U})$ is a projection of the unit matrix on one of the halves of our total space. $\frac{1}{2}(1-{}^\dagger\mathbf{U})$ is the complementary projection; that is:

$$\tfrac{1}{2}(1+{}^\dagger\mathbf{U})+\tfrac{1}{2}(1-{}^\dagger\mathbf{U})=\mathbf{1},\quad \tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot\tfrac{1}{2}(1-{}^\dagger\mathbf{U})=0.$$

Also:

$$[\tfrac{1}{2}(1+{}^\dagger\mathbf{U})]^2=\tfrac{1}{2}(1+{}^\dagger\mathbf{U});$$

similarly for $\frac{1}{2}(1-{}^\dagger\mathbf{U})$. As a result,

$$\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}=\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}\cdot\tfrac{1}{2}(1+{}^\dagger\mathbf{U})=\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}^+\cdot\tfrac{1}{2}(1+{}^\dagger\mathbf{U})$$

is the upper square of ${}^\dagger\mathbf{V}$ (also of ${}^\dagger\mathbf{V}^+$), whereas $\frac{1}{2}(1-{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}$ is the lower square of ${}^\dagger\mathbf{V}$ (or ${}^\dagger\mathbf{V}^-$):

$${}^\dagger\mathbf{V}=\begin{pmatrix}\mathbf{a} & \mathbf{0}\\ \mathbf{0} & \mathbf{b}\end{pmatrix}\equiv\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{a} & \mathbf{0}\\ \mathbf{0} & \mathbf{b}\end{pmatrix}+\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}\begin{pmatrix}\mathbf{a} & \mathbf{0}\\ \mathbf{0} & \mathbf{b}\end{pmatrix}.$$

Now, we may diagonalize the upper and lower squares of ${}^\dagger\mathbf{V}$ independently of each other. That is to say, we apply a transformation to $\frac{1}{2}(1+{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}^+$ which brings it into diagonal form, and another transformation—to diagonalize $\frac{1}{2}(1-{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}^-$.

It will be seen below, when the actual transformations are constructed, that both of these will be representatives of orthogonal rotations. But while the rotation $\mathbf{T}^+$ involved in diagonalizing $\mathbf{V}^+$ is a proper rotation, the other one, $\mathbf{T}^-$, is improper. As we have seen (Eq. (33)), this means that

$$\mathbf{T}^+\colon {}^\dagger\mathbf{U}\rightarrow+{}^\dagger\mathbf{U},\quad \mathbf{T}^-\colon {}^\dagger\mathbf{U}\rightarrow-{}^\dagger\mathbf{U}.$$

Therefore

$$\mathbf{S}(\mathbf{T}^+)\cdot\{\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}^+\}\cdot\mathbf{S}(\mathbf{T}^+)^{-1}=\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot\mathbf{S}(\mathbf{T}^+)\cdot({}^\dagger\mathbf{V}^+)\cdot\mathbf{S}(\mathbf{T}^+)^{-1}=\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot\mathbf{S}[(\mathbf{T}^+)\cdot(\mathbf{R}^+)\cdot(\mathbf{T}^+)^{-1}];$$

but

$$\mathbf{S}(\mathbf{T}^-)\cdot\{\tfrac{1}{2}(1-{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}^-\}\cdot\mathbf{S}(\mathbf{T}^-)^{-1}=\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot\mathbf{S}(\mathbf{T}^-)\cdot({}^\dagger\mathbf{V}^-)\cdot\mathbf{S}(\mathbf{T}^-)^{-1}=\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot\mathbf{S}[(\mathbf{T}^-)\cdot(\mathbf{R}^-)\cdot(\mathbf{T}^-)^{-1}]. \tag{37}$$

Suppose that we had already found the $2n$-eigenvalues of $\mathbf{R}^-$, and we denoted them by $\exp(\pm\gamma_{2r})$, $1\leqslant r\leqslant n$. By (29) this means that

$$\mathbf{S}(\mathbf{T}^-)\cdot({}^\dagger\mathbf{V}^-)\cdot\mathbf{S}(\mathbf{T}^-)^{-1}=\mathbf{S}(\mathbf{K})=\prod_{r=1}^n \exp\left(\frac{i}{2}\gamma_{2r}\mathbf{P}_r{}^*\mathbf{Q}_r{}^*\right)\equiv\mathbf{\Lambda}^-.$$

This is a diagonal matrix with components (eigenvalues):

$$\exp[\tfrac{1}{2}(\pm\gamma_2\pm\gamma_4\pm\cdots\pm\gamma_{2n})].$$

The particular combination of plus and minus signs appearing in each eigenvalue depends on the value of the components of $(\mathbf{P}_r{}^*\mathbf{Q}_r{}^*)$. Now, the factor $\frac{1}{2}(1+{}^\dagger\mathbf{U})$ eliminates half of these eigenvalues, keeping only those which fall into the upper square. In that square

$${}^\dagger\mathbf{U}_{ii}=+1=\prod_{r=1}^n(\mathbf{P}_r{}^*\mathbf{Q}_r{}^*)_{ii}.$$

On the other hand $(\mathbf{P}_r{}^*\mathbf{Q}_r{}^*)_{ii}$ may be either $+1$ or -1 for any $1\geqslant r\geqslant n$. In order to maintain ${}^\dagger\mathbf{U}_{ii}$ equal to $+1$, only an *even* number of factors $(\mathbf{P}_r{}^*\mathbf{Q}_r{}^*)_{ii}$ may equal -1. That is to say, only those eigenvalues remain in which an *even* number of angles appear with a minus sign.

