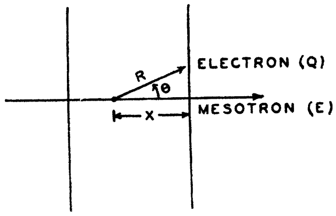

FIG. 7.

which $R(Q)$ can have, viz., $R(Q_0) \sim x_0$. Beyond this point $R(Q)$ is constant and equal to x_0. Thus Eq. (6) becomes the sum of two integrals,

$$\nu_\eta(E) = \langle\cos\theta(E)\rangle_{\text{Av}} \times\left\{\int_\eta^{Q_0} \chi(E, Q)R(Q)dQ + x_0\int_{Q_0}^{Q_M} \chi(E, Q)dQ\right\}. \quad (7)$$

Now for energies less than Q_0 (3 Mev in our case) the range energy curve for electrons can be closely approximated by,[14]

$$R(Q) = 0.4Q^{1.5} \qquad (Q \text{ in Mev}).$$

In view of this relation and that of Eq. (1) one obtains by performing the integrations in Eq. (7) and setting $x_0 = 2$ g/cm² for our particular case,

$$\nu_\eta(E) = \frac{2Cm}{\beta^2}\langle\cos\theta(E)\rangle_{\text{Av}}\left[k\left(\sqrt{Q_0} - \sqrt{\eta} - \frac{\beta^2 Q_0^{\frac{3}{2}}}{3Q_M}\right) + \frac{1}{Q_0} - \frac{1}{Q_M} - \frac{\beta^2}{Q_M}\log\frac{Q_M}{Q_0}\right], \quad (8)$$

where $\langle\cos\theta(E)\rangle_{\text{Av}}$ is given by Eq. (5), Q_M by Eq. (2), and $k = 8\times10^{-10}$ g/cm². Secondary electrons of energy less than 10^4 ev will not contribute appreciably to the values of $\nu_\eta(E)$. Hence, if one sets $\eta = 10^4$ ev, takes the value of the constant C appropriate for brass, and converts from energy to momentum of the incident mesotron, $\nu(p)$ can be plotted from Eq. (8). This curve is shown in Fig. 3 of the text for mesotron momenta between 10^8 and 1.2×10^9 ev/c. It should be noted that through the choice of the constant C appropriate to brass the number of secondaries produced in the glass counter envelopes is overestimated. Furthermore, scattering losses reduce the number reaching the sensitive volume of the counter. The computed values of $\nu(p)$ are consequently high which makes the estimated proton intensity (Fig. 6) conservatively low.

[14] L. E. Glendenin, Nucleonics 2, 12 (1948).

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Reflection and Refraction of Plane Shear Waves in Viscoelastic Media

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The shear elasticity and viscosity of liquids have been measured at ultrasonic frequencies by utilizing plane shear waves in an elastic solid and measuring the reflection loss and phase shift caused by reflection at a plane interface of the solid and a liquid. The first measurements of this type involved normal incidence. In a recent modification of the method, oblique incidence results in an enhanced effect. This paper derives the theoretical relations between the constants of the two media, the complex reflection coefficient and the angle of incidence. The theory describes some of the general properties of reflected and refracted shear waves in isotropic viscoelastic media.

INTRODUCTION

THIS paper is concerned with some theoretical relations involved in a method for measuring the shear wave parameters of viscous liquids at ultrasonic frequencies. The experimental details are described in a companion paper by W. P. Mason *et al.*, in this issue. The method is based on the measurement of a reflection coefficient for plane shear waves in an elastic solid, reflected obliquely from a plane interface of the solid and the liquid which is being investigated. A pulsing technique is used to separate the reflections of different orders at the detecting crystal, and the primary shear waves are suitably polarized so that reflection does not produce compressional waves in addition to the reflected shear waves. The complex reflection coefficient is measured by comparing the amplitudes and phases of waves reflected from the solid-liquid interface and from the same surface when the liquid is absent. Compared with normal incidence methods which had been developed previously, the oblique incidence method possesses some experimental advantages, resulting in greater accuracy of measurement, but oblique incidence involves more complicated relationships between the reflection coefficient and the constants of the two media. Solutions for these relations are derived in this paper. The theory describes some of the general properties of plane shear waves in viscoelastic media.

The elements of the system, omitting the electronic equipment, are shown in Fig. 1. A fused quartz rod with obliquely cut ends has a plane upper surface, which is covered in one part of the test by a thin layer of the liquid under investigation. This surface will be taken as the plane $y=0$ in a system of rectangular x, y, z coordinates, with the z axis perpendicular to the plane of Fig. 1. A suitably oriented Y cut quartz crystal at one end of the rod generates plane shear waves with the displacements in the z direction, and a similar crystal at the other end is used as a detector. The waves are reflected repeatedly at the upper surface and at both ends of the rod, but a pulsing system separates the reflections of different order at the detector. The primary displacement is parallel to the reflecting surfaces and perpendicular to the direction of propagation, and does not vary appreciably across the width of the rod. Hence, the displacements in the reflected and refracted waves also have these properties, and there are practically no compressional components. At ultrasonic frequencies the attenuation in the liquid is so large that a relatively thin layer of the liquid is equivalent to an infinite medium, as regards the effects in the quartz. The measurements indicate that the amplitude and phase of the detector output are practically independent of the thickness of the liquid until the thickness is reduced to a value of the order of 0.001 cm.

