the adjoining medium. When the electron is near the surface one is not allowed to smooth out the charge and consider that we have a uniform layer of charge. The reason for this is that the electron or the hole sees its image in the adjoining medium and the image follows the detailed motion of the particle. This effect will cause a raising or a lowering of the potential of 0.1 ev at a distance of 1A° from the surface. If the adjoining medium is a dielectric of lower κ, as in the case of silicon and air, the image charge will have the same sign as the charged particle.[22]

For these two reasons one cannot rely completely on the derivation just given when φ is larger than 0.5 ev. Most probably the equations are correct to within an order of magnitude but should not be trusted further. For $\varphi = 0.3$ ev or less, the surface effects are not so important and the theory should be fairly reliable. Most fortunately, the maximum value of φ we used for the diffusion layer in the case of silicon is less than 0.3 ev, and only one point is greater than this value for germanium.

Since the potential layer is very thin, tunneling effects can be very important. This would be true of a semiconductor metal junction. For the experiments of Smith this does not play a major role, since there is a large gap between the two surfaces which creates a large potential barrier between the conduction band of the metal and the semiconductor.

[22] M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Blackie and Son, Limited, London, 1937), p. 76.

PHYSICAL REVIEW VOLUME 75, NUMBER 6 MARCH 15, 1949

Relativistic Field Theories

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(Received August 9, 1948)

The study of the change of the Schroedinger wave functionals on space-like surfaces as the surface changes is formulated simply. In this formulation, we let S be any space-like surface with the curvilinear coordinates u, introduce $\langle q(u)S|\rangle$ as the Schroedinger functional on it, with the nature of the functions $q(u)$ not specified for the moment, and write the Schroedinger equation as

$$\hbar i\{\langle q(u),\ S'|\rangle - \langle q(u),\ S|\rangle\} = J\langle q(u),\ S|\rangle, \tag{1}$$

where S' is a surface slightly different from S, and J is an operator operating on $q(u)$ and depending on S, S', and the choice of their coordinates u. Obviously, the only necessary conditions for J is that J is Hermitian and that (1) is integrable. To get a theory resembling the existing ones, we require further that if we construct the expectation value of a certain field quantity at a point P from $\langle q(u),\ S|\rangle$, S being a surface passing through P, the expectation value is independent of choice of S and its coordinates u, and satisfies certain differential equations.

When these differential equations follow from a Langrangian principle, an expression for J was effectively given by Weiss. His work is given here in a complete form and the proof of the satisfaction of all requirements completed.

Finally, by a simple transformation on the wave functions and the observables, we deduce from the Weiss's formalism the Tomonaga's formalism. As a consequence, it is pointed out that the Tomonaga's formalism will encounter essentially the same difficulties as the orthodox Heisenberg-Pauli formalism.

1. INTRODUCTION

A QUANTUM theory of fields was given by Heisenberg and Pauli[1] as early as 1929. The theory given by them was completely apart from one aspect, i.e., there was no investigation of the transformation between the wave functionals for two different Lorentz observers. It is obvious, though never pointed out explicitly in the literature, that the operator transforming the Schroedinger functional for one Lorentz-frame to that for another must be related to the integral over space of the $\mu\nu 4$ component of the angular momentum tensor $M_{\mu\nu\rho}$. It is also obvious that as the Schroedinger functionals form a representation basis of a representation of the homogeneous and the inhomogeneous Lorentz group, these operators together with the space integral of the $\mu 4$ component of the energy momentum tensor $T_{\mu\nu}$ must satisfy certain well-known commutation laws.

Recently, Dirac introduced the study of the Schroedinger wave functionals on any arbitrary space-like surface.[2] He introduced certain deformation operators to describe the deformation of the surfaces and studied their commutation laws. Though the theory is the most general quantum mechanics of localizable dynamical systems and can be employed to check whether any given theory is relativistic, the explicit construction of the operators

[1] W. Pauli and W. Heisenberg, Zeits. f. Physik 56, 1 (1929).

[2] P. A. M. Dirac, Phys. Rev. 73, 1092 (1948).

Π corresponding to a given parametrization of the surface, and the operators P corresponding to a given dynamical system is not simple work. A simpler and equivalent formulation of Dirac's work is as follows.