The same considerations apply to ${}^\dagger\mathbf{V}^+$. Let the eigenvalues of $\mathbf{R}^+$ be denoted by $\exp(\pm\gamma_{2r-1})$ $(1\leqslant r\leqslant n)$. Then the eigenvalues of $\mathbf{V}^+$ are $\exp[\frac{1}{2}(\pm\gamma_1\pm\gamma_3\pm\cdots\pm\gamma_{2n-1})]$. But from (37), only those eigenvalues occur which fall into the subspace $\frac{1}{2}(1+{}^\dagger\mathbf{U})$. So that again we must take only such sign combinations in the eigenvalues in which an even number of angles appear with minus signs.

To summarize: Half of the 2^n eigenvalues of $\mathbf{V}$ are of the form $\exp[\frac{1}{2}(\pm\gamma_2\pm\gamma_4\pm\cdots\pm\gamma_{2n})]$; the other half are of the form $\exp[\frac{1}{2}(\pm\gamma_1\pm\gamma_3\pm\cdots\pm\gamma_{2n-1})]$. In each eigenvalue an *even* number of minus signs appears in the exponent. The γ_{2r} are the angles of rotation in $\mathbf{R}^-$, the γ_{2r-1}—those of $\mathbf{R}^+$, and they are found by diagonalizing the respective $2n$-dimensional matrices.

Inasmuch as diagonalization of $\mathbf{R}^-$ and $\mathbf{R}^+$ does not uniquely determine the order of their eigenvalues, the question might arise whether or not the selection of eigenvalues in the subspaces $\frac{1}{2}(1+{}^\dagger\mathbf{U})$ and $\frac{1}{2}(1-{}^\dagger\mathbf{U})$ is unambiguous. And indeed, in the equation $(\mathbf{T}^+)\cdot(\mathbf{R}^+)\cdot(\mathbf{T}^+)^{-1}=\mathbf{K}$, one may replace $\mathbf{T}^+$ by the transformation $\mathbf{P}\cdot\mathbf{T}^+$, where $\mathbf{P}$ is any permutation, which interchanges the order of eigenvalues in $\mathbf{K}$, provided it leaves $\mathbf{K}$ in canonical form (i.e., $\mathbf{P}$ may interchange the pair γ_r, $-\gamma_r$ with the pair γ_s, $-\gamma_s$, or it may interchange γ_r with $-\gamma_r$).

However, if $\mathbf{P}$ is an odd permutation (i.e., $|\mathbf{P}|=-1$), we have $\mathbf{PT}$: ${}^\dagger\mathbf{U}\rightarrow-{}^\dagger\mathbf{U}$, since $\mathbf{T}^+$ was a proper rotation. Then:

$$\mathbf{S}(\mathbf{PT}^+)\cdot\{\tfrac{1}{2}(1+{}^\dagger\mathbf{U})\cdot{}^\dagger\mathbf{V}^+\}\cdot\mathbf{S}(\mathbf{PT}^+)^{-1}=\tfrac{1}{2}(1-{}^\dagger\mathbf{U})\cdot\mathbf{S}[(\mathbf{PT}^+)\cdot(\mathbf{R}^+)\cdot(\mathbf{PT}^+)^{-1}],$$

so that we must take those eigenvalues which fall into the subspace $\frac{1}{2}(1-{}^\dagger\mathbf{U})$. There is nevertheless no difference between the set of eigenvalues selected by $\frac{1}{2}(1+{}^\dagger\mathbf{U})$ out of $\mathbf{S}[(\mathbf{T}^+)\cdot(\mathbf{R}^+)\cdot(\mathbf{T}^+)^{-1}]$ and the set selected by $\frac{1}{2}(1-{}^\dagger\mathbf{U})$ out of $\mathbf{S}[(\mathbf{PT}^+)\cdot(\mathbf{R}^+)\cdot(\mathbf{PT}^+)^{-1}]$.

This is so, because the odd $\mathbf{P}$, which is an improper rotation, will interchange an *odd* number of γ_r with $-\gamma_r$; but this interchange is counteracted by the selection operator $\frac{1}{2}(1-{}^\dagger\mathbf{U})$, which keeps only those sign combinations which have an *odd* number of minus signs,

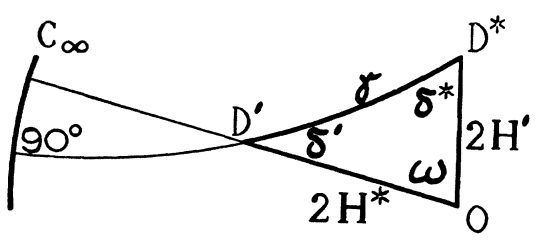

FIG. 2. Hyperbolic triangle. Stereographic projection, conformal. (Circles are represented by circles, geodetics by circles invariant toward inversion in the limiting circle C_∞ of the projection.)

