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Breakdown Probability of a Low Pressure Gas Discharge

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The probability distribution of the number of electrons in an "avalanche" in a low pressure gas discharge is calculated and the result applied to the calculation of the breakdown probability. Under some simplifying assumptions the probability P_0 that the avalanche caused by one initial electron, liberated from the cathode, leads to breakdown is found to be

$$P_0=1-1/q \quad \text{if } q>1$$
$$P_0=0 \quad \text{if } q<1,$$

in which $q=\gamma\,(\exp[\int_0^d\alpha(x')dx']-1)$ is the average number of secondary electrons liberated at the cathode as a result of the avalanche of the initial electron. The calculation has been generalized to the case that the initial electron is liberated in the gas. Finally the relation between the breakdown probability P_0 and the statistical distribution of sparking time lags is discussed.

I. INTRODUCTION

A SELF-SUSTAINING gas discharge is possible only if the voltage between the electrodes exceeds a certain critical value. The transition from the non-conducting to the conducting state is called a breakdown or a spark, and the smallest value of the applied voltage at which breakdown can occur is called breakdown- or sparking-potential. The question has been raised in the past as to whether a sharp criterion for breakdown can be applied and if so, whether the breakdown potential has a theoretically well defined value. These questions have been answered in the affirmative by Braunbeck[1] and Hertz[2] for the case of a gas at low pressure, using some simplifying assumptions.

In a recent article, L. B. Loeb[3] has indicated that in general most sparking thresholds are determined by an equation of the form $\gamma\exp(\alpha d)=1$, where $\exp(\alpha d)$ represents the electron multiplication resulting from ionization by collision in the gap length d, and γ is related to the probability of liberation of secondary electrons (the definition of α and γ will be given in Sections II and III). In this article Loeb has indicated that in the prototype equation both γ and $\exp(\alpha d)$ represent the average values of quantities which are subject to considerable statistical fluctuations. The magnitude of these fluctuations has a bearing on the interpretation of breakdown-potential measurements, and a statistical treatment of the spark formation is therefore important. Such a treatment has been given previously[1,2] in an incomplete way insofar as the fluctuations from the average value $\exp(\alpha d)$ of the electron multiplication factor were neglected. In Section II it will be shown that this neglect is not justified, and it is the purpose of this paper to take these fluctuations into account.

The fact that most processes in a gas discharge, such as ionization by collision, etc., are chance phenomena, has the result that an initial electron, liberated in the gas or at the cathode by an external agency (e.g., radiation), does not invariably lead to breakdown, even when the applied voltage exceeds the breakdown potential. In order to understand this qualitatively we shall give a rough sketch of the mechanism of conduction by a gas.

The initial electron will be accelerated in the electric field and will undergo elastic and inelastic collisions with gas molecules, ionizing some of the molecules. The newly created electrons will take part in the process of ionization and thus a so-called "avalanche" of electrons and positive ions is formed. It has already been mentioned that the number of ion pairs in an avalanche is subject to considerable fluctuations. The electrons eventually reach the anode, leaving behind the positive ions, which drift back toward the cathode. In order to maintain the discharge new electrons must be liberated in the gas or at the

[1] W. Braunbeck, Zeits. f. Physik. **39**, 6 (1926); **107**, 180 (1937).
[2] G. Hertz, Zeits. f. Physik. **106**, 102 (1937).
[3] L. B. Loeb, Rev. Mod. Phys. **20**, 151 (1948).

cathode by action of positive ions, photons, or atoms in metastable states, created by the electron avalanche. Because of the statistical character of these processes it may occur at a certain stage in the development of the spark that there is no supply of new electrons, as a result of which the discharge breaks off. If the probability of interruption is denoted by Q, then $P=1-Q$ is the probability that the discharge is not interrupted, which can be defined as the breakdown probability.

Since it is not yet possible to give a rigorous calculation in which all secondary processes and the influence of space charge formation are taken into account, we limit ourselves to the case of a gas at low pressure in which (1) the principal means of maintenance of the discharge is through liberation of secondary electrons from the cathode by positive ions, and (2) the effects of space charge accumulation on the breakdown probability may be assumed to be negligible. As shown by Schade,[4] space charge does not play an appreciable role in the first and most important stage of the development of a spark in a gas at low pressure. Furthermore we restrict our treatment to cases in which the geometry of the discharge tube is such that the electric field is a function of one coordinate only, which we denote by x. This restriction is satisfied in common cases, such as a uniform field between plane parallel electrodes, and the field between coaxial cylindrical electrodes.

Within the simplifying restrictions given above the probability distribution of the number of electrons in an avalanche will be calculated in Section II, and the result will be applied to the calculation of the breakdown probability in Section III. The breakdown probability is denoted by P_0 if the initial electron is liberated at the cathode. In general, if the initial electron is liberated at some point x in the gas, the breakdown probability is denoted by P_x. Besides the dependence on x, the breakdown probability will also depend on operational conditions such as the nature and the pressure of the gas, and the voltage between the electrodes.

[4] R. Schade, Zeits. f. Physik. **104**, 487 (1937).

II. PROBABILITY DISTRIBUTION OF THE NUMBER OF ELECTRONS IN AN AVALANCHE

To every point in the discharge is assigned a coordinate x. Let the cathode be at $x=0$, the anode at $x=d$. The energy distribution of the electrons will depend in general upon x. Therefore the probability $\alpha(x)dx$ that an electron ionizes while drifting through a region between x and $x+dx$ also depends upon x. α is Townsends first ionization coefficient.