SHEAR WAVES IN A VISCOELASTIC MEDIUM

The response of a liquid to shearing stress is an essentially viscous flow if the stress is constant or varies slowly enough, but a liquid may also have a partly elastic character in shear which affects the behavior at high frequencies. The combined viscous and elastic properties can be represented by distributed mechanical admittances or impedances which are complex functions of frequency. T. Alfrey has described the impedance method of representing such properties for a general class of linear viscoelastic media such that the stress-strain relations obey the superposition principle.[1–3] When a medium satisfies this restriction, it is characterized by a set of linear relations between the stresses and the strains and their time derivatives, which may be of various orders. The linear relations are analogous to the voltage-current relations for a passive linear electrical network and they can be expressed in forms similar to the network equations, involving operators which become equivalent to complex impedances or admittances when the disturbance is assumed to be proportional to a complex time factor $e^{i\omega t}$. In particular, for an isotropic linear viscoelastic medium, the typical relation between shearing stress and shearing strain can be expressed in either of the forms

$$T_{xz}=\zeta\dot{S}_{xz} \quad \text{or} \quad YT_{xz}=\dot{S}_{xz}=\partial S_{xz}/\partial t, \tag{1}$$

T_{xz} denotes the shearing stress corresponding to the shearing strain $S_{xz}=\partial\xi_x/\partial z+\partial\xi_z/\partial x$, where ξ_z is the displacement in the z direction, and Y and ζ are operators which are reciprocal functions of the operator $\partial/\partial t$. For a disturbance proportional to $e^{i\omega t}$, Y and ζ are functions of $i\omega$, as $\partial/\partial t$ is then equivalent to $i\omega$. A special case of Eqs. (1) represents purely viscous flow; in this case, Y and ζ are real and $\zeta=\eta=1/Y$, where η is the (shear) viscosity. For a strictly elastic solid of rigidity $\mu=1/s$,

$$T_{xz}=\mu S_{xz}=\zeta\partial S_{xz}/\partial t, \quad \zeta\partial/\partial t=\mu, \quad Y=s\partial/\partial t,$$

and, when $\partial/\partial t=i\omega$, $\zeta=\mu/i\omega$, $Y=i\omega s=1/\zeta$.

In general, when $\partial/\partial t$ is equivalent to $i\omega$, ζ and Y are complex and can be represented by

$$\begin{aligned} \zeta(i\omega)&=\eta+\mu/i\omega=\eta-i\mu/\omega=1/Y(i\omega),\\ Y(i\omega)&=\nu+i\omega s=1/\eta'+i\omega/\mu'=1/\zeta(i\omega), \end{aligned} \tag{2}$$

where η, μ, ν, s, η', μ' are real and may all be functions of frequency. If stress is considered to be analogous to voltage and rate of change of strain as analogous to current, ζ corresponds to an impedance operator or complex impedance and Y is the corresponding admittance operator or complex admittance. At a given frequency, η and η' are two different types of effective viscosities and μ and μ' are different types of effective rigidities, such that η and $1/\mu$ correspond, respectively, to resistance and capacitance in series, and η' and $1/\mu'$ correspond to resistance and capacitance in parallel. The values of η, μ, η', μ' may vary with frequency, but they are always ≥ 0. If the phase angles of the complex ζ and Y are denoted by $\arg\zeta$ and $\arg Y$,

$$\begin{gathered} \zeta(i\omega)=|\zeta|e^{i\arg\zeta}, \quad Y(i\omega)=|Y|e^{i\arg Y}\\ -\arg\zeta=\arg Y=\tan^{-1}Q,\\ Q=\mu/\omega\eta=\omega s/\nu=\eta'\omega/\mu'\geq 0\\ -\pi/2\leq\arg\zeta\leq 0\leq\arg Y\leq\pi/2. \end{gathered} \tag{3}$$

For different materials, or for a given material at varying frequency, the properties become relatively more elastic and less viscous as Q increases. As shown above, Q increases with μ/ω and with η' but

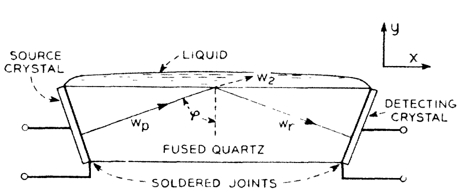


FIG. 1. Experimental arrangement.

[1] T. Alfrey, Jr., *Mechanical Behavior of High Polymers* (Interscience Publishers, Inc., New York, 1948).
[2] T. Alfrey, Jr., "Non-homogeneous stresses in viscoelastic media," Quart. App. Math. **2**, 113 (1944).
[3] T. Alfrey, Jr., "Methods of representing the properties of viscoelastic materials," Quart. App. Math. **3**, 143 (1945).

decreases as μ'/ω and η increase. If η' and μ' are independent of frequency, Q is proportional to frequency and is equal to unity at the relaxation frequency $f_r=\mu'/2\pi\eta'$; the viscous effect tends to predominate at frequencies below f_r and the elastic effect at frequencies above f_r.

The stress subscripts can be assigned so that positive T_{xz} represents a stress directed toward increasing z, acting on a part of the medium which has an outward normal directed toward increasing x. The stresses T_{xz} and T_{zx} are equal at any point, and positive T_{zz} denotes a tension. If ρ is the density and $D_x=\partial/\partial x$, the equation of motion in the z direction is

$$D_xT_{xz}+D_yT_{yz}+D_zT_{zz}=\rho D_t^2\xi_z=\rho D_t w=\rho\partial w/\partial t,$$

where w is the particle velocity in the z direction.

