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On Nucleon Moments and the Neutron-Electron Interaction

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Using the newly developed computational techniques and the covariant formalism, the nucleon magnetic moments and the neutron-electron interaction are calculated. A pseudoscalar meson theory is assumed. Finite and unambiguous results are obtained to the lowest non-vanishing order in the coupling constants. These depend on the unknown strength of the meson coupling. While a not unreasonable value may be used to fit the proton moment, the corresponding neutron moment comes out much too large. With this fitting, the neutron-electron potential turns out to be attractive with an equivalent well depth of the order of 5 to 10 kilovolts.

I. INTRODUCTION

ONE of the simplest of the many properties predicted by meson theory is the additional magnetic moment to be ascribed to a nucleon in an external magnetic field. The anomalous neutron and proton moments hence serve as a test for the theory. In the past the theory has failed this test. Weak coupling theories in particular led to divergent results.

In the light of recent advances in quantum electrodynamics, this question has been re-examined to see whether utilization of charge and mass renormalization concepts would alleviate the situation. This has indeed been found.[1] In fact, it seems as if a completely relativistic calculation is all that is needed to obtain finite magnetic moments[2] (at least in a pseudoscalar theory). While agreement with experiment has not been obtained, this could easily be due to the use of an incorrect model or to the inadequacy of the weak coupling approximation.

In the following the anomalous magnetic moments are calculated in the Schwinger-Tomonaga formalism. This serves, firstly, to place previous results on a firmer theoretical basis. Secondly, the use of the greatly improved computational techniques make the derivation considerably simpler and more elegant.

With the present method, another quantity of experimental interest is obtained simultaneously with the magnetic moments. This is the neutron-electron potential. An expression for this quantity is given below. In particular the spatial integral of the potential, which is the quantity measured, is expressed in closed form.

II. EQUATIONS OF MOTION

To describe the meson field we choose the simplest possibility which may be expected to yield interesting results—namely the pseudoscalar theory. Of the two conventional meson-nucleon couplings for this case, the pseudoscalar coupling was chosen. It can be shown that this does not imply any restriction for the magnetic moments. To the first non-vanishing order, the anomalous magnetic moments are the same with an arbitrary linear combination of the couplings, except for a simple multiplicative factor.[3]

To treat the neutron, the proton, and all possible combinations of charged and neutral mesons simultaneously, a four-dimensional isotopic spin formalism has been used. τ_1, τ_2, τ_3 are the conventional isotopic spin matrices chosen so that the eigenvalue

[1] K. M. Case, Phys. Rev. **74**, 1884 (1948).

[2] J. M. Luttinger, Helv. Phys. Acta, **21**, 483 (1948). I am indebted to Dr. Luttinger for the opportunity of seeing his paper prior to publication. It is gratifying to note that the results obtained with the present method agree with those he has previously found.

[3] This equivalence also holds for the nuclear forces predicted by the pseudoscalar meson theory with the same equivalent pseudoscalar coupling constant.

+1 of τ_3 corresponds to neutrons, −1 to protons. τ_4 is merely the unit operator in isotopic spin space. The four real wave functions ϕ_1 to ϕ_4 describe the meson field. f_1 to f_4 are the corresponding pseudoscalar coupling constants. ϕ_1 and ϕ_2 describe a charged field, ϕ_3 the neutral field to be added in a symmetrical theory, and ϕ_4 a purely neutral field. Gauge invariance requires the equations to be invariant with respect to rotation around the 3 axis in isotopic spin space and hence, that $f_1=f_2$. This specialization will not be made until the final formulas are obtained.

In the interaction representation, the covariant Schwinger-Tomonaga form of the equations may be stated as follows.

The field operators satisfy the equations of motion:

$$(\Box^2-\kappa^2)\phi_\nu=0, \tag{1}$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\kappa_0\right)\psi=\left(\tilde{\gamma}_\mu\frac{\partial}{\partial x_\mu}-\kappa_0\right)\bar{\psi}=0. \tag{2}$$

ψ, $\bar{\psi}$ are the spinor operators of the nucleon field as defined by Schwinger.[4] The γ_μ, $\tilde{\gamma}_\mu$ are the usual Dirac γ's and their transposes. Summation from 1 to 4 is implied by repeated indices.

$$\kappa=\mu c/\hbar,\quad \kappa_0=Mc/\hbar, \tag{3}$$

where μ and M are the meson and nucleon masses, respectively.

The commutation relations are

$$[\phi_\mu(x),\phi_\nu(x')]=ic\hbar\delta_{\mu\nu}\Delta(x-x'), \tag{4}$$

$$\begin{aligned}&\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}=(1/i)S_{\alpha\beta}(x-x'),\\&\{\psi_\alpha(x),\psi_\beta(x')\}=\{\bar{\psi}_\alpha(x),\bar{\psi}_\beta(x')\}=0.\end{aligned} \tag{5}$$

In (5) the index α refers to both spinor and isotopic spin components. Δ and S are Schwinger's[4] Δ and S functions for mass μ and M, respectively, except that S is also a unit matrix with respect to isotopic spin indices.

The Schrödinger equation for the wave functional is

$$ic\hbar(\delta\Psi/\delta\sigma(x))=H(x)\Psi, \tag{6}$$

where $\delta/\delta\sigma(x)$ denotes the variational derivative with respect to the surface σ at the point x.

$$H(x)=R_\nu\phi_\nu+H_1^{\text{ext}}+H_2^{\text{ext}}, \tag{7}$$

$$R_\nu=if_\nu\bar{\psi}\gamma_5\tau_\nu\psi\phi_\nu, \tag{8}$$

$$H_1^{\text{ext}}=-ie\bar{\psi}\frac{(1-\tau_3)}{2}\gamma_\mu\psi A_\mu, \tag{9}$$

$$H_2^{\text{ext}}=-\frac{e}{\hbar c}A_\mu\left[\phi_1\frac{\partial\phi_2}{\partial x_\mu}-\phi_2\frac{\partial\phi_1}{\partial x_\mu}\right]+\frac{e^2}{2\hbar^2c^2}[A_\mu^2+(n_\mu A_\mu)^2](\phi_1^2+\phi_2^2). \tag{10'}$$

A_μ denotes the four-vector potential of the external electromagnetic field. For the problems of interest here, the second term in (10) is of higher order in e. Hence, (10′) may be replaced by

$$H_2^{\text{ext}}=-\frac{e}{\hbar c}A_\mu\left(\phi_1\frac{\partial\phi_2}{\partial x_\mu}-\phi_2\frac{\partial\phi_1}{\partial x_\mu}\right). \tag{10}$$

Here e is the proton's charge. The various special theories are given by the substitutions:

Charged theory

$$f_1=f_2=f\neq 0;\quad f_3=f_4=0. \tag{11}$$

Symmetrical theory

$$f_1=f_2=f_3=f\neq 0;\quad f_4=0. \tag{12}$$

Neutral theory

$$f_1=f_2=f_3=0;\quad f_4=f\neq 0. \tag{13}$$

The above formulation is slightly incomplete since it fails to satisfy the requirement of invariance under electric charge conjugation. This defect may be remedied by defining charge conjugate operators to ψ and $\bar{\psi}$ which satisfy equations similar to (2). The charge conjugate expression should then be added to (8) and the result divided by two. For the present purposes the effect of these manipulations would be manifested by the vacuum expectation value of $\bar{\psi}\gamma_5\tau_\nu\psi$ being replaced by zero. Agreeing that this replacement shall be made wherever the indicated expression occurs makes it possible to use the present formulation without explicit mention of charge conjugate operators.

III. METHOD

In this application of the covariant formalism, the greatly simplified method of calculation discussed by Dyson[5] will be be taken as basic. Practically all of his arguments may be carried over from electrodynamics to meson theory, with the exception of some trivial modifications which are indicated when encountered.

The effects of interest are of order ef^2A_μ. Terms of higher order in e or f will be considered negligible.

For the one-nucleon system the effective Hamiltonian may be written (by transforming away the $R_\nu\phi_\nu$ term which describes the meson-nucleon

[4] J. S. Schwinger, Phys. Rev. **74**, 1439 (1948).

[5] F. J. Dyson, Phys. Rev. **75**, 486 (1949). I would like to thank Dr. Dyson for telling me of this formulation before it was published.

coupling) as:

$$H_{\text{eff}}=\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar c}\right)^n\frac{1}{n!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n \times P(H^{\text{ext}}(x_0)\cdots H_i(x_1)\cdots H_i(x_n)), \quad (14)$$

where P denotes the ordering operator which arranges the field quantities chronologically.