Let S be any space-like surface, (u_1, u_2, u_3) any arbitrary set of curvilinear coordinates on the surface, so that on the surface, we have

$$x_\mu = b_\mu(u_1, u_2, u_3). \tag{1}$$

Now let us introduce the Schroedinger wave functional Ψ, which depends on S and its coordinate system u (i.e., is a functional of the functions $b_\mu(u_1, u_2, u_3)$), and is further a functional of certain functions $q(u)$ whose nature we shall not specify at the moment. Such a functional may be considered as a vector depending on $b_\mu(u)$ with components (or coordinates) labelled by functions $q(u)$, and will be denoted by

$$\langle q(u) ; b_\mu(u) | \rangle. \tag{2}$$

In a deformation of S to S' given by

$$x_\mu = b_\mu'(u) = b_\mu(u) + \Delta x_\mu(u), \tag{3}$$

the corresponding change in Ψ is defined by

$$\Delta\Psi = \langle q(u) ; b_\mu'(u) | \rangle - \langle q(u) ; b_\mu(u) | \rangle. \tag{4}$$

Obviously, we must introduce an equation

$$\hbar i \Delta\Psi = J\Psi, \tag{5}$$

where J is an operator operating on the coordinates $q(u)$ of Ψ and may be looked upon as a matrix with both rows and columns labelled by functions $q(u)$. Obviously, (i) Eq. (5) considered as a total differential equation for Ψ must be integrable. This condition replaces the study of the Poisson brackets between the operators Π's and P's in Dirac's paper. (ii) J must be Hermitian, so that the length of Ψ does not change. As Ψ is a functional, its length can be best pictured by first letting (u_1, u_2, u_3) take discrete values (m_1a, m_2a, m_3a) only, $(m_1, m_2, m_3 = 0, \pm 1, \pm 2, \cdots)$ so that the square of the length is

$$\int dq(u') \int dq(u'') \cdots | \langle q(u'), q(u''), \cdots ; b_\mu(u'), b_\mu(u''), \cdots | \rangle |^2, \tag{6}$$

where $u', u'', \cdots$ denote the different discrete values of (u_1, u_2, u_3), and then passing to the limit $a \rightarrow 0$.

The above gives us rules for the determination of Ψ, but to get a physically sensible theory, we must construct expressions for the different "expectation values" in the state Ψ. For this purpose, let us consider any given point P with coordinates x_μ, draw a space-like surface S passing through P, introduce a coordinate system u on the surface so that $x_\mu = b_\mu(u)$ on it, and assume

$$\int dq(u') \int dq(u'') \cdots | \langle q(u'), q(u''), \cdots ; b_\mu(u'), b_\mu(u''), \cdots | \rangle |^2 q(u_P) \tag{7}$$

divided by the length (6) be the expectation value of a certain field quantity $q(x)$ at P, u_P in (7) being the u-coordinates of P on this surface S. Under this assumption, we naturally expect (iii) the expectation value given by (7) divided by (6) is independent of the choice of S and that of the coordinate system u on S, and (iv) the expectation value will satisfy certain differential equations, known as the field equations for $q(x)$. The last two conditions are not absolutely necessary, they may be discarded or replaced by similar ones, but their satisfaction will bring the present theory very close to the usual field theory of Heisenberg and Pauli.

For fields $q(x)$ whose field equations are deduced from the variation of a Lagrangian, an equation of the same nature as Eq. (5) was given some time ago by Weiss.[3] Unfortunately he has effectively only considered a one-parameter family of surfaces and thus we do not have the requirements (i) and (iii). In §2 we shall extend his work to general deformations of surfaces and point out that all conditions mentioned above are satisfied. We shall only indicate how the proof of the integrability is carried out and leave out all mathematical details.

The restriction to fields satisfying the Lagrangian principle is a serious one, though this does not exclude us from considering any of the existing fields in quantum mechanics. At the same time, it is difficult to see how one can get rid of the Lagrangian principle. As soon as one has the Schroedinger equation,

$$\hbar i (\partial\Psi/\partial t) = H\Psi,$$

one has the Heisenberg equation of motion,

$$df/dt = -(\hbar i)^{-1}[H, f]. \tag{8}$$

If H contains only operators q and $\partial/\partial q$, (8) are the classical Hamiltonian equations of motion, and these are derivable from a Lagrangian.