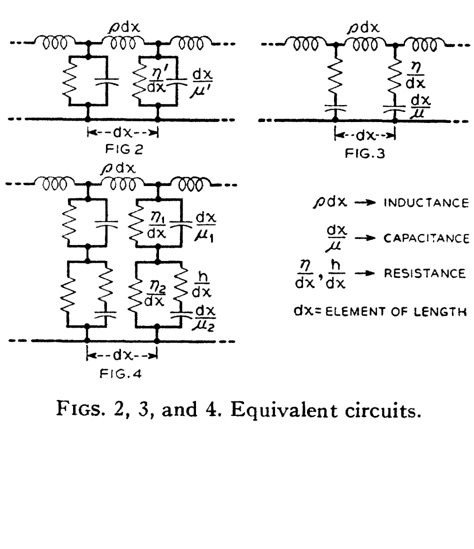
For a disturbance in an isotropic viscoeleastic medium such that the displacement is in the z direction and independent of z, the strains which do not vanish are $S_{xz}=D_x\xi_z$ and $S_{yz}=D_y\xi_z$, the tension T_{zz} is equal to zero, and the above equations result in the relations

$$\begin{gathered}\dot{S}_{xz}=D_xw=YT_{xz},\quad \dot{S}_{yz}=D_yw=YT_{yz},\\ D_xT_{xz}+D_yT_{yz}=\rho D_tw,\quad (D_x^2+D_y^2)w=\rho D_tYw.\end{gathered}\tag{4}$$

If w is independent of both y and z in Eqs. (4), the stress T_{yz} vanishes and

$$D_xw=YT_{xz},\quad D_xT_{xz}=\rho D_tw.$$

These equations are analogous to the voltage-current relations for an electrical transmission line. If current is now regarded as analogous to particle velocity instead of to strain rate, with voltage corresponding to stress as before, the density ρ corresponds to series inductance per unit length of the equivalent line and Y corresponds to the shunt admittance operator or complex shunt admittance per unit length. In Eqs. (2), the reciprocal viscosity $\nu=1/\eta'$ corresponds to shunt conductance per unit length and the compliance $s=1/\mu'$ to shunt capacitance per unit length. If η' and μ' are independent of frequency, the elements of the equivalent line are as indicated in Fig. 2. If η and μ are constant, the equivalent line is as shown in Fig. 3.



FIGS. 2, 3, and 4. Equivalent circuits.

Measurements described in a recent paper[4] indicate that certain long chain polymer liqiuds have shear properties at frequencies below 10^5 c.p.s. which correspond approximately to equivalent lines of the type shown in Fig. 2, with constant η' and μ'. Later measurements have been extended to 60 megacycles and show more complicated effects, which are described in the companion paper in this issue. In the larger frequency range, the shear properties of the long chain polymer liquids correspond more nearly to the equivalent line shown in Fig. 4, instead of the one in Fig. 2. The numerical values involved are such that the two circuits are nearly equivalent in the lower part of the frequency range. In Fig. 4, h denotes a resistance element which is inversely proportional to frequency; it represents approximately some effects due to a hysteresis type of nonlinearity in the stress-strain relations.[5]

When the motion is assumed to be proportional to $e^{i\omega t}$, the last Eq. (4) becomes

$$(D_x^2+D_y^2)w=\Gamma^2w,$$

where $\Gamma^2=i\omega\rho Y$. For motion independent of both y and z, this equation reduces to $D_x^2w=\Gamma^2w$, which has the general solution $w=(Ke^{-\Gamma x}+K'e^{\Gamma x})e^{i\omega t}$. From the first Eq. (4), the corresponding stress is

$$T_{xz}=Z(-Ke^{-\Gamma x}+K'e^{\Gamma x})e^{i\omega t},\text{ where } Z=\Gamma/Y.$$

Γ is the characteristic propagation constant of the medium and Z is the characteristic impedance for plane shear waves. They satisfy the relations

$$\begin{gathered}\Gamma=YZ=i\omega\rho/Z=(i\omega\rho Y)^{\frac{1}{2}}=\alpha+i\beta,\\ \beta=\omega/c=2\pi/\lambda,\\ Z=\Gamma/Y=i\omega\rho/\Gamma=(i\omega\rho/Y)^{\frac{1}{2}}=R+iX,\end{gathered}\tag{5}$$

where α, β, R, and X are real; α is an attenuation constant, β is a phase constant, c is a phase velocity, and λ is a wave-length. The velocity c is always different from zero, and the sign of Γ is chosen so that β and c are positive. The phase angle of Γ is therefore in the range 0 to π and, as a result of (3),

$$\begin{gathered}0\leq\arg Z\leq\pi/4\leq\arg\Gamma\leq\pi/2,\\ 0\leq X\leq R,\quad 0\leq\alpha\leq\beta.\end{gathered}\tag{6}$$

One limiting case corresponds to a strictly elastic medium, for which

$$\begin{gathered}Y=i\omega/\mu,\quad \alpha=0,\quad \beta=\omega/c,\quad c=(\mu/\rho)^{\frac{1}{2}},\\ Z=R=\rho c,\quad X=0.\end{gathered}$$

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[4] W. P. Mason, W. O. Baker, H. J. McSkimin, and J. H. Heiss, "Mechanical properties of long chain molecule liquids at ultrasonic frequencies," Phys. Rev. **73**, 1074 (1948).

[5] W. P. Mason, *Piezoelectric Crystals and Applications to Ultrasonics*, Chap. 12, Appendices A.8, A.9; in process of publication.