Strictly speaking H_i is given by

$$H_i(x_i)=R_\nu(x_i)\phi_\nu(x_i)-\delta Mc^2\bar{\psi}\psi, \quad (15)$$

where δM is the change in nucleon mass due to the coupling with the meson field. Dyson has shown, however, that for practical purposes, the second term in (15) may be omitted. This is done with the understanding that all mass renormalization terms are to be dropped when encountered. Afterwards all these terms are to be considered separately. Hence, instead of (15), we use

$$H_i(x_i)=R_\nu(x_i)\phi_\nu(x_i). \quad (16)$$

To the specified order only the first three terms of (14) contribute. The first term represents the ordinary interaction between a proton and an external field. The second term has no diagonal elements for the one nucleon, no meson state. It will be shown below that this term does not contribute to the problems considered here.

Thus, the modification of the electromagnetic properties of the nucleons are due to

$$H_{\text{eff}}'=\frac{1}{2}\left(\frac{-i}{\hbar c}\right)^2\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2 \times P(H^{\text{ext}}(x_0),H_i(x_1),H_i(x_2)). \quad (17)$$

This may be split into a sum of two terms:

$$H_1=\tfrac{1}{2}\left(\frac{-i}{\hbar c}\right)^2\iint_{-\infty}^{\infty}dx_1dx_2\, P(H_{\text{ext}}^{(1)}(x_0),H_i(x_1),H_i(x_2)), \quad (18)$$

and

$$H_2=\tfrac{1}{2}\left(\frac{-i}{\hbar c}\right)^2\iint_{-\infty}^{\infty}dx_1dx_2 \times P(H_{\text{ext}}^{(2)}(x_0),H_i(x_1),H_i(x_2)). \quad (19)$$

Physically H_1 is due to the proton having an electric charge, while H_2 is due to the occurrence of charged mesons.

Inserting the definition of (16), and using the commutativity of nucleon and meson fields gives for (18) and (19)

$$H_1=-\frac{ief_\nu f_\sigma}{2\hbar^2c^2}A_\mu\iint_{-\infty}^{\infty}dx_1dx_2 \times P\left\{\bar{\psi}(x_0)\frac{(1-\tau_3)}{2}\gamma_\mu\psi(x_0),\,\bar{\psi}(x_1)\tau_\nu\gamma_5\psi(x_1),\, \times\bar{\psi}(x_2)\tau_\sigma\gamma_5\psi(x_2)\right\}P\{\phi_\nu(x_1)\phi_\sigma(x_2)\}, \quad (20)$$

$$H_2=-\frac{ef_\nu f_\sigma}{2\hbar^3c^3}A_\mu\iint_{-\infty}^{\infty}dx_1dx_2 \times P\{\bar{\psi}(x_1)\tau_\nu\gamma_5\psi(x_1),\,\bar{\psi}(x_2)\tau_\sigma\gamma_5\psi(x_2)\} \times P\left\{\phi_1(x_0)\frac{\partial\phi_2(x_0)}{\partial x_\mu{}^0}-\phi_2(x_0)\frac{\partial\phi_1(x_0)}{\partial x_\mu{}^0},\, \phi_\nu(x_1),\,\phi_\sigma(x_2)\right\}. \quad (21)$$

The one-nucleon, zero-meson portions of (20) and (21) are now to be calculated. This means that the vacuum expectation value of the P-bracket involving meson fields and the one-nucleon portion of the P-bracket of nucleon fields are to be inserted. Using the given commutation relations calculations similar to those of Schwinger[6] and Dyson[5] give

$$\langle P\{\phi_\nu(x_1)\phi_\sigma(x_2)\}\rangle_{\text{vac}}=\tfrac{1}{2}c\hbar\delta_{\nu\sigma}\Delta_F(x_1-x_2), \quad (22)$$

where Δ_F is the function whose Fourier transform is

$$\Delta_F(k)=\frac{-i}{2\pi^2}\left[P_r\frac{1}{k_\mu{}^2+\kappa^2}+i\pi\delta(k^2+\kappa^2)\right], \quad (23)$$

with P_r indicating that the principal value of the four-dimensional Fourier integral is to be taken.

A similar though more tedious calculation shows

$$\left\langle P\left\{\phi_1(x_0)\frac{\partial\phi_2(x_0)}{\partial x_\mu{}^0}-\phi_2(x_0)\frac{\partial\phi_1(x_0)}{\partial x_\mu{}^0},\,\phi_\nu(x_1),\,\phi_\sigma(x_2)\right\}\right\rangle_{\text{vac}}$$
$$=\frac{h^2c^2}{4}\delta_{1\nu}\delta_{2\sigma}\left\{\Delta_F(x_0-x_1)\frac{\partial\Delta_F}{\partial x_\mu{}^0}(x_0-x_2)-\Delta_F(x_0-x_2)\frac{\partial}{\partial x_\mu{}^0}\Delta_F(x_0-x_1)\right\}$$
$$+\frac{h^2c^2}{4}\delta_{1\sigma}\delta_{2\nu}\left\{\Delta_F(x_0-x_2)\frac{\partial\Delta_F}{\partial x_\mu{}^0}(x_0-x_1)-\Delta_F(x_0-x_1)\frac{\partial}{\partial x_\mu{}^0}\Delta_F(x_0-x_2)\right\}. \quad (24)$$

[6] J. S. Schwinger, Phys. Rev. **75**, 1912 (1949).

Equation (24) is to be inserted in (21). It is seen that the two terms on the right-hand side of (24) differ only by a minus sign and the interchange of indices 1 and 2, i.e.,

$$\langle P\{\ \}\rangle=\frac{\hbar^2c^2}{4}(\delta_{1\nu}\delta_{2\sigma}-\delta_{1\sigma}\delta_{2\nu})\times\left\{\Delta_F(x_0-x_1)\frac{\partial\Delta_F}{\partial x_\mu{}^0}(x_0-x_2)-\Delta_F(x_0-x_2)\frac{\partial\Delta_F}{\partial x_\mu{}^0}(x_0-x_1)\right\}. \quad (25)$$

The one-nucleon portions of the nucleon P-brackets may be found in precisely the same manner as has been done by Dyson[5] for the electrodynamic case. Essentially, the procedure is the following. One ψ and one $\bar{\psi}$ are chosen to be left as operators. Neither of these may be functions of x_0, for these terms are the self-energy contributions we agreed to omit. A particular pairing of the remaining ψ and $\bar{\psi}$ is then selected. The vacuum expectation value of the P-bracket of each pair is substituted for the pair. If $\bar{\psi}$ and ψ refer to the same coordinate, this expectation value is zero—since all terms have been implicitly charge-symmetrized with a resulting vacuum expectation value of zero. For $\bar{\psi}$ and ψ of different coordinates we have

$$\langle P\{\bar{\psi}_\alpha(x)\psi_\beta(y)\}\rangle_{\text{vac}}=\tfrac{1}{2}(S_F)_{\beta\alpha}(x-y), \quad (26)$$

where

$$(S_F)_{\beta\alpha}(x)=-\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}+\kappa_0\right)_{\beta\alpha}\Delta_F{}^0(x), \quad (27)$$

with $\Delta_F{}^0$ the same as Δ_F but with the replacement $\kappa\rightarrow\kappa_0$. Multiplying by a factor of -1 for each S_F to take account of the anticommutativity of the spinors, and summing over the contributions of all pairings, gives the one nucleon portions of the brackets. For a much more detailed description of this procedure the reader is referred to reference 5.

The nucleon P-bracket in (20) gives two distinct terms, each of which occurs twice. The two occurrences differ merely by an interchange of x_1 and x_2, and so the effect of the second is merely a factor of 2. $H_1{}^a$ and $H_1{}^b$ are the contributions of the two essentially different arrangements. There is only one type of term from the P-bracket in (21).