In the last section (§3), we show that it is possible to get from our Eq. (5) Tomonaga's formalism[4] by a simple transformation. The result obtained here is slightly more general than the original Tomonaga's formalism which does not allow points on the surface S to interact. The possibility of obtaining Tomonaga's formalism from the Weiss's formalism implies that the Tomonaga's formalism will encounter essentially the same difficulties as the orthodox Heisenberg-Pauli formalism and will not give us anything essentially now.

[3] P. Weiss, Proc. Roy. Soc. **A156**, 192 (1936).
[4] S. Tomonaga, Prog. Theoretical Phys. **1**, 27 (1946).

2. THE THEORY OF WEISS IN COMPLETE FORM

Let x_μ be (x, y, z, ict) as usual, and let q^α be the field quantities. Let $q_\mu{}^\alpha$ denote $\partial q^\alpha/\partial x_\mu$ and let q^α satisfy equations obtained from the variation of a Lagrangian L. We have thus

$$\partial L/\partial q^\alpha - \partial/\partial x_\mu(\partial L/\partial q_\mu{}^\alpha) = 0. \qquad (9)$$

Let us introduce

$$N_\mu = \epsilon_{\nu\rho\theta\mu}\frac{\partial x_\nu}{\partial u_1}\frac{\partial x_\rho}{\partial u_2}\frac{\partial x_\theta}{\partial u_3}, \qquad (10)$$

where $\epsilon_{\nu\rho\theta\mu}$ is plus or minus one according to whether $\nu\rho\theta\mu$ is an even or odd permutation of 1 2 3 4 and is zero otherwise. Let us introduce

$$p_\alpha = N_\mu(\partial L/\partial q_\mu{}^\alpha), \qquad (11)$$

$$G_\mu = LN_\mu - \sum_\alpha p_\alpha q_\mu{}^\alpha, \qquad (12)$$

$$q_r{}^\alpha = q_\mu{}^\alpha(\partial x_\mu/\partial u_r). \qquad (13)$$

Then, following the spirit of Weiss's method, we let J in (5) be

$$i\int du_1 du_2 du_3 (G_\mu \Delta x_\mu), \qquad (14)$$

where G_μ are functions $G_\mu(q, q_r, p, \partial x_\mu/\partial u_r)$ [5] of q, q_r, p, $\partial x_\mu/\partial u_r$ as given by (10), (11), (12), (13) and q, q_r, p are operators satisfying

$$\left.\begin{aligned} q(u^a)\langle q'(u);\ |\rangle &= q'(u^a)\langle q'(u);\ |\rangle,\\ q_\gamma(u^a)\langle q'(u);\ |\rangle &= (\partial q'(u^a)\partial u_\gamma{}^a)\langle q'(u);\ |\rangle,\\ p(u^a)\langle q'(u);\ |\rangle &= \hbar\delta\langle q'(u);\ |\rangle/\delta q'(u^a), \end{aligned}\right\}^{6} \qquad (15)$$

u^a being a given value of u. The form (14) for J is slightly more general than the corresponding one in Weiss's original formulation, since we do not restrict ourselves to a one-parameter family of surfaces.

To prove the integrability of (5) with J given by (14), we calculate by means of (5), (14), the change $\Delta\Psi$ of Ψ as the functions b_μ change from initial values $b_\mu{}^{(0)}(u)$ to the final values

$$b_\mu{}^{(f)}(u) = b_\mu{}^{(0)}(u) + \Delta_1 x_\mu(u) + \Delta_2 x_\mu(u) \qquad (16)$$

in two ways, one by letting $b_\mu(u)$ pass through the intermediate functions

$$b_\mu{}^{(1)}(u) = b_\mu{}^{(0)}(u) + \Delta_1 x_\mu(u), \qquad (17)$$

and the other by letting $b_\mu(u)$ pass through the intermediate functions

$$b_\mu{}^{(2)}(u) = b_\mu{}^{(0)}(u) + \Delta_2 x_\mu(u), \qquad (18)$$

and compare the two results for $\Delta\Psi$ to the second order in $\Delta_1 x_\mu$, $\Delta_2 x_\mu$. Let us say that their difference is $B\Psi^{(0)}$ where $\Psi^{(0)}$ is the initial wave function. Equation (5) is integrable if $B\Psi^{(0)} = 0$.