In the other extreme, for strictly viscous flow,

$$Y=1/\eta,\quad \alpha=\beta=(\omega\rho/2\eta)^{\frac{1}{2}},\quad c=(2\eta\omega/\rho)^{\frac{1}{2}},\quad R=X=(\omega\rho\eta/2)^{\frac{1}{2}}.$$

For a progressive plane shear wave advancing toward increasing x, the particle velocity and stress are (the real parts of)

$$w=w_0e^{-\Gamma x}=Ke^{-\alpha x+i(\omega t-\beta x)},\quad T_{xz}=-Zw. \tag{7}$$

The negative sign in the last equation is a consequence of the stress sign conventions described previously. T_{xz} is a stress across a plane $x=$constant, parallel to the wave fronts, and the part of the medium on the forward or increasing x side of this plane is acted upon by a stress which has the same direction as positive w when T_{xz} is negative; this corresponds to a positive value of (the real part of) $-T_{xz}=Zw$.

For shear waves such that the motion in the z direction may vary with both x and y, the particle velocity and stresses satisfy Eqs. (4). When this motion is assumed to be proportional to $e^{i\omega t}$, ρD_tY is equivalent to Γ^2 and $Y=\Gamma/Z$. Hence, Eqs. (4) are then equivalent to

$$(D_x^2+D_y^2)w=\Gamma^2w, \tag{8}$$

$$T_{xz}=Z\Gamma^{-1}D_xw,\quad T_{yz}=Z\Gamma^{-1}D_yw. \tag{9}$$

A typical solution of Eq. (8) is

$$w_p=w_0e^{-Gx-Hy}=w_0e^{-\Gamma(mx+ny)}, \tag{10}$$

where

$$G^2+H^2=\Gamma^2,\quad m^2+n^2=1.$$

G and H are auxiliary propagation constants for the x and y directions and can have arbitrary complex values, subject to the boundary conditions and the restriction $G^2+H^2=\Gamma^2$. The corresponding restriction on m and n is $m^2+n^2=1$, and because of this equation, m and n can be represented by $m=\sin\varphi$ and $n=\cos\varphi$. However, φ is not necessarily a real angle, as m and n can, in general, be complex. If φ is real, $mx+ny$ is a real distance which increases in the direction of propagation and Eqs. (7) and (10) then represent waves of the same type traveling in different directions. The characteristics of waves such that φ is complex will be considered later.

REFLECTION AND REFRACTION OF PLANE SHEAR WAVES

Assume that the plane $y=0$ is a common boundary of two isotropic viscoelastic media. One of these may be an essentially elastic solid and the other a liquid, as in the system described in the introduction. Let subscripts 1 and 2 refer to the negative y and positive y regions, respectively, and with Γ replaced by Γ_1, let Eqs. (10) represent a primary wave generated by suitable motion of a boundary in a plane $mx+ny=$constant, where $m=\sin\varphi$, $n=\cos\varphi$, and $0\leq\varphi\leq\pi/2$. The real angle φ is then equal to the angle of incidence relative to the normal to the $y=0$ interface. The primary wave is partially reflected at the interface, and a refracted wave is transmitted into the second medium. Since the displacement of the interface caused by the primary wave is in the plane of the interface and is independent of distance in the direction of the displacement, the reflected and refracted disturbances are non-dilatational and the displacements in these waves are also parallel to the z axis and independent of z. Hence, in the absence of reflections from other surfaces, the resultant particle velocities and stresses satisfy Eqs. (8) and (9), when the appropriate values are assigned to Γ and Z for each medium. The sum of the primary and reflected particle velocities must be equal to the refracted particle velocity at the $y=0$ interface if there is no slip at this surface, and the stress across planes parallel to the interface must also be continuous at the interface.

The boundary conditions at $y=0$ can be satisfied if the particle velocities in the two media are proportional to the same function of x, and the wave equation for the first medium will be satisfied if the primary and reflected particle velocities are proportional to the same function of x and to functions of y in which the coefficients of y differ only in sign. The resultant particle velocity in the first medium will then have the form

$$w_1=w_0e^{-Gx}(e^{-Hy}+re^{Hy}),\quad G=\Gamma_1m,\quad H=\Gamma_1n. \tag{11}$$

At $y=0$,

$$w_1=(1+r)w_0e^{-Gx}.$$

The refracted particle velocity w_2 will satisfy the wave equation for the second medium and the boundary condition $w_1=w_2$ at $y=0$ if

$$w_2=\tau w_0e^{-Gx-H_2y},\quad \tau=1+r, \tag{12}$$

$$H_2=(\Gamma_2^2-G^2)^{\frac{1}{2}}=(\Gamma_2^2-\Gamma_1^2m^2)^{\frac{1}{2}}=\alpha_y+i\beta_y, \tag{13}$$

where α_y and β_y are real, representing an attenuation constant and a phase constant of the refracted wave for the y direction.

To represent attenuation and propagation of phase in the direction away from the interface, toward increasing y, α_y and β_y must both be positive, and therefore the imaginary part of H_2^2 must also be positive. In terms of the "characteristic" attenuation and phase constants of the two media,

$$\begin{aligned}H_2^2&=(\alpha_2+i\beta_2)^2-(\alpha_1+i\beta_1)^2m^2\\&=(\beta_1^2-\alpha_1^2)m^2-(\beta_2^2-\alpha_2^2)+i2(\alpha_2\beta_2-\alpha_1\beta_1m^2)\\&=\alpha_y^2-\beta_y^2+i2\alpha_y\beta_y,\end{aligned}$$

$$\alpha_y\beta_y=\alpha_2\beta_2-\alpha_1\beta_1m^2=2\pi(\alpha_2/\lambda_2-m^2\alpha_1/\lambda_1). \tag{14}$$

To make $\alpha_y\beta_y$ positive for all angles of incidence, it will be assumed that the refracting medium has the larger attenuation per wave-length, $\alpha_2/\lambda_2 > \alpha_1/\lambda_1$. This restriction is satisfied by the experimental system indicated in Fig. 1, as the attenuation per wave-length in fused quartz is negligible in comparison with the attenuation per wave-length in a liquid. (Other cases are discussed in an appendix of this paper.)