Substituting the various expressions in (20) and (21) gives the operator in the Schrödinger equation which describes the modified nucleon electromagnetic properties. This is

$$H_{\text{eff}}'=H_1{}^a+H_1{}^b+H_2 \quad (28)$$

$$H_1{}^a=\frac{ief_\sigma{}^2}{8\hbar c}A_\mu\int\int_{-\infty}^{\infty}dx_1dx_2\times\text{Tr}\left[\tau_\sigma\gamma_5S_F(x_0-x_1)\gamma_\mu\frac{(1-\tau_3)}{2}S_F(x_1-x_0)\right]\times\Delta_F(x_1-x_2)\bar{\psi}(x_2)\gamma_5\tau_\sigma\psi(x_2), \quad (29)$$

$$H_1{}^b=\frac{-ief_\nu{}^2}{8\hbar c}A_\mu\int\int_{-\infty}^{\infty}dx_1dx_2\times\left\{\bar{\psi}(x_1)\gamma_5\tau_\nu S_F(x_0-x_1)\gamma_\mu\frac{(1-\tau_3)}{2}\times S_F(x_2-x_0)\gamma_5\tau_\nu\psi(x_2)\Delta_F(x_1-x_2)\right\}, \quad (30)$$

$$H_2=\frac{ef_1f_2}{8\hbar c}A_\mu\int\int_{-\infty}^{\infty}dx_1dx_2\times\bar{\psi}(x_1)\tau_1\gamma_5S_F(x_2-x_1)\tau_2\gamma_5-\gamma_5\tau_2S_F(x_2-x_1)\gamma_5\tau_1\psi(x_2)\}\times\left\{\Delta_F(x_0-x_1)\frac{\partial}{\partial x_\mu{}^0}\Delta_F(x_0-x_2)-\Delta_F(x_0-x_2)\frac{\partial}{\partial x_\mu{}^0}\Delta_F(x_0-x_1)\right\}. \quad (31)$$

$H_1{}^a$ describes the polarization of the vacuum. It is due to the production of nucleon pairs by mesons emitted by a proton. These pairs might be expected to give a contribution to the electric current density even at points at a finite distance from the nucleon. Indeed, it is just these terms which give rise to charge renormalization and the attendant polarization phenomena in the electrodynamic case.[7] $H_1{}^b$ is the analog of the principal term in the Lamb Shift calculations. From the analogy this term would be expected to describe anomalous moments even for a purely neutral meson field. H_2 is due to charged mesons and describes the modification of nucleon properties due to the non-spherically symmetric charged cloud surrounding the nucleon.

IV. EVALUATION

The isotopic spin dependence of H_{eff}' may be considerably simplified by noting that spin and isotopic spin operators commute and act on different coordinates. Thus, the trace in (29) may be written

$$(\text{Tr}\tau_\sigma(1-\tau_3)/2)\ \text{Tr}\Gamma_\mu, \quad (32)$$

[7] This term will be shown to vanish in the present theory.

where

$$\Gamma_\mu = \gamma_5 S_F(x_0 - x_1)\gamma_\mu S_F(x_1 - x_0). \quad (33)$$

Similarly, the matrix operator in (30) is

$$f_\nu^2 \tau_\nu \gamma_5 S_F(x_0-x_1)\gamma_\mu \frac{(1-\tau_3)}{2} S_F(x_2-x_0)\gamma_5\tau_\nu$$

$$= f_\nu^2 \tau_\nu \frac{(1-\tau_3)}{2} \tau_\nu \gamma_5 S_F(x_0-x_1)\gamma_\mu S_F(x_2-x_0)\gamma_5. \quad (34)$$

But, from the relations satisfied by the τ matrices,

$$f_\nu^2 \tau_\nu \frac{(1-\tau_3)}{2}\tau_\nu = (f_3^2+f_4^2)\frac{(1-\tau_3)}{2} + (f_1^2+f_2^2)\frac{(1+\tau_3)}{2}. \quad (35)$$

In (31) the isotopic spin operator is

$$\tau_1\tau_2 - \tau_2\tau_1 = 2i\tau_3. \quad (36)$$

With these substitutions, (29)–(31) become

$$H_1^{(a)} = \frac{ief_\sigma^2}{8hc}\int\int_{-\infty}^{\infty} dx_1 dx_2 \,\mathrm{Tr}\,\tau_\sigma \frac{(1-\tau_3)}{2}(\mathrm{Tr}\Gamma_\mu) \times \Delta_F(x_1-x_2)\bar{\psi}(x_2)\gamma_5\tau_\sigma\psi(x_2), \quad (37)$$

$$H_1^{(b)} = \frac{-ie}{8hc}A_\mu \int\int_{-\infty}^{\infty} dx_1 dx_2 \Bigg\{ \bar{\psi}(x_1) \times \left[(f_3^2+f_4^2)\frac{(1-\tau_3)}{2} + (f_1^2+f_2^2)\frac{(1+\tau_3)}{2}\right] \times \gamma_5 S_F(x_0-x_1)\gamma_\mu S_F(x_2-x_0) \times \gamma_5\psi(x_2)\Delta_F(x_1-x_2)\Bigg\}, \quad (38)$$

$$H_2 = \frac{ief_1f_2}{4hc}A_\mu \int\int_{-\infty}^{\infty} dx_1dx_2 \times \bar{\psi}(x_1)\tau_3\gamma_5 S_F(x_2-x_1)\gamma_5\psi(x_2) \times \left\{ \Delta_F(x_0-x_1)\frac{\partial}{\partial x_\mu^0}\Delta_F(x_0-x_2) - \Delta_F(x_0-x_2)\frac{\partial \Delta_F}{\partial x_\mu^0}(x_0-x_1)\right\}. \quad (39)$$

As might be expected, H_2 is equal and opposite for neutron and proton (being proportional to τ_3). This merely states that the charged cloud around the neutron and proton is distributed the same but in one case is made up of negative mesons and in the other of positive ones.

For the evaluation of (37), $\mathrm{Tr}\Gamma_\mu$ is needed. Expressing S_F in terms of γ's by means of (27) gives

$$\mathrm{Tr}\Gamma_\mu = -\mathrm{Tr}\gamma_5\gamma_\lambda\gamma_\mu\gamma_\rho \frac{\partial}{\partial x_\lambda^0}\Delta_F(x_0-x_1)\frac{\partial}{\partial x_\rho^0}\Delta_F(x_0-x_1) + \kappa_0\,\mathrm{Tr}\gamma_5\gamma_\lambda\gamma_\mu \frac{\partial}{\partial x_\lambda^0}\Delta_F(x_0-x_1)\Delta_F(x_0-x_1) - \kappa_0\,\mathrm{Tr}\gamma_5\gamma_\mu\gamma_\rho\left(\frac{\partial}{\partial x_\rho^0}\Delta_F(x_0-x_1)\right)\Delta_F(x_0-x_1) + \kappa_0^2\,\mathrm{Tr}\gamma_5\gamma_\mu\Delta_F(x_0-x_1)\Delta_F(x_0-x_1). \quad (40)$$

Now,

$$\mathrm{Tr}\gamma_5\gamma_\lambda\gamma_\mu\gamma_\rho = \tfrac{1}{2}\,\mathrm{Tr}[\gamma_5\gamma_\lambda\gamma_\mu\gamma_\rho + \gamma_\lambda\gamma_\mu\gamma_\rho\gamma_5] = 0 \quad (41)$$

(since γ_5 anticommutes with γ_μ). Similarly,

$$\mathrm{Tr}\,\gamma_5\gamma_\mu = \tfrac{1}{2}\,\mathrm{Tr}(\gamma_5\gamma_\mu + \gamma_\mu\gamma_5) = 0. \quad (42)$$

On changing the name of the summation variable in the third term of (40) from λ to ρ, we have

$$\mathrm{Tr}\Gamma_\mu = \kappa_0\,\mathrm{Tr}\gamma_5(\gamma_\lambda\gamma_\mu - \gamma_\mu\gamma_\lambda) \times \left(\frac{\partial}{\partial x_\lambda^0}\Delta_F(x_0-x_1)\right)\Delta_F(x_0-x_1). \quad (43)$$

Equation (43) is certainly zero if $\mu=\lambda$. If $\mu\neq\lambda$, $\gamma_\lambda\gamma_\mu\gamma_5$ reduces to the product of two different γ's. But these anticommute, and so the trace is zero.

$$\therefore \quad \mathrm{Tr}\,\Gamma_\mu = 0, \quad (44)$$

and so

$$H_1^a \equiv 0. \quad (45)$$

This means that not only the charge renormalization but all other polarization terms vanish. A more general proof showing the physics involved is given in Appendix A.

To perform the integration indicated in H_1^b and H_2, it is convenient to use Fourier integral representations of the various functions. This integration is greatly facilitated by an observation of R. P. Feynman.