Let us consider the Hamilton-Jacobi equation from the Lagrangian L, which is

$$\Delta I = \int G_\mu\left(q, \frac{\partial q}{\partial u_r}, \frac{\delta I}{\delta q(u)}, \frac{\partial x_\mu}{\partial u_r}\right)du. \qquad (du = du_1 du_2 du_3) \qquad (19)$$

This is always integrable provided that the field equations are consistent and admit solutions, and the solution $I(q(u), b_\mu(u))$ of (19) is

$$\int L(q, q_\mu) dx_1 dx_2 dx_3 dx_4, \qquad (20)$$

where the integral extends over a volume bounded by two space like parametrized surfaces, $b_\mu{}^{(00)}$ and b_μ, and the surfaces at infinity, q and q_μ, in L satisfy the field equations, and the whole integral is considered as a functional of $b_\mu{}^{(00)}$ and b_μ and the functions $q^{(00)}(u)$ and $q(u)$, which are the values q, take on the surfaces. From (19), we calculate the change ΔI of I as b_μ changes from $b_\mu{}^{(0)}$ to $b_\mu{}^{(f)}$ in the above two ways, and the difference of these two ΔI's must be zero, since (19) is integrable. It can be easily verified that this difference is precisely B if we replace p contained in B by $\delta I^{(0)}/\delta q(u)$, $I^{(0)}$ being the value of I on the surface $b^{(0)}$. Thus $B\Psi^{(0)} = 0$ and the integrability of (5) is proved.[7]

It is possible to verify directly that the operator B is zero, but the direct verification will be left out. The above proof indicates incidentally what changes in the formalisms are necessary for Fermi-Dirac fields.

The other requirements are easily seen to be satisfied. For (ii) we note that the Hermitian character of J can be ensured with a proper choice of the Lagrangian. For (iii), let us first consider two surfaces passing through the same point P with u_P taking the same value on them. Thus

$$\Delta x_\mu = 0 \text{ at } u = u_P. \qquad (21)$$

The Heisenberg equations of motion,

$$(\Delta q)(u) = -(\hbar i)^{-1}[J, q(u)], \qquad (22)$$

gives us immediately $\Delta q = 0$. Next we consider a change of parameters from u to some new parameters, say v. If a change of parametrization is considered as a change of $b_\mu(u)$, we may employ (22) again and we prove readily that there is no change of q at the point P as this b_μ changes. Finally, (iv) must be satisfied, as it must be in a Weiss theory.

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[5] We drop the suffix α.

[6] Equation (15) restricts us to Einstein-Bose fields, but this restriction is really not essential.

[7] That the integrability of (5) may follow from that of (19) is learned by the author from Professor Dirac.

3. TOMONAGA'S FORMALISM

Another study of the change of Schroedinger wave functionals on space-like surfaces was given by Tomonaga and has been found useful in formulating a relativistic quantum electrodynamics.[8] Let us see if this formalism can be obtained from our formulation (5).

Let us consider for definiteness only electrons and the Maxwell field. Then

$$G_\mu = G_\mu{}^0 + eG_\mu{}^1, \tag{23}$$

where both $G_\mu{}^0$ and $G_\mu{}^1$ do not contain the charge constant e. Let $\bar{G}^0$, $\bar{G}^1$ denote $\int G_\mu{}^0 \Delta x_\mu du$ and $\int G_\mu{}^1 \Delta x_\mu du$, and let us write the wave functions $\langle q; b|\rangle$ as $\langle q|b\rangle$. Obviously, the equation

$$\hbar\Delta\Psi = \bar{G}^0\Psi \tag{24}$$

is integrable and the solution may be written in the form

$$\langle q(u)|b\rangle = \langle q(u)|R(bb_0)|b_0\rangle, \tag{25}$$

where b_0 is any given b and R is an unitary operator. With respect to variations Δx_μ of b, we have