The solution for the refracted particle velocity can be expressed in another form by letting $G=\Gamma_2 M$ and $H_2=\Gamma_2 N$. Then $M^2+N^2=1$. Hence, M and N are equivalent to the sine and cosine of an angle, not necessarily real. Separating the primary and reflected components of w_1, Eqs. (11) and (12) are equivalent to the following set of equations:

$$\begin{aligned} w_p &= w_0 e^{-\Gamma_1(mx+ny)}, & w_r &= r w_0 e^{-\Gamma_1(mx-ny)}, \\ w_1 &= w_p + w_r, & w_2 &= \tau w_0 e^{-\Gamma_2(Mx+Ny)}, \end{aligned} \tag{15}$$

$$\begin{aligned} m &= \sin\varphi, & n &= \cos\varphi, \\ M &= \sin\psi = m\Gamma_1/\Gamma_2, & N &= \cos\psi = H_2/\Gamma_2. \end{aligned} \tag{16}$$

The angle of reflection is equal to the angle of incidence, φ, and in special cases such that ψ is real, ψ is the angle of refraction. However, ψ is generally complex.

Equations (9) and (15) result in the following formulas for the stresses in the two media, and the values of T_{yz} lead to a second relation between the reflection coefficient r and the transmission coefficient $\tau = 1+r$.

$$\begin{aligned} &\text{When } y\leq 0, \\ &\quad T_{xz} = -mZ_1 w_1, \quad T_{yz} = -nZ_1(w_p - w_r). \\ &\text{When } y\geq 0, \\ &\quad T_{xz} = -MZ_2 w_2, \quad T_{yz} = -NZ_2 w_2. \end{aligned} \tag{17}$$

As T_{yz} is a stress across a plane $y=$ constant, parallel to the interface, this stress must be continuous at $y=0$, and as a result of Eqs. (11), (12), and (17), the stress across the interface itself is given by either of the formulas

$$T_{yz}(x,0) = -nZ_1(1-r)w_0 e^{-Gx} = -NZ_2 \tau w_0 e^{-Gx}. \tag{18}$$

Hence,

$$nZ_1(1-r) = NZ_2\tau = NZ_2(1+r). \tag{19}$$

The reflection and transmission coefficients therefore have the following values:

$$\begin{aligned} r &= (nZ_1 - NZ_2)/(nZ_1+NZ_2) \\ &= (Z_1\cos\varphi - Z_2\cos\psi)/(Z_1\cos\varphi + Z_2\cos\psi), \end{aligned} \tag{20}$$

$$\tau = 1+r = 2nZ_1/(nZ_1+NZ_2).$$

The stress T_{xz} is not necessarily continuous at the interface. Since $w_1=w_2$ at $y=0$, Eqs. (17) indicate that as the interface is approached from opposite sides, the values of T_{xz} approach two limits the ratio of which is mZ_1/MZ_2. Because of Eqs. (16) and (5), this ratio is also equal to $\Gamma_2 Z_1/\Gamma_1 Z_2 = Y_2/Y_1$, which is generally different from unity. The value of this ratio does not depend on the direction of propagation.

The product NZ_2 which appears in the solution for the reflection coefficient is the impedance of the second medium for the refracted wave per unit area of the interface $y=0$ or of any plane $y=$ constant. This is shown by the last Eq. (17). Similarly, if there were no reflection at $y=0$, nZ_1 would be the impedance of the first medium for the primary wave per unit area of a plane $y=$ constant. The negative signs in Eqs. (17) are due to reasons similar to those discussed in connection with the sign of the stress in the last Eq. (7).

When ψ is complex, the refracted wave is a disturbance of a different type from an ordinary plane wave such as the one represented by Eqs. (7). Some of the properties of the refracted wave can be investigated by resolving the quantity $Gx+H_2y$ into its real and imaginary parts. Some of the sign relations involved will be utilized later in the application of the theory to the experimental measurements. With α_y and β_y both positive in Eqs. (13), the phase angle of H_2 is between zero and $\pi/2$, and from (3) and (16) it then follows that

$$-\pi/4 \leq \arg M \leq \pi/4, \quad -\pi/2 \leq \arg N \leq \pi/4.$$

Hence, the real parts of M and N are ≥ 0. Let $\psi = A+iB$ where A and B are real. Then, from (16),

$$N = \cos\psi = \cos A\,\cosh B - i\,\sin A\,\sinh B;$$

$$\begin{aligned} M &= \sin\psi = \sin A\,\cosh B + i\,\cos A\,\sinh B \\ &= m\Gamma_1/\Gamma_2 = m[(\alpha_1+i\beta_1)/(\alpha_2+i\beta_2)] \\ &= m\{[(\alpha_1\alpha_2+\beta_1\beta_2)+i\beta_1\beta_2(\alpha_2/\beta_2-\alpha_1/\beta_1)]/[\alpha_2^2+\beta_2^2]\}. \end{aligned} \tag{21}$$