If $\Delta_F(k_\mu)$ be given by (23), then

$$\int_{-\infty}^{\infty} F(k_\mu)\Delta_F(k_\mu)dk_0 = \int_c F(k_\mu)\frac{-i}{2\pi^2}\frac{dk_0}{k_\mu^2+\kappa^2}, \quad (46)$$

where c is a contour in the complex k_0 plane going from $-\infty$ on the real axis below the pole at $k_0 = -\kappa$, above the pole at $k_0 = \kappa$, and then to $+\infty$ on the real axis. For functions F which vanish properly at infinity (which is true of those found below), the integral along c is equal to the integral taken along the imaginary axis from $-i\infty$ to $i\infty$. Hence, with the understanding that integrals over k_0 are to be taken on this path, the Fourier transform of $\Delta_F(x)$ is

$$\Delta_F(k_\mu)=\frac{-i}{2\pi^2}\frac{1}{k_\mu^2+\kappa^2}. \tag{47}$$

Setting

$$T=(f_3^2+f_4^2)((1-\tau_3)/2)+(f_1^2+f_2^2)((1+\tau_3)/2), \tag{48}$$

$$\xi=x_1-x_0;\quad \xi'=x_0-x_2. \tag{49}$$

H_1^b becomes

$$H_1^b=\frac{-ie}{8\hbar c}A_\mu\int\int_{-\infty}^{\infty} d\xi d\xi'\bar{\psi}(x_0+\xi)T\gamma_5 S_F(-\xi)\times\gamma_\mu S_F(-\xi')\gamma_5\psi(x_0-\xi')\Delta_F(\xi+\xi'). \tag{50}$$

After an integration by parts and the change of variables,

$$\eta=x_0-x_1;\quad \eta'=x_0-x_2. \tag{51}$$

H_2 becomes

$$H_2=\frac{ief_1f_2}{2\hbar c}A_\mu\int\int_{-\infty}^{\infty} d\eta d\eta'\bar{\psi}(x_0-\eta)\tau_3\gamma_5 S_F(\eta-\eta')\times\gamma_5\psi(x_0-\eta')\Delta_F(\eta)\frac{\partial}{\partial\eta_\mu'}\Delta_F(\eta'). \tag{52}$$

As in earlier electromagnetic calculations, we consider a typical Fourier component of these operators.

Let

$$\psi(x)=a\exp(iP_\mu x_\mu);\quad \bar{\psi}(x)=\bar{b}\exp(-iP_\mu' x_\mu). \tag{53}$$

Since ψ, $\bar{\psi}$ satisfy (2),

$$(iP_\mu\gamma_\mu+\kappa_0)a=\bar{b}(iP_\mu'\gamma_\mu+\kappa_0)=0, \tag{54}$$

and

$$P_\mu^2=P_\mu'^2=-\kappa_0^2. \tag{55}$$

In (50) we insert the Fourier representation

$$\Delta_F(\xi+\xi')=\frac{1}{(2\pi)^2}\int\exp[i(k_\mu\xi_\mu+k_\mu\xi_\mu')]\times\Delta_F(k)(dk), \tag{56}$$

where $\Delta_F(k)$ is the Fourier transform of $\Delta_F(x)$.

In (52) we set

$$S_F(\eta-\eta')=\frac{1}{(2\pi)^2}\int\exp[ik_\mu(\eta_\mu'-\eta_\mu)]S_F(k)(dk) \tag{57}$$

where

$$S_F(k)=\text{Fourier transform of } S_F(-x). \tag{58}$$

With (53) there results

$$H_1^b=\frac{-ie}{8\hbar c}A_\mu\frac{1}{(2\pi)^2}\int\int\int d\xi d\xi' dk\bar{\psi}(x_0)\, T\gamma_5\exp[i(k_\mu-P_\mu')\xi_\mu]S_F(-\xi)\gamma_\mu S_F(-\xi')\times\exp[i(k_\mu-P_\mu)\xi_\mu']\gamma_5\psi(x_0)\Delta_F(k). \tag{59}$$

$$H_2=\frac{ief_1f_2}{2\hbar c}\frac{1}{(2\pi)^2}\int\int\int d\eta d\eta' dk\times\bar{\psi}(x_0)\tau_3\gamma_5 S_F(k)\gamma_5\psi(x_0)\times\exp[i(P_\lambda'-k_\lambda)\eta_\lambda]\Delta_F(\eta)\times\exp[i(k_\nu-P_\nu)n_\nu'](\partial/\partial\eta_\mu')\Delta_F(\eta'). \tag{60}$$

The integrals over ξ, ξ', η, η' are readily done since they are just the transforms of the functions involved. (59) and (60) become

$$H_1^b=\frac{-ieA_\mu}{8\hbar c}4\pi^2\int dk\bar{\psi}(x_0)T\gamma_5 S_F(P_\mu'-k_\mu)\times\gamma_\mu S_F(P_\mu-k_\mu)\gamma_5\psi(x_0)\Delta_F(k), \tag{61}$$

$$H_2=\frac{ief_1f_2A_\mu}{2\hbar c}4\pi^2\int dk\bar{\psi}(x_0)\tau_3\gamma_5 S_F(k)\gamma_5\psi(x_0)\times\Delta_F(k_\lambda-P_\lambda')i(k_\nu-P_\nu)\Delta_F(k_\nu-P_\nu). \tag{62}$$

Inserting the transforms gives

$$H_1^b=\frac{-ieA_\mu}{8\hbar c}\frac{i}{2\pi^4}\int dk\bar{\psi}(x_0)T\gamma_5\frac{[i\gamma_\nu(P_\nu'-k_\nu)-\kappa_0]\gamma_\mu[i\gamma_\lambda(P_\lambda-k_\lambda)-\kappa_0]\gamma_5\psi(x_0)}{[(k_\lambda-P_\lambda')^2+\kappa_0^2][(k_\lambda-P_\lambda)^2+\kappa_0^2][k_\lambda^2+\kappa^2]}, \tag{63}$$

$$H_2=\frac{ief_1f_2}{2\hbar c}A_\mu\frac{1}{2\pi^4}\int dk\bar{\psi}(x_0)\tau_3\gamma_5(-i\gamma_\nu k_\nu+\kappa_0)\psi(x_0)\frac{(P_\mu-k_\mu)}{(k_\mu^2+\kappa_0^2)[(k_\mu-P_\mu)^2+\kappa^2][(k_\mu-P_\mu')^2+\kappa^2]}. \tag{64}$$

Utilizing the anticommutivity of γ_5 with γ_μ and the relations (54) give

$$H_1^b=\frac{-ieA_\mu}{8\hbar c}\left(\frac{-i}{2\pi^4}\right)\int dk\frac{\bar{\psi}(x_0)T[i\gamma_\nu(P_\nu'-k_\nu)+\kappa_0]\gamma_\mu[i\gamma_\lambda(P_\lambda-k_\lambda)+\kappa_0]\psi(x_0)}{[(k_\mu-P_\mu')^2+\kappa_0^2][(k_\mu-P_\mu)^2+\kappa_0^2][k_\mu^2+\kappa^2]}, \tag{65}$$

$$H_2=\frac{ief_1f_2A_\mu}{2\hbar c}\frac{1}{2\pi^4}\int dk\frac{\bar{\psi}(x_0)\tau_3 i\gamma_\nu(k_\nu-P_\nu')(P_\mu-k_\mu)\psi(x_0)}{[k_\mu^2+\kappa_0^2][(k_\mu-P_\mu)^2+\kappa^2][(k_\mu-P_\mu')^2+\kappa^2]}. \tag{66}$$

With a few of the conventional tricks of integrating and differentiations with respect to parameters, (65) and (66) may be integrated over (k), and expressed as:

$$H_1^{(b)}=\frac{-ieA_\mu}{8\hbar c}\left(-\frac{1}{4\pi^2}\right)\int_0^1 dx\int_0^x dy\int_{\kappa^2}^\infty da\int_{\kappa^2}^\infty db$$
$$\times\frac{4y(x-y)}{(K_1^2)^3}\bar{\psi}(x_0)T\gamma_\nu\gamma_\mu\gamma_\lambda[y(P_\nu'-P_\nu)+xP_\nu]$$
$$\times[y(P_\lambda'-P_\lambda)+xP_\lambda]\psi(x_0)$$
$$+\frac{(x-y)y}{(K_1^2)^3}\delta_{\lambda\nu}\bar{\psi}(x_0)T\gamma_\nu\gamma_\mu\gamma_\lambda\psi(x_0), \quad (67)$$

with

$$K_1^2=(x-y)y(\Delta P_\mu)^2+x^2k_0^2$$
$$-ay-b(x-y)+\kappa^2(1-x), \quad (68)$$
$$\Delta P_\nu=P_\nu-P_\nu'. \quad (69)$$

$$H_2=\frac{ief_1f_2}{2\hbar c}A_\mu\left(\frac{-i}{8\pi^2}\right)\int_0^1 dx\int_0^x dy\int_{\kappa^2}^\infty da\int_{\kappa^2}^\infty db$$
$$\times\frac{8(x-y)(1-x)}{(K_2^2)^3}\bar{\psi}(x_0)\tau_3 i\gamma_\nu[(x-y)P_\nu-xP_\nu']$$
$$\times[(x-y-1)P_\mu+(1-x)P_\mu']\psi(x)$$
$$+\frac{2(x-y)(1-x)}{(K_2^2)^2}\delta_{\nu\mu}\bar{\psi}(x_0)i\gamma_\nu\tau_3\psi(x_0), \quad (70)$$

$$K_2^2=(x-y)(1-x)(\Delta P_\mu)^2+y^2\kappa_0^2$$
$$+a(x-y)+b(1-x). \quad (71)$$