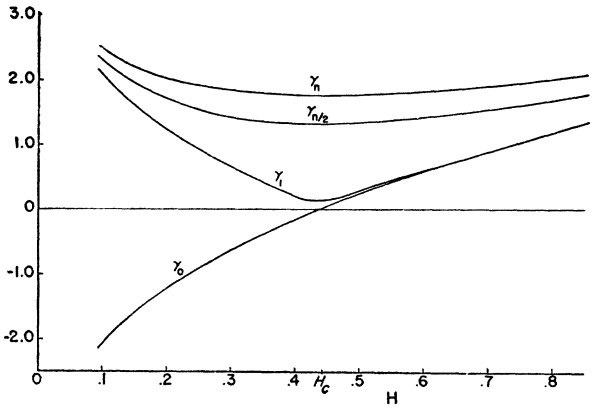

FIG. 3. Dependence of the angles γ_r on temperature. For $H<H_c$ (high temperatures) $\gamma_1\sim-\gamma_0$. For $H>H_c$ (low temperatures) $\gamma_1\sim+\gamma_0$.

so that finally we get again our old angles with an *even* number of minus signs. If $\mathbf{P}$ were a proper rotation, it would interchange an even number of γ_r with $-\gamma_r$, and would remain in the subspace $\frac{1}{2}(1+{}^\dagger\mathbf{U})$, so that once more we have our old angles with an *even* number of minus signs.

Thus we see that any choice of transformation, $\mathbf{PT}^+$, will give the same selection of eigenvalues, provided we take the appropriate subspace to go with $\mathbf{PT}^+$. Similar considerations apply of course to $\mathbf{V}^-=\mathbf{S}(\mathbf{R}^-)$.

The Complete Partition Function

The complete partition function for the lattice can now be formally written down:

$$Z=(2\sinh 2H)^{mn/2}\cdot\sum_{i=1}^{2^n}\lambda_i{}^m=(2\sinh 2H)^{mn/2}\cdot\{\sum\exp[m/2(\pm\gamma_2\pm\gamma_4\pm\cdots)]+\sum\exp[m/2(\pm\gamma_1\pm\gamma_3\pm\cdots)]\}. \quad (38)$$

The summations are performed over the permitted sign combinations. A more compact form is:

$$Z=\tfrac{1}{2}(2\sinh 2H)^{mn/2}\cdot\left\{\prod_{r=1}^{n}\left(2\cosh\frac{m}{2}\gamma_{2r}\right)+\prod_{r=1}^{n}\left(2\sinh\frac{m}{2}\gamma_{2r}\right)+\prod_{r=1}^{n}\left(2\cosh\frac{m}{2}\gamma_{2r-1}\right)+\prod_{r=1}^{n}\left(2\sinh\frac{m}{2}\gamma_{2r-1}\right)\right\}. \quad (39)$$

The Rotation Represented by $\mathbf{V}^-$, its Eigenvalues and Eigenvectors

We now proceed to determine explicitly the eigenvalues and eigenvectors of $\mathbf{V}^+$ and $\mathbf{V}^-$. $\mathbf{V}^-$ is the easier one to handle from the point of view of notation, and we will treat it first. It is convenient to deal with the symmetrical operator

$$\mathbf{V}_0\equiv\mathbf{V}_1{}^{\frac{1}{2}}\cdot\mathbf{V}_2\cdot\mathbf{V}_1{}^{\frac{1}{2}}=\mathbf{V}_1{}^{-\frac{1}{2}}\cdot\mathbf{V}\cdot\mathbf{V}_1{}^{\frac{1}{2}},$$

We will find the eigenvectors of $\mathbf{V}_0$ and obtain the eigenvectors of $\mathbf{V}$ from these by a further transformation with $\mathbf{V}_1^{\frac{1}{2}}=\exp(\frac{1}{2}H'\mathbf{A})$. The eigenvalues of $\mathbf{V}$ and $\mathbf{V}_0$ are of course the same.

We have

$$\mathbf{V}_0^-\equiv\prod_{r=1}^{n}\exp\left(\frac{i}{2}H^*\mathbf{P}_r\mathbf{Q}_r\right)\cdot\prod_{r=1}^{n}\exp(-iH'\mathbf{P}_{r+1}\mathbf{Q}_r)\cdot\prod_{r=1}^{n}\exp\left(\frac{i}{2}H^*\mathbf{P}_r\mathbf{Q}_r\right)\equiv\mathbf{S}(\mathbf{R}_0^-). \quad (41)$$

The first (and last) product represents the rotation:

$$\begin{pmatrix}\cosh H^* & i\sinh H^* & & & \\ -i\sinh H^* & \cosh H^* & & & \\ & & \cosh H^* & i\sinh H^* & \\ & & -i\sinh H^* & \cosh H^* & \\ & & & & \ddots\end{pmatrix}. \quad (42)$$

The middle product represents the rotation:

$$\begin{pmatrix}\cosh 2H' & & & & -i\sinh 2H' \\ & \cosh 2H' & i\sinh 2H' & & \\ & -i\sinh 2H' & \cosh 2H' & & \\ & & & \ddots & \\ i\sinh 2H' & & & & \cosh 2H'\end{pmatrix} \quad (43)$$

Each of these $2n$-dimensional matrices is "cyclic" in two-row strips. Their product, $\mathbf{R}_0^-$, is therefore also cyclic. Cyclic matrices have very convenient properties which permit the determination of eigenvalues and eigenvectors without much difficulty. More complicated matrices than $\mathbf{R}^-$ (corresponding to different physical models) can also be handled in the same way.