Since $\cosh B \geq 1$ and the real parts of M and N are ≥ 0, $\sin A$ and $\cos A$ are also ≥ 0 and $0 \leq A \leq \pi/2$. B has the same sign as $m(\alpha_2/\beta_2 - \alpha_1/\beta_1)$. For the case corresponding to the experimental measurements, $\alpha_1/\beta_1 \ll \alpha_2/\beta_2 < 1$, hence B and the imaginary part of M are positive and the imaginary part of N is negative, providing $m \neq 0$. For normal incidence, $m=0=M$. An additional restriction on A results from the relations $\sin A \leq |M| < |\Gamma_1|/\beta_2$, and when α_1 is small, $|\Gamma_1|/\beta_2 \approx \beta_1/\beta_2 = c_2/c_1$. In the experimental system, the velocity ratio c_2/c_1 is small, certainly less than 0.7, hence $0 \leq A < \pi/4$. This result will be useful later.

Multiplying the above values of M and N by $\Gamma_2 = \alpha_2 + i\beta_2$, it can be shown that

$$\begin{aligned} G &= \Gamma_2 M = \alpha'\sin(A-a) + i\beta'\sin(A+b), \\ H_2 &= \Gamma_2 N = \alpha'\cos(A-a) + i\beta'\cos(A+b), \\ \alpha' &= [\alpha_2^2 + (\alpha_2^2+\beta_2^2)\sinh^2 B]^{\frac{1}{2}}, \\ \beta' &= [\beta_2^2 + (\alpha_2^2+\beta_2^2)\sinh^2 B]^{\frac{1}{2}}, \\ a &= \tan^{-1}[(\beta_2/\alpha_2)\tanh B], \\ b &= \tan^{-1}[(\alpha_2/\beta_2)\tanh B]. \end{aligned}$$

Then

$$\begin{aligned} Gx + H_2 y &= \alpha' x' + i\beta' y', \\ x' &= x\sin(A-a) + y\cos(A-a), \\ y' &= x\sin(A+b) + y\cos(A+b). \end{aligned}$$

Substituting these results in (12) and letting $w_0 = Ke^{i\omega t}$, the refracted particle velocity has the form

$$w_2 = \tau K e^{-\alpha' x' + i(\omega t - \beta' y')}.$$

These equations indicate that α' is the effective attenuation constant, β' the effective phase constant, and ω/β' the effective phase velocity of the refracted wave; x' and y' are real distances measured in certain directions such that x' = constant represents a plane of uniform amplitude and y' = constant represents a plane of uniform phase. In general, these two sets of planes are not parallel. Hence, the amplitudes are generally non-uniform over any wave front; x' increases in the direction of the greatest rate of attenuation and y' increases in the direction of propagation of phase. The angles between these directions and the direction of increasing y are $A-a$ and $A+b$, respectively. Thus when ψ is complex there are two angles of refraction, one for the direction of propagation, $A+b$, and one for the direction of greatest attenuation, $A-a$. The angles a and b are of like sign, with the same sign as B and therefore the same sign as $\alpha_2/\beta_2 - \alpha_1/\beta_1$. The effective attenuation α' per unit distance is greater than the "characteristic" attenuation α_2 and the velocity of propagation is ω/β', which is less than the characteristic velocity of propagation ω/β_2.

EVALUATION OF THE CONSTANTS OF THE REFRACTING MEDIUM

In the system of measurements described in the introduction, the reflection coefficient r is measured experimentally and the preceding theory then leads to solutions for the constants of the refracting medium in terms of the known quantities. With reflections of different orders separated by means of a pulsing method, the reflection coefficient is measured by comparing waves reflected from the interface between the two media with waves reflected from the same surface when the refracting medium is absent. In the latter case, the stress across the reflecting surface is zero and Eq. (18) indicates that the reflection coefficient is then equal to unity. Hence, from Eqs. (15) the particle velocity in the wave reflected from the free boundary is $w_f = w_0 e^{-\Gamma_1(mx-ny)}$. With the refracting medium present, r is different from unity and the particle velocity in the reflected wave is $w_r = rw_f$ for equivalent primary waves in the two cases. Hence the complex reflection coefficient r can be evaluated by comparing the amplitudes and phases of w_r and w_f. If $p = mx - ny$ = distance in the direction of propagation of the reflected waves,

$$w_f = w_0 e^{-\Gamma_1 p} = Ke^{i\omega t - (\alpha_1 + i\beta_1)p} = |w_f| e^{i(\omega t - \beta_1 p)},$$
$$w_r = rw_f = |w_r| e^{i(\omega t - \beta_1 p - \theta)},$$
$$r = w_r/w_f = Re^{-i\theta}, \text{ where } R = |w_r/w_f| = |r|. \quad (22)$$

R is the relative reduction in amplitude and θ the phase shift of the reflected waves caused by the presence of the refracting medium. If θ is positive, the phase shift is equivalent to an increase of $\beta_1 p$ and therefore corresponds either to an increase of the effective transmission distance p, or to an increase of frequency, as β_1 increases with frequency.