Equations (67) and (70) may be expressed in terms of ΔP alone on using (54) and the relations between the γ-matrices. Three types of terms result. These are

$$\text{(const.) } A_\mu\bar{\psi}(x_0)\gamma_\mu\psi(x_0)f_1((\Delta P_\mu)^2), \quad (72)$$
$$\text{(const.) } A_\mu\bar{\psi}(x_0)\Delta P_\nu\sigma_{\mu\nu}\psi(x_0)f_2((\Delta P_\mu)^2), \quad (73)$$

with $\sigma_{\mu\nu}=-\frac{1}{2}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$

$$\text{(const.) } A_\mu\bar{\psi}(x_0)\Delta P_\mu\bar{\psi}(x_0)f_3((\Delta P_\mu)^2). \quad (74)$$

Since these expressions are those for a typical Fourier component given by (53), it is apparent that $i\Delta P_\mu$ corresponds to the operator $\partial/\partial x_\mu$ acting on a general wave function. Hence, (74) is

$$\text{(const.) } A_\mu(x_0)(\partial/\partial x_\mu)f(\Box^2)\bar{\psi}(x_0)\psi(x_0). \quad (75)$$

On integrating by parts, this expression is seen to be proportional to $\partial A_\mu/\partial x_\mu$, which is zero by the Lorentz condition. Thus, all terms of the form (74) may be omitted.

When these various manipulations have been done in the above formulas, one more consideration is necessary. This concerns all the "renormalization" terms which have so far been omitted. These terms will be independent of ΔP. Similar terms occur in (67) and (70). It might be expected that these terms will cancel. In any event it is obvious that such terms should not contribute to the quantities calculated here. Hence, we will just omit these contributions.

On setting

$$\phi(x)=x^2\kappa_0^2+\kappa^2(1-x), \quad (76)$$

this gives

$$H_1^b=\frac{ieA_\mu}{16\pi^2\hbar c}\int_0^1 dx\int_0^x dy\bar{\psi}(x_0)T\gamma_\mu\psi(x_0)$$
$$\times\left\{\frac{(x^2\kappa_0^2+\phi(x))y(x-y)(\Delta P_\mu)^2}{\phi(x)[y(x-y)(\Delta P_\mu)^2+\phi(x)}\right.$$
$$\left.+\ln\frac{(x-y)y(\Delta P_\mu)^2+\phi(x)}{\phi(x)}\right\}+\Delta P_\nu\bar{\psi}(x_0)$$
$$\times T\sigma_{\mu\nu}\psi(x_0)\frac{\kappa_0x^2}{y(x-y)(\Delta P_\mu)^2+\phi(x)}, \quad (77)$$

$$H_2=\frac{ief_1f_2}{8\pi^2\hbar c}A_\mu\int_0^1 dx\int_0^x dy\bar{\psi}(x_0)\tau_3\gamma_\mu\psi(x_0)$$
$$\times\left\{2\kappa_0^2y^2\frac{(x-y)(1-x)(\Delta P_\mu)^2}{\phi(y)[(x-y)(1-x)(\Delta P_\mu)^2+\phi(y)]}\right.$$
$$\left.-\ln\frac{(x-y)(1-x)(\Delta P_\mu)^2+\phi(y)}{\phi(y)}\right\}$$
$$+\frac{\kappa_0y^2\Delta P_\nu\bar{\psi}\sigma_{\mu\nu}\tau_3\psi}{(x-y)(1-x)(\Delta P_\mu)^2+\phi(y)}. \quad (78)$$

The sum of (77) and (78) contains the anomalous electromagnetic properties. Specializing to various types of electromagnetic field as done below gives the magnetic moments and the neutron-electron interaction.

V. MAGNETIC MOMENTS

Consider a constant magnetic field. Since $i\Delta P_\nu$ represents the operator $\partial/\partial x_\nu$, an integration by parts shows that only the $\sigma_{\mu\nu}$ terms in (77) and (78) remain. These simplify to

$$H_1^b=\frac{ieA_\mu}{16\pi^2\hbar c}\Delta P_\nu\bar{\psi}(x_0)T\sigma_{\mu\nu}\psi(x_0)$$
$$\times\int_0^1 dx\int_0^x dy\frac{\kappa_0x^2}{\phi(x)}. \quad (79)$$

Here the $(\Delta P)^2$ in the denominator has been set equal to zero since the external field is assumed constant in space and time.

$$H_2=\frac{ief_1f_2A_\mu}{8\pi^2\hbar c}\Delta P_\nu\bar{\psi}(x_0)\tau_3\sigma_{\mu\nu}\psi(x_0)\times\int_0^1 dx\int_0^x dy\frac{\kappa_0 y^2}{\phi(y)}. \quad (80)$$

Using the differential operator interpretation of ΔP_ν and integrating by parts gives

$$H_1{}^b=\frac{e}{16\pi^2\hbar c}B_1\sum_{\mu<\nu}F_{\mu\nu}\bar{\psi}T\sigma_{\mu\nu}\psi, \quad (81)$$

$$H_2=\frac{ef_1f_2}{8\pi^2\hbar c}B_2\sum_{\mu<\nu}F_{\mu\nu}\bar{\psi}\tau_3\sigma_{\mu\nu}\psi, \quad (82)$$

with

$$B_1=\int_0^1 dx\int_0^x dy\kappa_0 x^2/\phi(x), \quad (83)$$

$$B_2=\int_0^1 dx\int_0^x dy\kappa_0 y^2/\phi(y), \quad (84)$$

where $F_{\mu\nu}$ is the field tensor.

For a proton,

$$T=f_3{}^2+f_4{}^2, \quad \tau_3=-1. \quad (85)$$

For a neutron,

$$T=f_1{}^2+f_2{}^2, \quad \tau_3=+1. \quad (86)$$

Setting these formulas in (81) and (82), equating f_1 to f_2 to f (as necessary for a charged field), and adding, we find

Proton

$$H_{\text{eff}}'=+\left\{\frac{-e(f_3{}^2+f_4{}^2)}{16\pi^2\hbar c}B_1+\frac{ef^2B_2}{8\pi^2\hbar c}\right\}\times\{-\psi^*\sigma\cdot\mathcal{H}\psi\}. \quad (87)$$

Neutron

$$H_{\text{eff}}'=+\left\{\frac{-ef^2}{8\pi^2\hbar c}B_1-\frac{ef^2B_2}{8\pi^2\hbar c}\right\}\{-\psi^*\sigma\cdot\mathcal{H}\psi\}, \quad (88)$$

where $\mathcal{H}$ is the constant magnetic field.

The brackets are obviously the magnetic moments. Physically these correspond just to what we would expect. The B_2 term is equal and opposite for neutron and proton since it is due to meson clouds of opposite signs. Similarly, the B_1 term, which is due to the proton's having a charge, is zero for a proton when only a charged field is present. This is a consequence of the proton's becoming a chargeless neutron when it emits a meson. It does contribute to the neutron moment since the neutron becomes a proton on emitting a meson. The factor of two is due to the fact that a charged meson field really consists of two real fields.

It is interesting to note that from the calculation for a purely charged field that for all possible combinations of neutral and charged fields may be derived—since B_1 and B_2 are needed for charged mesons and no new integrals intervene in the general case.