We start with the simple n-dimensional cyclic matrix

$$\alpha=\begin{pmatrix}a & b & c & d & \cdots & g & h\\ h & a & b & c & \cdots & & g\\ g & h & a & b & \cdots & & \\ \cdots & & & & & & \\ b & c & d & \cdots & & & a\end{pmatrix}, \quad (44)$$

where a, b, c, $\cdots h$ are any n scalar numbers. The normalized eigenvectors of this matrix are

$$1/(n)^{\frac{1}{2}}\begin{pmatrix}\epsilon^{2r}\\ \epsilon^{4r}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \epsilon^{2(n-1)r}\\ 1\end{pmatrix}, \quad \text{where} \quad \epsilon\equiv e^{i(\pi/n)}, \quad 1\leqslant r\leqslant n. \quad (45)$$

The corresponding eigenvalues are

$$a+\epsilon^{2r}\cdot b+\epsilon^{4r}\cdot c+\cdots+\epsilon^{2(n-1)r}\cdot h. \quad (46)$$

This can be verified by construction.

In the general case, the members of α may themselves be matrices, say of dimension f. α is now $n\cdot f$-dimensional. Its eigenvectors will be of the form

$$\begin{pmatrix}\epsilon^{2r}\cdot\mathbf{W}_{2r}\\ \epsilon^{4r}\cdot\mathbf{W}_{2r}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \epsilon^{2nr}\cdot\mathbf{W}_{2r}\end{pmatrix}, \quad (47)$$

where $\mathbf{W}_{2r}$ is an eigenvector of the f-dimensional matrix

$$\alpha_{2r}=\mathbf{a}+\epsilon^{2r}\cdot\mathbf{b}+\epsilon^{4r}\cdot\mathbf{c}+\cdots+\epsilon^{2(n-1)r}\cdot\mathbf{h}. \quad (48)$$

If λ_{2r} is an eigenvalue of α_{2r} then it is also an eigenvalue of α. Thus the eigenvalue problem for the matrix α is only f-dimensional, instead of being $n\cdot f$-dimensional.

In applying the above to our matrix $\mathbf{R}_0^-$, we see that $f=2$. Upon multiplying the three factors of the rotation $\mathbf{R}_0^-$ (see (42), (43)), we find that $\mathbf{R}_0^-$ can be written schematically as

$$\mathbf{R}_0^-=\begin{pmatrix}\mathbf{a} & \mathbf{b} & 0 & 0 & \cdots & 0 & 0 & \mathbf{b}^*\\ \mathbf{b}^* & \mathbf{a} & \mathbf{b} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \mathbf{b}^* & \mathbf{a} & \mathbf{b} & 0 & \cdots & & & \\ \cdots & & & & & & & & \\ \mathbf{b} & 0 & \cdots & & & & 0 & \mathbf{b}^* & \mathbf{a}\end{pmatrix},$$

where

$$\mathbf{a}=\begin{pmatrix}\cosh 2H'\cdot\cosh 2H^* & -i\cosh 2H'\cdot\sinh 2H^*\\ i\cosh 2H'\cdot\sinh 2H^* & \cosh 2H'\cdot\cosh 2H^*\end{pmatrix},$$

$$\mathbf{b}=\begin{pmatrix}-\frac{1}{2}\sinh 2H'\cdot\sinh 2H^* & i\sinh 2H'\cdot\sinh^2 H^*\\ -i\sinh 2H'\cdot\cosh^2 H^* & -\frac{1}{2}\sinh 2H'\cdot\sinh 2H^*\end{pmatrix},$$

$$\mathbf{b}^*=\begin{pmatrix}-\frac{1}{2}\sinh 2H'\cdot\sinh 2H^* & i\sinh 2H'\cdot\cosh^2 H^*\\ -i\sinh 2H'\cdot\sinh^2 H^* & -\frac{1}{2}\sinh 2H'\cdot\sinh 2H^*\end{pmatrix}.$$

The $2n$-eigenvalues of $\mathbf{R}_0^-$ are the eigenvalues of the

n 2-dimensional matrices

$$\boldsymbol{\alpha}_{2r}=\mathbf{a}+\epsilon^{2r}\cdot\mathbf{b}+\epsilon^{2(n-1)r}\cdot\mathbf{b}^*. \tag{50}$$

The determinant of this matrix is $+1$. Its eigenvalues may therefore be written as $\exp(\pm\gamma_{2r})$, and γ_{2r} is determined by[13]

$$\begin{aligned}\tfrac{1}{2}\text{ trace }(\boldsymbol{\alpha}_{2r})&=\tfrac{1}{2}(e^{\gamma_{2r}}+e^{-\gamma_{2r}})\\&=\cosh\gamma_{2r}=\cosh 2H^*\cdot\cosh 2H'\\&\quad-\sinh 2H^*\cdot\sinh 2H'\cdot\cos(2r\pi/n).\end{aligned} \tag{51}$$

Geometrically, γ_{2r} is the third side of a hyperbolic triangle whose other two sides, $2H'$ and $2H^*$, include the angle $\omega_{2r}=(2r\pi/n)$. Introducing the angle δ_{2r}' (between $2H^*$ and γ_{2r}), we can simplify the matrix $\boldsymbol{\alpha}_{2r}$. (See Fig. 2.)