When the reflection coefficient r, angle of incidence φ, and characteristic impedance Z_1 are known, the value of the "refracted wave impedance" NZ_2 can be calculated by means of Eqs. (19) and (22):

$$NZ_2 = Z_1 n[(1-r)/(1+r)] = Z_1 nq, \quad (23)$$

where

$$\begin{aligned} q &= (1-r)/(1+r) \\ &= (1-R^2+i2R\sin\theta)/(1+R^2+2R\cos\theta) \\ &= [(1+R^2-2R\cos\theta)/(1+R^2+2R\cos\theta)]^{\frac{1}{2}} \\ &\quad \times \exp[i\tan^{-1}(2R\sin\theta/(1-R^2))]. \quad (24) \end{aligned}$$

The value of q corresponding to a given complex value of r can also be derived in other ways. Let the amplitude ratio in nepers be denoted by

$$L = \log_e |w_f/w_r| = -\log_e R.$$

Then

$$r = Re^{-i\theta} = e^{-L-i\theta},$$
$$q = \tanh[(L+i\theta)/2] = i\tan[(\theta - iL)/2].$$

Thus q can be evaluated by means of Eqs. (24) or by means of charts of complex hyperbolic or circular tangents, such as Kennelly's charts.[6]

If the densities of the two media are known, besides the values of r, φ, and Z_1, separate solutions for Z_2 and N and the other constants of the system can be derived by the following method, based on preliminary evaluation of $\sin 2\psi$ and $\cos 2\psi$ from the known data. From Eqs. (5), (16), and (23):

$$m = \sin\varphi, \quad n = \cos\varphi, \quad 2mn = \sin 2\varphi;$$
$$M = \sin\psi = m\Gamma_1/\Gamma_2 = mkZ_2/Z_1,$$

where $k = \rho_1/\rho_2$;

$$N = \cos\psi = qnZ_1/Z_2 = qnk\Gamma_2/\Gamma_1;$$
$$2MN = \sin 2\psi = 2mnkq = qk\sin 2\varphi. \quad (25)$$

Except for an ambiguity of sign, the value of $\cos 2\psi$ is then given by $\pm(1-\sin^2 2\psi)^{\frac{1}{2}}$. The sign can be determined with the aid of the formula,

$$\cos 2\psi = \cos 2A \cosh 2B - i \sin 2A \sinh 2B.$$

It was shown previously that in the experimental system, $0 \leq A \leq \pi/4$ and $B \geq 0$. Hence the real part of $\cos 2\psi$ is ≥ 0 and the imaginary part is ≤ 0.

The different steps involved in the solutions can

[6] A. E. Kennelly, *Chart Atlas of Complex Hyperbolic and Circular Functions* (Harvard University Press, Cambridge, 1914).

now be summarized as follows:

$$r=|w_r/w_f|e^{-i\theta},\quad q=(1-r)/(1+r),\quad k=\rho_1/\rho_2;$$
$$\sin 2\psi = qk\sin 2\varphi.$$

If $\varphi>0$:

$$\cos 2\psi=[1-\sin^2 2\psi]^{\frac{1}{2}},$$
$$\text{real part}>0,\ \text{imaginary part}<0; \quad (26)$$
$$M=\sin\psi=[(1/2)(1-\cos 2\psi)]^{\frac{1}{2}},$$
$$\text{real part}>0,\ \text{imaginary part}>0;$$
$$N=\cos\psi=[(1/2)(1+\cos 2\psi)]^{\frac{1}{2}},$$
$$\text{real part}>0,\ \text{imaginary part}<0.$$

If $\varphi=0$, $\psi=0$, $M=0$, and $N=1$. The characteristic impedance and propagation constant of the refracting medium can now be evaluated by means of the alternative formulas:

$$\begin{aligned} Z_2&=Z_1M/mk=Z_1nq/N=i\omega\rho_2/\Gamma_2,\\ \Gamma_2&=\Gamma_1 m/M=\Gamma_1 N/nkq=i\omega\rho_2/Z_2.\end{aligned}\quad (27)$$

The "characteristic" attenuation constant is the real part of $\Gamma_2=\alpha_2+i\beta_2$, the "characteristic" velocity of propagation is ω/β_2, and the viscoelastic admittance Y_2 and impedance ζ_2 have the values $Y_2=\Gamma_2/Z_2$ and $\zeta_2=Z_2/\Gamma_2$. Separating these into their real and imaginary parts leads to the values of the effective viscosities and effective elastic parameters defined by Eqs. (2). For most viscous liquids, the "parallel" viscosity η' and stiffness μ', and their reciprocals ν and s, will probably be approximately independent of frequency in a fairly wide frequency range; this will not be true of the "series" viscosity η and stiffness μ.

Measurements have been made on a number of viscous liquids covering a large range of viscosities. In each case, $|\sin 2\psi|$ was small relative to unity, and therefore some simple approximations could be used in Eqs. (26) and (27). With the rules given previously to determine the signs, it follows that when $|\sin 2\psi|\ll 1$:

$$\cos 2\psi=[1-\sin^2 2\psi]^{\frac{1}{2}}=1-(1/2)\sin^2 2\psi-\cdots,$$
$$N=\cos\psi=[1/2(1+\cos 2\psi)]^{\frac{1}{2}}=1-(1/8)\sin^2 2\psi-\cdots,$$
$$M=\sin\psi=(\sin 2\psi)/(2N)=(1/2)\sin 2\psi+\cdots.$$

For all of the preliminary measurements, the values of $|\sin 2\psi|$ did not exceed 0.2. Hence, N differed from unity by less than two percent. In such cases, Eqs. (27) are very nearly equivalent to the approximations

$$\begin{aligned} Z_2&\approx Z_1nq=[(1-r)/(1+r)]Z\cos\varphi,\\ \Gamma_2&\approx\Gamma_1/nkq=[\rho_2/\rho_1][\Gamma_1/\cos\varphi][(1+r)/(1-r)].\end{aligned}\quad (28)$$

These approximations will probably be sufficiently accurate for most of the materials likely to be investigated by the oblique incidence method. If the factors $\cos\varphi$ and $1/\cos\varphi$ are omitted in Eqs. (28), these equations have the same forms as the rigorous solutions for normal incidence, but the reflection coefficient r will, of course, have different values for oblique and normal incidence.