Evaluating the elementary integrals B_1 and B_2 and expressing the results in terms of nuclear magnetons, we obtain (with λ the square of the ratio of meson to nucleon masses):

Neutral theory

$$\mu_p=-\frac{f_4{}^2}{4\pi^2\hbar c}\left\{\frac{1}{4}+\frac{\lambda}{2}-\frac{\lambda(1-\lambda)}{4}\ln\frac{1}{\lambda}-\frac{\lambda^2(3-\lambda)\cos^{-1}\lambda^{\frac{1}{2}}/2}{2(4\lambda-\lambda^2)^{\frac{1}{2}}}\right\}, \quad (89)$$

$$\mu_n=0.$$

Charged theory

$$\mu_p=\frac{f^2}{4\pi^2\hbar c}\left\{\frac{1}{2}-\lambda+\frac{\lambda(2-\lambda)}{2}\ln\frac{1}{\lambda}-\lambda\frac{(2-4\lambda+\lambda^2)}{(4\lambda-\lambda^2)^{\frac{1}{2}}}\cos^{-1}\frac{\lambda^{\frac{1}{2}}}{2}\right\}, \quad (90)$$

$$\mu_n=\frac{-f^2}{4\pi^2\hbar c}\left\{1+\frac{\lambda\ln(1/\lambda)}{2}-\frac{\lambda(2-\lambda)}{(4\lambda-\lambda^2)^{\frac{1}{2}}}\cos^{-1}\frac{\lambda^{\frac{1}{2}}}{2}\right\}.$$

The symmetrical theory results are obtained by adding those for a neutral theory with $f_4=f$ to those for a charged theory. These magnetic moments are identical (when notation is adjusted) to those obtained by Luttinger with his "patent state" method.[2]

Since λ is quite small, these formulas are well approximated to by:

Neutral theory

$$\mu_p=-(f^2/4\pi\hbar c)1/\pi\{\tfrac{1}{4}+0(\lambda)\}, \quad (91)$$
$$\mu_n=0.$$

Charged theory

$$\mu_p=(f^2/4\pi\hbar c)1/\pi\{\tfrac{1}{2}+0(\lambda^{\frac{1}{2}})\}, \quad (92)$$
$$\mu_n=-(f^2/4\pi\hbar c)1/\pi\{1+0(\lambda^{\frac{1}{2}})\}.$$

Symmetrical theory

$$\mu_p=(f^2/4\pi\hbar c)1/\pi\{\tfrac{1}{4}+0(\lambda^{\frac{1}{2}})\}, \quad (93)$$
$$\mu_n=-(f^2/4\pi\hbar c)1/\pi\{1+0(\lambda^{\frac{1}{2}})\}.$$

Comparison of these results with experiment is greatly hampered by our lack of knowledge of the coupling constant. The very questionable values obtained from trying to fit the predicted nuclear forces to the deuteron binding energy or other two-particle interactions give much too large a value for the anomalous neutron moment (a factor of two or more). However, even omitting this consideration it can be seen that these moments are inadequate. Experimentally, the anomalous proton and neutron moments are almost equal in magnitude. The best of the above theories (charged) has the predicted values differing by a factor of two. The lack of agreement with experiment may be due to the several causes discussed later.

VI. THE NEUTRON-ELECTRON INTERACTION*

Specializing (77) and (78) to the case of the neutron, and taking the four-vector potential in (77) and (78) to be that due to a stationary charge $-e$ at the origin, gives the neutron-electron interaction operator. The $\sigma_{\mu\nu}$ terms may be omitted. The space-space components vanish because there is no space vector potential. The space time components describe the interaction of the electric dipole moment of the neutron with the electron. This dipole moment is merely that resulting from the motion of the particle with the previously determined magnetic moment. As such it is proportional to the velocity and hence, negligible in the slow neutron region in which the experiments are performed.

With the above results, and interchanging x and y and the order of integration in (77), the neutron-electron interaction operator is given by

$$H_{N-e} = -\frac{ef^2}{8\pi^2\hbar c} V^0(\mathbf{r})\bar{\psi}\gamma_4\psi$$
$$\times \int_0^1 dx \int_x^1 dy \frac{[y^2\kappa_0^2+\phi(y)]x(y-x)(\Delta P_\mu)^2}{\phi(y)[x(y-x)(\Delta P_\mu)^2+\phi(y)]}$$
$$+\int_0^1 dx \int_x^1 dy \ln \frac{[(y-x)x(\Delta P_\mu)^2+\phi(y)]}{\phi(y)}$$
$$+\int_0^1 dx \int_0^x dy \frac{2\kappa_0^2 y^2(x-y)(1-x)(\Delta P_\mu)^2}{\phi(y)[(x-y)(1-x)(\Delta P_\mu)^2+\phi(y)]}$$
$$-\int_0^1 dx \int_0^x dy \ln \frac{[(x-y)(1-x)(\Delta P_\mu)^2+\phi(y)]}{\phi(y)}. \quad (94)$$

Here $V_0(\mathbf{r})$ is the Coulomb potential due to the stationary electron at the origin, i.e., in our rationalized units

$$V_0(\mathbf{r}) = -(e/4\pi|\mathbf{r}|). \quad (95)$$

The operator interpretation of ΔP_μ shows that

$$(\Delta P_\mu)^2 \rightarrow -\Box^2, \quad (96)$$

where the operator acts on $\bar{\psi}\gamma_4\psi$.

Integrating by parts results in this operator being applied to $V^0(\mathbf{r})$. Since $V^0(\mathbf{r})$ is independent of time, the D'Alembertian is equivalent to the Laplacian acting on $V^0(\mathbf{r})$. To evaluate the resultant of the operator on $V^0(\mathbf{r})$, the latter is expanded as a three-dimensional Fourier Integral. Thus,

$$V^0(\mathbf{r}) = (1/(2\pi)^{\frac{3}{2}}) \int V_p{}^0 e^{ip\cdot r}(dp). \quad (97)$$

The Laplacian acting on this is equivalent to multiplication by $-p^2$. These replacements may be summarized by saying that $(\Delta P_\mu)^2$ in (94) is to be replaced by P^2, and that this function multiplied by $V_p{}^0$ is the Fourier transform of the potential.

This yields

$$H_{N-e} = \psi^*\psi V(r), \quad (98)$$

where

$$V(r) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{ip\cdot r} V_p(dp), \quad (99)$$

$$V_p = \frac{-ef^2}{8\pi^2\hbar c} \int_0^1 dx \int_x^1 dy \frac{[y^2\kappa_0^2+\phi(y)]x(y-x)P^2 V_p{}^0}{\phi(y)[x(y-x)P^2+\phi(y)]}$$
$$+\int_0^1 dx \int_0^x dy \frac{2\kappa_0^2 y^2(x-y)(1-x)P^2 V_p{}^0}{\phi(y)[(x-y)(1-x)(P^2)+\phi(y)]}$$
$$+\int_0^1 dx \int_x^1 dy \ln \frac{[(y-x)xP^2+\phi(y)]}{\phi(y)} V_p{}^0$$
$$-\int_0^1 dx \int_0^x dy \ln \frac{[(x-y)(1-x)P^2+\phi(y)]}{\phi(y)} V_p{}^0. \quad (100)$$

Here H_{N-e} is in a form such that $V(r)$ is obviously the neutron-electron potential.

An elementary integration gives

$$V_p{}^0 = [-e/(2\pi)^{\frac{3}{2}}]1/p^2. \quad (101)$$

Inserting (101) in (100) and then putting the resulting expression in (99) gives the neutron-electron potential. At least some of the integrations may be performed—at considerable labor. However, the only quantity of experimental interest is the integral of the potential over-all space. This may, of course, be obtained by finding the indicated function of $\mathbf{r}$ and integrating directly. A simpler method

* This work was stimulated by learning of a similar calculation performed by Dr. M. Slotnick using more conventional methods. M. Slotnick and W. Heitler, Phys. Rev. **75**, 1645 (1949).

is found by noting that

$$\int V(\mathbf{r})d\tau=(2\pi)^{\frac{3}{2}}V_p(p=0). \quad (102)$$

Letting P go to zero in (100), and performing the elementary x and y integration which remain, gives

$$\int V(\mathbf{r})d\tau=\frac{+e^2f^2}{8\pi^2\kappa_0^2\hbar c}\left\{\frac{1}{4-\lambda}\left[\frac{13}{3}-\lambda 4\right]-\left[\frac{1}{4}-\frac{2\lambda}{3}\right]\log\frac{1}{\lambda}+\lambda^2\left[\frac{-35}{3}+\frac{17\lambda}{2}-\frac{4\lambda^2}{3}\right]\frac{\cos^{-1}\lambda^{\frac{1}{2}}/2}{(4\lambda-\lambda^2)^{\frac{3}{2}}}\right\}. \quad (103)$$

It is convenient as well as conventional to express the integrated potential in terms of the depth of a square well of width r_0 which gives the same volume integral. r_0 is the classical electron radius. From (98) we find

$$V_0=-\left(\frac{f^2}{4\pi\hbar c}\right)\frac{1}{\alpha^2}\left(\frac{m}{M}\right)^2 mc^2 \times\frac{3}{2\pi}\left\{\frac{1}{4-\lambda}\left[-\frac{13}{3}+4\lambda\right]+\left[\frac{1}{4}-\frac{2\lambda}{3}\right]\log\frac{1}{\lambda}+\lambda^2\left[\frac{35}{3}-\frac{17\lambda}{2}+\frac{4\lambda^2}{3}\right]\frac{\cos^{-1}\lambda^{\frac{1}{2}}/2}{(4\lambda-\lambda^2)^{\frac{3}{2}}}\right\}, \quad (104)$$

where α is the fine structure constant and m the electron mass.