We have from the geometry:

$$\begin{aligned}\sinh 2H^*\cdot\cosh 2H'-\cosh 2H^*\cdot\sinh 2H'\cdot\cos\omega&=\sinh\gamma\cdot\cos\delta',\\ \sinh 2H'\cdot\sin\omega&=\sinh\gamma\cdot\sin\delta'.\end{aligned} \tag{52}$$

Thus

$$\begin{aligned}\boldsymbol{\alpha}_{2r}&=\cosh\gamma_{2r}\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\\&\quad+\sinh\gamma_{2r}\cdot\begin{pmatrix}0&\sin\delta_{2r}'-i\cos\delta_{2r}'\\ \sin\delta_{2r}'+i\cos\delta_{2r}'&0\end{pmatrix}\\&=\cosh\gamma_{2r}-i\sinh\gamma_{2r}\cdot\begin{pmatrix}0&e^{i\delta_{2r}'}\\-e^{i\delta_{2r}'}&0\end{pmatrix}.\end{aligned}$$

The normalized eigenvectors of $\boldsymbol{\alpha}_{2r}$ are

$$1/(2)^{\frac{1}{2}}\begin{pmatrix}e^{(i/2)\delta_{2r}'}\\ ie^{-(i/2)\delta_{2r}'}\end{pmatrix},\quad 1/(2)^{\frac{1}{2}}\begin{pmatrix}ie^{(i/2)\delta_{2r}'}\\ e^{-(i/2)\delta_{2r}'}\end{pmatrix}, \tag{53}$$

corresponding to the eigenvalues $\exp(\gamma_{2r})$, $\exp(-\gamma_{2r})$. By (47), the $2n$-normalized eigenvectors of $\mathbf{R}_0^-$ are

$$\mathbf{u}_{2r}\equiv 1/(2n)^{\frac{1}{2}}\begin{pmatrix}e^{i(\omega_{2r}+\frac{1}{2}\delta_{2r}')}\\ ie^{i(\omega_{2r}-\frac{1}{2}\delta_{2r}')}\\ e^{i(\omega_{4r}+\frac{1}{2}\delta_{2r}')}\\ ie^{i(\omega_{4r}-\frac{1}{2}\delta_{2r}')}\\ \cdot\\ \cdot\\ \cdot\\ ie^{i(\omega_{2nr}-\frac{1}{2}\delta_{2r}')}\end{pmatrix}$$

and

$$\mathbf{v}_{2r}\equiv 1/(2n)^{\frac{1}{2}}\begin{pmatrix}ie^{i(\omega_{2r}+\frac{1}{2}\delta_{2r}')}\\ e^{i(\omega_{2r}-\frac{1}{2}\delta_{2r}')}\\ ie^{i(\omega_{4r}+\frac{1}{2}\delta_{2r}')}\\ \cdot\\ \cdot\\ \cdot\\ e^{i(\omega_{2nr}-\frac{1}{2}\delta_{2r}')}\end{pmatrix}. \tag{54}$$

Let the matrix of these eigenvectors be denoted by $\mathbf{t}$, so that

$$\mathbf{t}\cdot\mathbf{R}^-\cdot\mathbf{t}^{-1}=\boldsymbol{\lambda}^-, \tag{55}$$

[13] These are the γ_r given in I, Eq. (89). Figure 2 is the same as Fig. 4 of I.

where $\boldsymbol{\lambda}^-$ is the diagonal form of $\mathbf{R}_0^-$. Neither $\boldsymbol{\lambda}^-$ nor $\mathbf{t}$ are orthogonal, and so they cannot be represented in spin space. We therefore apply a transformation $\mathbf{I}$ to both sides of (55):

$$(\mathbf{It})\cdot\mathbf{R}^-\cdot(\mathbf{t}^{-1}\mathbf{I}^{-1})\equiv\mathbf{T}\cdot\mathbf{R}^-\cdot\mathbf{T}^{-1}=\mathbf{I}\cdot\boldsymbol{\lambda}^{-1}\cdot\mathbf{I}^{-1}\equiv\mathbf{K}. \tag{56}$$

$\mathbf{I}$ is so chosen that it brings $\boldsymbol{\lambda}^-$ into its canonical form, and at the same time makes $\mathbf{T}=\mathbf{I}\cdot\mathbf{t}$ orthogonal. The spin-representative of the canonical form $\mathbf{K}$ is given by (28). We shall not need to know the spin-representative of $\mathbf{T}$ explicitly, but we must make sure that $\mathbf{T}$ is orthogonal, so that it possesses a spin-representative.