$$\hbar\Delta R = \left(\int G_\mu{}^0 \Delta x_\mu du\right) R, \tag{26.1}$$

and with respect to variations Δx_μ of b_0, we have

$$\hbar\Delta(R^{-1}) = \left(\int G_\mu{}^0 \Delta x_\mu du\right) R^{-1},$$

or

$$\hbar\Delta R = -R\left(\int G_\mu{}^0 \Delta x_\mu du\right). \tag{26.2}$$

Let us introduce a wave function $\langle q(u)|b_1 b\rangle$ connected to the general wave function $\langle q|b\rangle$ in (5) by

$$\langle q(u)|b_1 b\rangle = \langle q(u)|R(b_1 b)|b\rangle. \tag{27}$$

Looked upon as a $\langle q(u)|b_1\rangle$, it satisfies the wave equation (5) without the interaction terms. Its variation with respect to b for a constant b_1 is given by

$$\begin{aligned}\hbar\Delta\langle q(u)|b_1 b\rangle &= \langle q(u)|R(b_1 b)[\bar{G}^0 + e\bar{G}^1]|b\rangle + \langle q(u)|\hbar\Delta R|b\rangle \\ &= \langle q(u)|R(b_1 b)e\bar{G}^1|b\rangle \\ &= \langle q(u)|e\bar{G}^{1*}(bb_1)|b_1 b\rangle,\end{aligned} \tag{28}$$

where operators with *'s are defined in $q(u)$ representation by

$$l^*(b_2 b_1) = R(b_1 b_2) l R^{-1}(b_1 b_2) = R^{-1}(b_2 b_1) l R(b_2 b_1). \tag{29}$$

Considered as a function of u and b_2, $q^*(u)(b_2 b_1)$ satisfies the Heisenberg's equation of motion without the interaction terms. It reduces further at $b_2 = b_1$ to $q(u)$.

For the interaction between the electrons and the electromagnetic field, $G_\mu{}^1$ is equal to $N_\mu L^1$, where eL^1 is the interaction term in the Lagrangian. Hence if Δx_μ are nearly zero everywhere except at a certain value of u, the operator in (28) is

$$eL^{1*}(bb_1)$$

evaluated at this value of u times the volume ΔV between the original surface S and the displaced surface S'. This fact has been employed by Schwinger to introduce a differentiation of Ψ with respect to the surface at a point on the surface.

(28) can be put in a nicer form by introducing a representation $\langle q^{*\prime}(b_1 b_2)|\rangle$. As q^* and q have the same eigenvalues, we may let $q^{*\prime}$, q' (and similarly $q^{*\prime\prime}$, q'') denote the same eigenvalues. Defining $\langle q^{*\prime}(b_1 b_2)|\rangle$ by

$$\langle q^{*\prime}(b_1 b_2)|\rangle = \langle q'|R(b_1 b_2)|\rangle, \tag{30}$$

we have

$$\begin{aligned}\langle q^{*\prime}(b_1 b_2)|q^*(b_1 b_2)|\rangle &= \langle q'|R(b_1 b_2)\{R^{-1}(b_1 b_2) q R(b_1 b_2)\}|\rangle \\ &= q'\langle q'|R(b_1 b_2)|\rangle = q'\langle q^{*\prime}(b_1 b_2)|\rangle\end{aligned} \tag{31}$$

and thus $q^*(b_1 b_2)$ is diagonal in this representation. From (28) and (30)

$$\begin{aligned}\hbar\Delta\langle q^{*\prime}(b_1 b)|b\rangle &= \hbar\Delta\langle q'|R(b_1 b)|b\rangle \\ &= \langle q'|R(b_1 b)e\bar{G}^{(1)}|b\rangle = \langle q^{*\prime}(b_1 b)|e\bar{G}^1|b\rangle.\end{aligned} \tag{32}$$

A further simplification is obtained by introducing $\langle q'(b_1)|b\rangle$ defined by

$$\langle q'(b_1)|b\rangle = \langle q'|R(b_1 b)|b\rangle = \langle q'|b_1 b\rangle, \tag{33}$$

so that

$$\langle q'(b_2)|b\rangle = \textstyle\sum_{q''}\langle q'|R(b_2 b_1)|q''\rangle\langle q''(b_1)|b\rangle,$$

etc., and an operator $l(b)$ by

$$\langle q'(b_1)|l(b_2)|q''(b_1)\rangle = \langle q'|l^*(b_2 b_1)|q''\rangle. \tag{34}$$