Since λ is small, an accurate approximation for V_0 is

$$V_0\cong-\left(\frac{f^2}{4\pi\hbar c}\right)\frac{1}{\alpha^2}\left(\frac{m}{M}\right)^2 mc^2 \times\frac{3}{2\pi}\left\{-\frac{13}{12}+\frac{1}{2}\log\frac{1}{\lambda^{\frac{1}{2}}}+\frac{35\pi\lambda^{\frac{1}{2}}}{48}\right\}. \quad (105)$$

Inserting numerical values for the universal constants gives

$$V_0\cong-(1.36)\left(\frac{f^2}{4\pi\hbar c}\right)\times\left(\frac{-13}{12}+\frac{1}{2}\log\frac{1}{\lambda^{\frac{1}{2}}}+\frac{35\pi\lambda^{\frac{1}{2}}}{48}\right)\text{ kev.} \quad (106)$$

For a mass ratio of 6, (106) becomes

$$V_0=-(270)(f^2/4\pi\hbar c)\text{ ev.} \quad (107)$$

For a ratio of 10:1,

$$V_0=-(410)(f^2/4\pi\hbar c). \quad (108)$$

Again, the unknown coupling constant occurs. This may be fitted in several arbitrary ways. If the neutron moment be fitted $(f^2/4\pi\hbar c)\sim 6$ and so the well depth is of the order 2 kev. Fitting the proton moment in the symmetrical theory would give $(f^2/4\pi\hbar c)\sim 24$ and a well depth of 8 kev. Fitting the singlet state nucleon scattering predicted by the pseudoscalar theory can give values as high as 50 for the coupling constant. The well depth is then between 10 and 20 kev. It is interesting, even if coincidental, that approximate agreement with experimental values for nuclear forces, the proton moment and the neutron-electron interaction may be obtained with one value of the coupling constant.

However, the most important fact to be gleaned from these numbers would seem to be that it is impossible to fit the neutron and proton anomalous moments and the neutron-electron interaction with the same constants. This is due to the much too large neutron moments obtained when proton moment and neutron-electron interaction are adjusted to agree with experiment.

At this point, a few words are in order concerning the neglect of second-order effects of the first-order corrections of the nuclear electromagnetic interaction caused by the meson field. In the somewhat analogous Lamb Shift calculation these terms are quite important and serve to avoid the infra-red catastrophe. For the neutron-electron interaction these terms are of higher order in the coupling constants and are to be omitted. Thus, if H_I denotes the first-order correction,

$$H_I\sim e^2g. \quad (109)$$

The second-order perturbation due to this term is

$$H_{\text{pert}}=\sum_A\frac{(H_I)_{OA}(H_I)_{AO}}{E_A-E_O}, \quad (110)$$

where A denotes the intermediate state.

The numerator in (110) is by (109) of order e^4g^2. In the Lamb Shift calculation the energy differences in the denominator are proportional to the coupling constants and so reduce the order of smallness of (110). The neutron, though, does not have any zeroth-order coupling with the electromagnetic field of the electron. Moreover, the meson has a finite rest mass. Hence, the energy difference in (110) has a zeroth-order term independent of e and g. Thus, for a neutron in the field of an electron (110) is of order e^4g^2. Terms of this order have been considered to be negligible throughout this calculation.

VII. SCALAR THEORY

To interpret the above results it is most desirable to know the predictions of some other meson theory. There would then be some possibility of separating out effects arising from the perturbation approximation from those due to the model chosen. Fortunately the results of the scalar meson theory with scalar coupling are readily obtainable. Thus, substituting i for γ_5 in the previous formulas gives the effective nucleon Hamiltonian as modified by interaction with scalar mesons. The calculations from Eq. (31) on are then very similar to those given above. It should be sufficient to quote the results.

In nuclear magnetons the anomalous magnetic moments are given as follows:

Neutral theory

$$\mu_p=+(f^2/4\pi\hbar c)c_1/2\pi, \quad \mu_n=0. \tag{111}$$

Charged theory

$$\mu_p=(f^2/4\pi\hbar c)c_2/\pi, \qquad \mu_n=-(f^2/4\pi\hbar c)[(-c_1+c_2)/\pi], \tag{112}$$

where

$$c_1=\frac{3}{2}-\lambda-\frac{\lambda(\lambda-3)}{2}\log\frac{1}{\lambda}-\frac{\lambda(\lambda^2-5\lambda+4)}{(4\lambda-\lambda^2)^{\frac{1}{2}}}\cos^{-1}\frac{\lambda^{\frac{1}{2}}}{2}, \tag{113}$$

$$c_2=\frac{5}{2}-\lambda-\frac{(\lambda^2-4\lambda+2)}{2}\log\frac{1}{\lambda}-\frac{\lambda(\lambda^2-6\lambda+8)}{(4\lambda-\lambda^2)^{\frac{1}{2}}}\cos^{-1}\frac{\lambda^{\frac{1}{2}}}{2}. \tag{114}$$

The symmetrical theory results may again be obtained by adding those for neutral and charged. For small mass ratios:

$$c_1\sim 3/2; \quad c_2\sim 5/2-\log 1/\lambda. \tag{115}$$

As the meson mass goes to zero, this gives:

Neutral

$$\mu_p=0.240(f^2/4\pi\hbar c); \quad \mu_n=0. \tag{116}$$

Charged

$$\mu_p\rightarrow-\infty; \quad \mu_n\rightarrow+\infty. \tag{117}$$

The divergence in (117) is logarithmic. The sum of neutron and proton moments in (117) is finite and is

$$\mu_p+\mu_n\rightarrow+0.48(f^2/4\pi\hbar c). \tag{118}$$

To see how important the logarithmic term is, we have calculated the moments for several values of the meson mass. The values given in Table I give the resultant moments divided by $(f^2/4\pi\hbar c)$.

TABLE I. Anomalous moments $\div f^2/4\pi\hbar c$.

Meson mass	1100	300	180
Neutral μ_p	+0.113	+0.183	+0.203
Neutral μ_n	0	0	0
Charged μ_p	+1.33	+0.03	−0.468
Charged μ_n	−1.1	+0.34	+0.870

It has conventionally been assumed that $(f^2/4\pi\hbar c)\sim 3\lambda^{\frac{1}{2}}$. This is derived on the basis of a rough fitting of the scalar theory nuclear forces to experimental data. This procedure has very little to recommend it since the scalar theory forces are not those actually observed, i.e., the scalar theory must have very little to do with nucleon interaction. However, we can, in this manner, obtain an order of magnitude estimate. Using such a fit gives much too small magnetic moments for all except the very largest meson masses. This is to be expected since the anomalous moments calculated in a non-relativistic approximation assuming an infinitely heavy nucleon are zero. All in all it can be said that these moments have little in common with experimental ones and should not be expected to agree. The one striking fact is the logarithmic dependence on mass. This point will be discussed more fully below. The above numbers do show that, although the moments tend to infinity as the mass goes to zero, this logarithmic increase is significant only for very, very small meson masses.

The equivalent well depth for the neutron-electron interaction[8] in the scalar theory is given by

$$V_0=-1.36(f^2/4\pi\hbar c)\left\{\frac{(4-17\lambda+73\lambda^2/2-31\lambda^3/2)}{3(4\lambda-\lambda^2)}+\tfrac{1}{6}(13/2-5\lambda+\lambda^2)\log\frac{1}{\lambda}+\frac{\cos^{-1}\lambda^{\frac{1}{2}}/2}{(4\lambda-\lambda^2)^{\frac{3}{2}}}\times(-28\lambda/3+67\lambda^2/3-9\lambda^3/2+11\lambda^4/3-\lambda^5/3)\right\}\text{ kev.} \tag{119}$$

The interesting point about (119) is that V_0 is here proportional to $1/\lambda$ for small λ (i.e., it is inversely proportional to the square of the meson mass). This is to be contrasted to (104) which is proportional to $\log\lambda$.

For a mass ratio of 6:1, (119) gives

$$V_0=-6.55(f^2/4\pi\hbar c)\text{ kev.} \tag{120}$$

To see the rate of increase with decreasing λ, the well depth for a 10:1 ratio has also been computed. This is

$$V_0=-26.5(f^2/4\pi\hbar c)\text{ kev.} \tag{121}$$

[8] A similar calculation has been done by Dancoff and Drell using ordinary perturbation theory. Dancoff and Drell, Phys. Rev. (to be published).