When this is carried through, we find that the transformation $\mathbf{T}$ is given by

$$\begin{aligned}\mathbf{T}\colon\ \mathbf{P}_a&\to\sum_{r=1}^{n}\sigma_{ra}\mathbf{P}_r+\sum_{r=1}^{n}\tau_{ra}\mathbf{Q}_r,\\ \mathbf{Q}_a&\to\sum_{r=1}^{n}\sigma_{ra}'\mathbf{P}_r+\sum_{r=1}^{n}\tau_{ra}'\mathbf{Q}_r,\end{aligned} \tag{57}$$

with

$$\begin{aligned}\sigma_{ra}&=\frac{1}{n^{\frac{1}{2}}}\cos\left(\frac{2ra\pi}{n}+\tfrac{1}{2}\delta_{2r}'\right),\\ \tau_{ra}&=\frac{-1}{n^{\frac{1}{2}}}\sin\left(\frac{2ra\pi}{n}+\tfrac{1}{2}\delta_{2r}'\right),\\ \sigma_{ra}'&=\frac{1}{n^{\frac{1}{2}}}\sin\left(\frac{2ra\pi}{n}-\tfrac{1}{2}\delta_{2r}'\right),\\ \tau_{ra}'&=\frac{-1}{n^{\frac{1}{2}}}\cos\left(\frac{2ra\pi}{n}-\tfrac{1}{2}\delta_{2r}'\right).\end{aligned} \tag{58}$$

$\mathbf{T}$ is now orthogonal,[14] and possesses a spin-representative, so that the equation

$$\mathbf{T}\cdot(\mathbf{R}^-)\cdot\mathbf{T}^{-1}=\mathbf{K} \tag{56}$$

leads to

$$\mathbf{S}(\mathbf{T})\cdot(\mathbf{V}_0^-)\cdot\mathbf{S}(\mathbf{T})^{-1}=\mathbf{S}(\mathbf{K})=\prod_{r=1}^{n}\exp[(i/2)\gamma_r\mathbf{P}_r\mathbf{Q}_r]. \tag{59}$$

Diagonalization of $\mathbf{V}^-$

$\mathbf{S}(\mathbf{K})$ is still not diagonal, because in our coordinate system $i\mathbf{P}_r\mathbf{Q}_r=\mathbf{C}_r$. However, this is as close as we can come to the diagonal form of $\mathbf{V}_0^-$ as long as we make transformations in $2n$-space. In order to diagonalize $\mathbf{S}(\mathbf{K})$ we have to use the transformation

$$\begin{aligned}\mathbf{g}=2^{n/2}\cdot(\mathbf{C}+\mathbf{s})\times(\mathbf{C}+\mathbf{s})\times\cdots\times(\mathbf{C}+\mathbf{s})=\mathbf{g}^{-1},\\ \mathbf{g}\mathbf{C}_r\mathbf{g}=\mathbf{s}_r,\end{aligned}$$

which is not the spin-representative of any rotation. Then we find

$$\mathbf{g}\cdot\mathbf{S}(\mathbf{T})\cdot(\mathbf{V}_0^-)\cdot\mathbf{S}(\mathbf{T})^{-1}\cdot\mathbf{g}=\mathbf{g}\cdot\mathbf{S}(\mathbf{K})\cdot\mathbf{g}=\boldsymbol{\Lambda}^-. \tag{60}$$

[14] The determinant of $\mathbf{T}$ is -1, so that it is an improper rotation.

Since $\mathbf{V}_0=\mathbf{V}_1^{-\frac{1}{2}}\cdot\mathbf{V}^-\cdot\mathbf{V}_1^{\frac{1}{2}}\equiv\mathbf{S}(\mathbf{H})\cdot(\mathbf{V}^-)\cdot\mathbf{S}(\mathbf{H})^{-1}$, we have

$$\mathbf{g}\cdot\mathbf{S}(\mathbf{TH})\cdot(\mathbf{V}^-)\cdot\mathbf{S}(\mathbf{TH})^{-1}\cdot\mathbf{g}\equiv\mathbf{\Psi}_-\cdot(\mathbf{V}^-)\cdot\mathbf{\Psi}_-^{-1}=\mathbf{\Lambda}^-, \quad (61)$$

with

$$\mathbf{\Psi}_-=\mathbf{g}\cdot\mathbf{S}(\mathbf{TH}). \quad (62)$$

Here $\mathbf{H}$ stands for the rotation represented by $\mathbf{V}_1^{-\frac{1}{2}}$ (the reciprocal of the rotation in (42)):

$$\mathbf{H}: \quad \mathbf{P}_r\rightarrow\cosh H^*\cdot\mathbf{P}_r-i\sinh H^*\cdot\mathbf{Q}_r, \quad \mathbf{Q}_r\rightarrow i\sinh H^*\cdot\mathbf{P}_r+\cosh H^*\cdot\mathbf{Q}_r. \quad (63)$$

Although the rotations involved in $\mathbf{\Psi}_-$ are completely known, it is nevertheless not feasible to write down explicitly the components of $\mathbf{\Psi}_-$ because of the complexity of $\mathbf{S}(\mathbf{T})$.

On the other hand, we can, without much difficulty, evaluate quantities of the form $\{\mathbf{\Psi}_-\cdot\mathbf{X}\cdot\mathbf{\Psi}_-^{-1}\}_{11}$ which will be seen in Section III to have important physical significance.