APPENDIX

In the theory of refraction derived in this paper, it is assumed that the refracting medium has the larger attenuation per wave-length, in order that the assumed solution for the refracted wave may have appropriate directions of attenuation and propagation of phase. This restriction is satisfied in the experimental system which has been described, as the attenuation in the fused quartz is practically negligible. However, the necessity for the restriction indicates that the equations which have been given do not represent a complete solution for all cases and a question arises concerning the conditions under which the assumed solution is sufficiently accurate. The distinction between the different cases can be explained by the existence of diverging waves which are unimportant in some cases but not in others. In order to derive a complete solution for all cases, it would be necessary to take account of the nature of the source and some associated features of other boundary conditions which were not mentioned previously.

The primary wave represented by the first Eq. (15) can be regarded as generated by motion of a boundary of the first medium occupying part or all of a semi-infinite plane $mx+ny=C=\text{constant}$, which meets the boundary $y=0$ at $x=x_0=C/m$. If the entire semi-infinite plane is assumed to have a suitable forced motion, the boundary conditions specified previously apply only to the part of the plane $y=0$ for which $x>x_0$. The second medium must then have an additional boundary where other conditions apply; this may be the rest of the plane $y=0$, or it may be some other boundary—for example, the plane $x=x_0$. In some cases it may be permissible to ignore the additional boundary conditions, but this will not always be true. If the source is assumed to occupy the entire semi-infinite plane $mx+ny=C$, there will be additional complications caused by secondary reflections at the source.

Under certain conditions the problem will be simplified if it is assumed that only part of the plane $mx+ny=C$ is driven, so that the primary disturbance consists of a beam in which the wave fronts are essentially plane, but are of finite width in the direction perpendicular to the z axis. This neglects the effects of divergence at the sides of the beam, which is permissible as regards the primary beam if the width of the driving surface is sufficiently large relative to the wave-length. Then in some cases, the reflected and refracted disturbances may also consist principally of non-divergent beams, in which the wave fronts are essentially plane. This will tend to be the case if the attenuation is small in the first medium, or in both media, the latter case being analogous to a well-known case of optical reflection and refraction. The solutions for such cases can be represented with sufficient accuracy by Eqs. (15), within the limits of the respective beams. However, a fundamentally different situation may occur if the attenuation in the first medium is relatively large. If the primary wave is represented by the first Eq. (15), within a limited beam, there will be a restricted part of the $y=0$ interface, say the region $x_a<x<x_b$, where the primary particle velocity is

$$w_0 e^{-\Gamma_1 m x} = w_0 e^{-(\alpha_1+i\beta_1)x\sin\varphi}.$$

This is proportional to the real function $e^{-\alpha_1 x\sin\varphi}$ and if $\alpha_1\sin\varphi$ is large enough, the forced motion of the interface in the vicinity of $x=x_a$ will have the predominant effect, resulting in diverging secondary disturbances in both media, with wave fronts which tend to be cylindrical. Furthermore, if the characteristic attenuation of the second medium is small, then in the vicinity of the plane $y=0$ at sufficient distance from $x=x_a$ the diverging disturbance in the second medium will tend to have a greater intensity than the primary intensity at nearby points on the other side of the interface. Hence, in such a region, the resultant direction of propagation in the second medium will have a component toward the interface instead of away from it. This direction of propagation cannot apply throughout the second medium, of course, especially near $x=x_a$, but in a limited region, the refracted disturbance may correspond approximately to Eqs. (12) and (13), with positive α_y and negative β_y. The region in which this situation exists will become larger as the distance between x_a and x_b increases, but x_a cannot be assumed to approach $-\infty$, as the amplitudes would then be infinite at $x=-\infty$. To derive a general rigorous solution it would be necessary to modify the boundary conditions, taking into account the fact that the primary wave function does not apply to all values of x. The rigorous solution would be very complicated at best.

Equations (14) indicate that the criteria which determine whether the divergent effects are important or not are the angle of incidence and the attenuations per wave-length in the two media. The equation can be expressed in the form

$$\alpha_y\beta_y=\alpha_1\beta_1(m_c^2-\sin^2\varphi),$$

where

$$m_c^2=\alpha_2\beta_2/\alpha_1\beta_1=(\alpha_2/\alpha_1)(c_1/c_2)=(\alpha_2/\lambda_2)/(\alpha_1/\lambda_1).$$

c_1 and c_2 denote the characteristic velocities of propagation in the two media. If the second medium has the larger attenuation per wave-length, $m_c^2>1$ and $\alpha_y\beta_y>0$ for all angles of incidence. For transmission in the reverse direction, the second medium has the smaller attenuation per wave-length, $m_c^2<1$, and $\alpha_y\beta_y$ is positive or negative according as the angle of incidence is less than or greater than the critical angle $\varphi_c=\sin^{-1}m_c$. It seems probable that the solutions derived in this memorandum will be reasonably accurate whenever m_c^2 is large relative to $\sin^2\varphi$, which will be true for all angles of incidence if m_c^2 is large relative to unity. As m_c^2 and $\sin^2\varphi$ approach equality, the formulas become inaccurate as a result of the increasing importance of the divergent effects.