With our wildly hypothetical fitting of the coupling constant the well depth is seen to be of the order of several kev.

VIII. CONCLUSION

To summarize, we may say that when provision is made for mass renormalization, the modifications in nucleon electromagnetic properties caused by interaction with a meson field are finite and unambiguous. (This is at least true for the scalar and pseudoscalar theories.) Some of the present results are uncertain because of our lack of knowledge of the coupling constants. The absolute values of all quantities calculated here are all indefinite, since any fit to nuclear force data is questionable in the present state of the theory. The relative values of neutron moment, proton moment, and neutron-electron interaction can be found independently of the exact value of the coupling constants. It can be concluded that the quantities calculated above on the basis of a pseudoscalar theory cannot fit the existing experimental data. It may be hoped that other models for the meson field and the coupling will give more favorable values. The results for the scalar theory which have been included throw some light on this point. There is also the possibility that large higher order corrections may significantly alter the theoretical predictions. Since the first-order effects that have been found are so large, this possibility is particularly likely.

The one serious difficulty encountered in the pseudoscalar theory is that the contributions to the magnetic moments due to the meson's electric charge is not large compared with those due to the proton's charge. Qualitatively one can argue in the following way. The magnetic moment from the proton's charge is due to a zero-point oscillation of a particle of charge e over a region of dimensions of the order of the nucleon Compton wave-length. Thus, it would be expected that in nuclear magnetons this contribution would be

$$\sim\left(\frac{f^2}{4\pi\hbar c}\right)\frac{e}{Mc/\hbar}\,\frac{1}{e\hbar/2Mc}=2\left(\frac{f^2}{4\pi\hbar c}\right). \qquad (122)$$

This expectation is indeed born out by the above detailed calculation. The meson charge contribution to the magnetic moments should be roughly that of a charge e spread out over a distance like that of a meson Compton wave-length, i.e., the contributions to the magnetic moment in nuclear magnetons should be

$$\left(\frac{f^2}{4\pi\hbar c}\right)\frac{e}{\mu c/\hbar}\,\frac{1}{e\hbar/2Mc}=2\left(\frac{f^2}{4\pi\hbar c}\right)\frac{M}{\mu}$$

$$=2\left(\frac{f^2}{4\pi\hbar c}\right)\frac{1}{\lambda^{\frac{1}{2}}}. \qquad (123)$$

Thus, the meson contributions should be of order M/μ times the proton contribution. Since the meson portions are equal and opposite for neutron and proton, approximate agreement with the almost equal and opposite character of the observed moments would be expected. However, (123) tells us that for small meson masses the meson contribution to the moments should be proportional to $\lambda^{-\frac{1}{2}}$. Equation (92) shows that this is not so for the pseudoscalar theory to our approximation.

The question arises as to what extent this failure to predict equal and opposite anomalous moments justifies throwing out the pseudoscalar theory and in what way higher order corrections may be expected to modify the present statements. Suggestions as to the answers to these questions are given by the neutron-electron interaction and the scalar theory results. In measuring the integral of the neutron-electron interaction and the magnetic moments, essentially two properties of the meson cloud are determined. The former measurement tells us about the spherically symmetric charge distribution while the latter gives information about the asymmetric portion of the cloud. *A priori* one would expect the symmetric part to extend out somewhat further but that the orders of magnitude of the two regions would not be appreciably different. In the pseudoscalar theory the portion of the integrated potential coming from the meson distribution can be seen to be proportional to $\log\lambda$. This corresponds to a potential of the form $\exp(-\kappa r/r^3)$ in the distant regions which contribute to the neutron-electron interaction. On the other hand, we have seen that the regions of space over which the asymmetric charge distributions which contributes to the magnetic moment exists are of linear dimensions $1/\kappa_0$. Thus, in the pseudoscalar theory we have, to first approximation, an anisotropic meson distribution extending to the nucleon Compton wave-length followed by a symmetrical distribution of the form $\exp(-\kappa r/r^3)$. In the scalar theory the integrated potential is inversely proportional to λ, corresponding to an $\exp(-\kappa r/r)$ symmetrical distribution. The meson portion of the magnetic moments are proportional to $\log\lambda$ for small λ. Hence, the asymmetric charge distribution is spread over a region of linear dimensions somewhere intermediate between the nucleon and meson Compton wave-lengths.

From the above, one can conclude that the failure of pseudoscalar theory to give approximately equal and opposite moments is not an indication that all models will similarly fail. The pseudoscalar theory is peculiar in that the meson cloud accompanying a nucleon is particularly closely bound to the nucleon. In other theories the mesons can be more widely spread out giving some hope of approximate agreement with experiment.

One can also make some estimate as to the relative changes in the quantities calculated here that will result from taking into account higher order corrections. From the general knowledge acquired in dealing with additional radiative corrections, it can be said that the general effect is to spread out density distributions. Thus, including further terms in the coupling constant would distribute the effects over a distance which might be something intermediate between nucleon and meson wave-lengths. Hence, quantities such as the neutron-electron interaction calculated here should be relatively unaffected. In particular, the scalar theory potential would be less affected than that found in the pseudoscalar theory. The magnetic moments, as calculated here, have their principal contributions from quite small distances. These would be expected to suffer substantial modification when radiative corrections are included. This is particularly true for the pseudosclar magnetic moments. Thus, precisely those calculated quantities which are in most flagrant contradiction with experiment are those which have been least accurately computed.

The objection to the pseudoscalar theory that the calculated moments are not even approximately equal in magnitude would still seem to be rather serious. Although the calculated moments may be expected to suffer considerable alteration in magnitude by the spreading out, it is difficult to see why the relative magnitudes of proton charge and meson charge contributions should be altered. In order for agreement with experiment to be achieved, the effects of the higher order terms must be highly selective. The proton "Zitterbewegung" must be but slightly changed while the anisotropic meson current is spread out to the meson Compton wavelength. That this will happen is rather debatable.

It is the author's personal opinion that the most hopeful approach is to continue to seek some model which gives rough qualitative agreement with experiment when calculated to the first order in the coupling constant. Including higher order corrections might then be expected to give quantitative agreement. The results found for the scalar theory indicate that the finding of such a qualitatively agreeing model is at least a possibility.

In conclusion I would like to thank Professor J. R. Oppenheimer for several very illuminating discussions.

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APPENDIX A

Vanishing of vacuum polarization terms

The result obtained in Section IV, that not only the charge renormalization but all other polarization terms vanish for the pseudoscalar coupling, may be based on a considerably simpler and more general physical proof suggested by Professor J. R. Oppenheimer. It is deduced from the theorem that polarization phenomena vanish when particle and antiparticle have the same "mesonic charge" (f). Vacuum polarization is due to the creation and separation of pairs. However, if particle and antiparticle have the same "mesonic charge," the force on the particles of a pair due to a virtual meson will be the same. Therefore, the meson field will not separate the pair and there will be no polarization.

The relation between the mesonic charges of particle and antiparticle may be derived from the twin postulates of invariance under time reversal and charge conjugation. Let us suppose an interaction term is to be used which contains some Dirac covariant. Omitting the meson field quantities, this term will be

$$H_{\text{int}} \equiv g\bar{\psi}0\psi, \tag{A1}$$

where 0 is some combination of γ's, and g is the "mesonic charge." For the expression to be covariant under the transformation

$$x_i \rightarrow x_i', \quad x_4 \rightarrow -x_4', \tag{A2}$$

(A1) must be modified by the introduction of the charge conjugate to B

$$H_{\text{int}}' = (g/2)[\bar{\psi}0\psi \pm \psi'0\psi'], \tag{A3}$$

where primes on the ψ denote charge conjugate operators. Whether the plus or minus sign is to be taken in A3 follows uniquely from the properties of 0.

Suppose the plus sign is found necessary for a given 0. Since $\psi \rightarrow \psi'$, $\bar{\psi} \rightarrow \psi$ under the transformation $e \rightarrow -e$, the requirement of invariance for H_{int}' requires that

$$e \rightarrow -e, \quad g \rightarrow g, \tag{A4}$$

i.e., particle and antiparticle have the same charge g.

On the other hand, if the minus sign holds in (A3), charge invariance requires

$$e \rightarrow -e, \quad g \rightarrow -g, \tag{A5}$$

and then particle and antiparticle have equal and opposite "mesonic charges."

On examining the five Dirac covariants, it is found that vector and tensor coupling require opposite charge. Scalar, pseudoscalar, and pseudovector coupling must have the same mesonic charge. Hence, for the last three types of coupling, all vacuum polarization terms vanish.