Analogous Results for $\mathbf{V}^+$

We still have to find the eigenvalues and eigenvectors of

$$\mathbf{V}^+=\prod_{r=1}^{n}\exp(iH^*\mathbf{P}_r\mathbf{Q}_r)\cdot\prod_{r=1}^{n-1}\exp(-iH'\mathbf{P}_{r+1}\mathbf{Q}_r)\cdot\exp(iH'\cdot\mathbf{P}_1\mathbf{Q}_n)=\mathbf{S}(\mathbf{R}^+). \quad (64)$$

$\mathbf{R}^+$ is identical with $\mathbf{R}^-$ except for a few sign changes. Schematically $\mathbf{R}_0^+$ is of the form

$$\begin{pmatrix}\mathbf{a}&\mathbf{b}&0&0&\cdot&\cdot&\cdot&0&-\mathbf{b}^*\\ \mathbf{b}^*&\mathbf{a}&\mathbf{b}&0&0&\cdot&\cdot&&0\\ 0&\mathbf{b}^*&\mathbf{a}&\mathbf{b}&0&\cdot&\cdot&&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&&0\end{pmatrix}. \quad (65)$$

The eigenvectors are:

$$1/(2n)^{\frac{1}{2}}\begin{pmatrix}\epsilon^{2r-1}\cdot\mathbf{W}_{2r-1}\\ \epsilon^{2(2r-1)}\cdot\mathbf{W}_{2r-1}\\ \cdot\\ \cdot\\ \cdot\\ \epsilon^{n(2r-1)}\cdot\mathbf{W}_{2r-1}\end{pmatrix}, \quad (66)$$

where $\mathbf{W}_{2r-1}$ is an eigenvector of the 2-dimensional matrix

$$\boldsymbol{\alpha}_{2r-1}=\mathbf{a}+\epsilon^{2r-1}\cdot\mathbf{b}-\epsilon^{(n-1)(2r-1)}\cdot\mathbf{b}^*=\mathbf{a}+\epsilon^{2r-1}\cdot\mathbf{b}+\epsilon^{-(2r-1)}\cdot\mathbf{b}^*. \quad (67)$$

The eigenvalues of $\boldsymbol{\alpha}_{2r-1}$ (and of $\mathbf{R}^+$) may be denoted by $\exp(\pm\gamma_{2r-1})$ and

$$\cosh\gamma_{2r-1}=\cosh 2H^*\cdot\cosh 2H'-\sinh 2H^*\cdot\sinh 2H'\cdot\cos((2r-1)\pi/n). \quad (68)$$

Thus, γ_{2r} and γ_{2r-1} are covered by the same formula (as can also be seen from the similarity between $\boldsymbol{\alpha}_{2r}$ and $\boldsymbol{\alpha}_{2r-1}$).

Again, if we introduce the angles δ'_{2r-1}, with the help of which the eigenvectors are expressed, we find just as in the case of $\mathbf{V}^-$

$$\mathbf{\Psi}_+=\mathbf{g}\cdot\mathbf{S}(\mathbf{TH}), \quad (69)$$

where $\mathbf{T}$ now involves the odd-indexed angles δ'_{2r-1} and $\omega_{2r-1}=[(2r-1)\pi/n]$.

4. DISCUSSION OF THE PARTITION FUNCTION

Comparison of the relative magnitudes of the eigenvalues of $\mathbf{V}$ yields important qualitative results about the behavior of the partition function. Since we have

$$Z=\sum_{i=1}^{2^n}\lambda_i^m,$$

we find for very large m

$$Z\sim f\cdot\lambda_{\max}{}^m, \quad (70)$$

where f is the degree of degeneracy of $\lambda_{\max}$.

We have two sets of eigenvalues,

$$\log\lambda^-\equiv\tfrac{1}{2}(\pm\gamma_2\pm\gamma_4\pm\cdots\pm\gamma_{2n}), \quad (71)$$

and

$$\log\lambda^+\equiv\tfrac{1}{2}(\pm\gamma_1\pm\gamma_3\pm\cdots\pm\gamma_{2n-1}), \quad (72)$$

with the permissible sign combinations. The largest eigenvalue in each set is the one in which all signs are positive. The two maximal eigenvalues are, then,

$$\lambda_{\max}{}^+=\exp[\tfrac{1}{2}(\gamma_1+\gamma_3+\cdots+\gamma_{2n-1})], \quad (73)$$

and

$$\lambda_{\max}{}^-=\exp[\tfrac{1}{2}(\gamma_0+\gamma_2+\cdots+\gamma_{2n-2})], \quad (74)$$

(we write γ_0 instead of γ_{2n}).

Since, from its definition, $\gamma_{2r}\sim\gamma_{2r-1}$ for large n, these two eigenvalues are almost equal for low temperatures, up to the critical point. However, γ_0 falls off very rapidly in comparison with γ_1, and, as a result, $(\lambda_{\max})^m$ becomes negligible compared to $(\lambda_{\max}{}^+)^m$, for temperatures higher than the critical point. This can be readily seen in the case of the quadratic crystal, where $J'=J$ or $H'=H$. There we have:

$$\cosh\gamma_r=\cosh 2H^*\cdot\cosh 2H-\sinh 2H^*\cdot\sinh 2H'\cdot\cos(r\pi/n)=\coth 2H\cdot\cosh 2H-\cos(r\pi/n), \quad (75)$$

since from the definition of H^*, $\sinh 2H^*\cdot\sinh 2H=1$. All the γ_r have a minimum at the point $H=H_c$ given by

$$\coth 2H_c\cdot\sinh 2H_c-\frac{\cosh 2H_c}{\sinh^2 2H_c}=0, \quad (76)$$

or

$$\sinh^2 2H_c=1. \quad (77)$$

H must be positive, and therefore $H_c=0.4407\cdots$.