Then (28) becomes

$$\begin{aligned}\hbar\Delta\langle q'(b_1)|b\rangle &= \textstyle\sum_{q''}\langle q'|e\bar{G}^{1*}(bb_1)|q''\rangle\langle q''|b_1 b\rangle \\ &= \textstyle\sum_{q''}\langle q'(b_1)|e\bar{G}^1(b)|q''(b_1)\rangle\langle q''(b_1)|b\rangle \\ &= \langle q'(b_1)|e\bar{G}^1(b)|b\rangle.\end{aligned} \tag{35}$$

When $G_\mu{}^1 = N_\mu L^1$, we obtain precisely Tomonaga's formalism. As it stands, (35) is slightly more general than Tomonaga's formalism, since the operator in (35) may be an integral over u of non-commuting quantities. Though the formulas giving the change of the wave functional with respect to the surface S still depend on the parametrization of S, the coordinates of the wave function do not, as pointed out by Dirac.

The expectation value of an operator $l(u)$ on the surface b is, in terms of the wave functions in (5),

[8] J. Schwinger, Conference of physics, Pocono Manor, National Academy of Science, April 1948.

$\langle b|l|b\rangle$ or

$$\sum_{q'}\langle b|q'\rangle\langle q'|l|b\rangle. \tag{36}$$

Transforming to wave functions of the type $\langle q'|b_1b\rangle$, it becomes

$$\langle b_1b|l^*(bb_1)|b_1b\rangle. \tag{37}$$

Transforming to wave functions of the type $\langle q'(b_1)|b\rangle$, it becomes

$$\sum_{q'q''}\langle b|q'(b_1)\rangle\langle q'(b_1)|l(b)|q''(b_2)\rangle\langle q''(b_2)|b\rangle \tag{38}$$

or simply $\langle b|l(b)|b\rangle$.

The passage to Tomonaga's formalism consists essentially of introducing $\langle q'(b_1)|b\rangle$ by (33) and operator $q(b)$ by (34). Thus all features in the orthodox Heisenberg-Pauli formalism are more or less retained in the new formalism. For example, we have in electrodynamics supplementary conditions on the wave functions in (5), and by introducing the transformation (33), (34), we deduce the corresponding supplementary conditions on the new wave functions. If the supplementary conditions on the wave functions in (5) are consistent and are satisfied always if satisfied for one surface b, so are the new supplementary conditions. Further, Schwinger has shown for electrons interacting with the Maxwell field that from (35), a certain transformation of the wave function produces a wave equation with the same left-hand side but with the operator on the right-hand side as a power series of e with the leading power e^2. Thus similarly, a certain transformation on the wave function in (5) will also produce the same result. In fact, if we let the wave function in (5) be subjected to the transformation

$$\langle q|b\rangle_{\text{new}}=e^{iS(b)}\langle q|b\rangle_{\text{old}}. \tag{39}$$

where $S(b)$ is an operator depending on b and satisfies

$$\hbar i\Delta S=-e\bar{G}^1-i[S,\bar{G}^0], \tag{40}$$

we find that the above is integrable, and admits the solution

$$S(b)=-(e/\hbar i)\int R^{-1}(b_1b)\bar{G}^1R(b_1b), \tag{41}$$

where Δx_μ in G^1 is the change of b_1 and the integration is carried out with respect to such increments from any initial surface to the final surface b. The wave equation for the new wave function is

$$\hbar\Delta\Psi=\left\{\bar{G}^0+\frac{ie}{2}[S(b),\bar{G}^1]+0(e^3)\right\}\Psi. \tag{42}$$

A further transformation enables us to get rid of $\bar{G}^0$.

The question of the self-energy of an electron or a photon will not be discussed here, as it has been discussed already by a number of physicists. The purpose of the present section is to show that the Tomonaga's formalism is essentially the same as the Heisenberg-Pauli formalism and it seems likely that it will encounter the same difficulties.

In conclusion, the writer wishes to thank Professor P. A. M. Dirac and Professor H. C. Corben for many stimulating discussions.