However, γ_0 behaves differently from the other γ_r. From

$$\cosh\gamma_0=\cosh 2H^*\cdot\cosh 2H-\sinh 2H^*\cdot\sinh 2H=\cosh 2(H^*-H), \quad (78)$$

we see that γ_0 changes sign[15] at the critical point $H=H_c$. Figure 3 shows the behavior of γ_r as a function of $H=(J/kT)$. All the γ_r are more or less symmetrical about the critical point, except for γ_0, which changes sign there. This causes λ_{max}^- to fall off very rapidly in comparison with λ_{max}^+ for temperatures above the critical point.

$$\frac{\lambda_{max}^-}{\lambda_{max}^+}=\begin{cases}1 & \text{for } H>H_c,\ T<T_c\\ e^{2(H-H^*)} & \text{for } H<H_c,\ T>T_c.\end{cases} \tag{79}$$

The existence of a degenerate eigenvalue implies long-range order:[5] this occurs for the low temperature range (twofold degeneracy) up to the critical point T_c. For the partition function we have approximately

$$\begin{aligned}&(2\sinh 2H)^{-mn/2}\cdot Z\sim 2\lambda_{max}{}^m\\ &\equiv 2\exp\left[\frac{m}{2}(\gamma_1+\gamma_3+\cdots+\gamma_{2n-1})\right]\\ &\qquad\text{when } H>H_c,\\ &(2\sinh 2H)^{-mn/2}\cdot Z\sim\lambda_{max}{}^m\\ &=\exp\left[\frac{m}{2}(\gamma_1+\gamma_3+\cdots+\gamma_{2n-1})\right]\\ &\qquad\text{when } H<H_c.\end{aligned} \tag{80}$$

The exact partition function, which takes into account all eigenvalues, does not differ much from the approximate result. In the exact solution we have

$$\begin{aligned}Z=&\tfrac{1}{2}(2\sinh 2H)^{mn/2}\\ &\cdot\left\{\prod_{r=1}^{n}\left(2\cosh\frac{m}{2}\gamma_{2r-1}\right)+\prod_{r=1}^{n}\left(2\sinh\frac{m}{2}\gamma_{2r-1}\right)\right.\\ &\left.+\prod_{r=1}^{n}\left(2\cosh\frac{m}{2}\gamma_{2r}\right)+\prod_{r=1}^{n}\left(2\sinh\frac{m}{2}\gamma_{2r}\right)\right\}.\end{aligned} \tag{39}$$

Everywhere, except in the immediate vicinity of the critical point, we may take $\gamma_{2r}=\gamma_{2r-1}$ $(1\leqslant r\leqslant n-1)$ for large enough n, and

$$\gamma_0=\begin{cases}\gamma_1 & \text{for } T<T_c\\ -\gamma_1 & \text{for } T>T_c.\end{cases}$$

Therefore the two products containing $\cosh(m\gamma/2)$ as factors may be taken as equal; while the two products with $\sinh(m\gamma/2)$ are equal below the critical temperature, but cancel each other above $T=T_c$. We have then:

$$\begin{aligned}&(2\sinh 2H)^{-mn/2}\cdot Z\cong\prod_{r=1}^{n}(2\cosh(m/2)\gamma_{2r-1}),\\ &\qquad\text{for } T>T_c,\\ &(2\sinh 2H)^{-mn/2}\cdot Z\\ &\cong\prod_{r=1}^{n}(2\cosh(m/2)\gamma_{2r-1})+\prod_{r=1}^{n}(2\sinh(m/2)\gamma_{2r-1})\\ &=\prod_{r=1}^{n}(2\cosh(m/2)\gamma_{2r-1})\cdot\{1+\prod_{r=1}^{n}(\tanh(m/2)\gamma_{2r-1})\}\\ &\qquad\text{for } T<T_c.\end{aligned} \tag{81}$$

Furthermore, since m is very large,

$$2\cosh(m/2)\gamma_{2r-1}\sim\exp[(m/2)\gamma_{2r-1}],$$

except for temperatures very near T_c ($\cosh x\sim e^x/2$ as soon as $x\sim 5$). Therefore, above the critical temperature

$$(2\sinh 2H)^{-mn}/2\cdot Z\sim\exp[(m/2)(\gamma_1+\gamma_3+\cdots+\gamma_{2n-1})]\equiv\lambda_{max}{}^m. \tag{82}$$

Below the critical temperature we find

$$(2\sinh 2H)^{-mn}/2\cdot Z\sim\eta\cdot\lambda_{max}{}^m, \tag{83}$$

where

$$\eta=1+\prod_r(\tanh(m/2)\gamma_{2r-1}).$$

The product of the factors $\tanh(m\gamma/2)$ is never larger than 1, so that $1<\eta<2$; and with large m, $\eta\sim 2$ rather close to T_c. The appearance of η here is equivalent to the twofold degeneracy of λ_{max} for $T<T_c$ in the approximate solution of Z.

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[15] (75) and (78) do not determine the sign of γ_r. To do this one must actually apply the chosen transformation **T** to **R**. With our choice, and the suitable selection operators (see p. 1239), we find easily that all $\gamma_r>0$, $(r\geqslant 1)$, and $2\gamma_0\equiv H-H^*$.