One may now ask what the probability $p(n, x)$ is that a given electron which started from the cathode has grown to an avalanche of n electrons at a distance x from the cathode. We assume that in every ionization only one additional electron is created.

The value of $p(1, x)$ is easily found. It is the probability that the electron has not ionized at all between $x=0$ and $x=x$, and is given by

$$p(1, x)=\exp[-S(x)], \tag{1}$$

in which

$$S(x)=\int_0^x \alpha(x')dx'. \tag{2}$$

In order to find an expression for $p(n, x)$ we reason as follows: the probability that the avalanche contains $n-1$ electrons at $x=x'$ is

$$p(n-1, x'). \tag{3a}$$

The probability that one and only one of these electrons will ionize in the region between x' and $x'+dx'$ is $(n-1)\alpha(x')dx'(1-\alpha(x')dx')^{n-2}$. This approaches, as $dx'\rightarrow 0$

$$(n-1)\alpha(x')dx'. \tag{3b}$$

The number of electrons in the avalanche has now increased from $n-1$ to n, and the probability that none of these n electrons will ionize in the region between $x'+dx'$ and x is

$$\exp[-n\{S(x)-S(x')\}]. \tag{3c}$$

If we take the product of the expressions (3a), (3b), and (3c) and integrate over x' we get (for $n>1$)

$$p(n, x)=\int_0^x p(n-1, x')(n-1)\alpha(x')dx' \times\exp[-n\{S(x)-S(x')\}]. \tag{4}$$

The solution of Eq. (4), in view of Eq. (1), can easily be shown to be:

$$p(n, x) = \exp[-nS(x)](\exp[S(x)]-1)^{n-1}. \quad (5)$$

In particular, for the avalanche at the anode we have,

$$p(n, d) = \exp[-nS(d)](\exp[S(d)]-1)^{n-1}. \quad (6)$$

The expectation value of n can be obtained from Eq. (6) to be

$$\bar{n} = \sum_{n=1}^{\infty} np(n, d) = \exp[S(d)] = \exp\left[\int_0^d \alpha(x')dx'\right]. \quad (7)$$

We then can write (6) as:

$$p(n, d) = 1/\bar{n}(1-(1/\bar{n})^{n-1}.^* \quad (8)$$

For $\bar{n} \gg 1$, which is the usual case, Eq. (8) can be approximated by:

$$p(n, d) \cong (1/\bar{n}) \exp[-n/\bar{n}], \quad (9)$$

which shows that $p(n, d)$ is exponentially decreasing with n. This formula also shows that the fluctuations of n from the average value $\bar{n}$ are large.

If the initial electron is not liberated at the cathode, but at some point x in the gas, the avalanche will be shorter and therefore on the average will not contain as many electrons. Let the average number in the avalanche be $\bar{n}_x$, then for the probability distribution we have by analogy to (8),

$$p_x(n, d) = \frac{1}{\bar{n}_x}\left(1-\frac{1}{\bar{n}_x}\right)^{n-1}. \quad (8a)$$

III. BREAKDOWN PROBABILITY

We may proceed to calculate the breakdown probability P. This calculation is an extension of the one given previously by Hertz.[2]

Suppose that the initial electron is liberated at the cathode. This electron creates an avalanche containing n electrons and $n-1$ positive ions. The positive ions drift back toward the cathode and liberate secondary electrons, which create new avalanches, etc. Since the liberation of electrons at the cathode is a chance phenomenon, at a certain stage in the process all the positive ions may have drifted out of the discharge without liberating any new electrons. In that case the succession of avalanches is interrupted, the probability of interruption being denoted by Q_0. The probability that the succession of avalanches goes on indefinitely is then given by

$$P_0 = 1 - Q_0. \quad (10)$$

We assume that each positive ion has a probability γ of liberating an electron at the cathode, while the probability that one positive ion liberates more than one electron is assumed to be negligibly small.

The positive ions from the first avalanche liberate ν secondary electrons with a probability $u_{0\nu}$ ($\nu = 0, 1, 2\cdots$). We want to investigate the probability that the discharge will break off for a certain value of ν. For $\nu = 0$ it is clearly 1, for $\nu = 1$ it is by definition Q_0, in general it is Q_0^ν in view of the fact that the processes caused by each secondary electron are independent of each other. Therefore it is immaterial also whether the secondary electrons are liberated at the same time or not. We then get

$$Q_0 = \sum_{\nu=0}^{\infty} u_{0\nu} Q_0^\nu. \quad (11)$$

The probability $u_{0\nu}$ can be calculated as follows. The first avalanche yields n electrons and $n-1$ positive ions with a probability $p(n, d)$, given by Eq. (8). The probability that $n-1$ positive ions liberate ν secondary electrons is

$$\binom{n-1}{\nu}\gamma^\nu(1-\gamma)^{n-1-\nu}.$$

Therefore:

$$u_{0\nu} = \sum_{n=1}^{\infty} p(n, d)\binom{n-1}{\nu}\gamma^\nu(1-\gamma)^{n-1-\nu}.$$

Making use of the relationship

$$\frac{1}{b!}\sum_{\lambda=0}^{\infty}\frac{(\lambda+b)!}{\lambda!}a^\lambda = (1-a)^{-b-1}, \quad \text{if } |a| < 1$$

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* After this article was written, it was kindly pointed out that Eq. (8) had previously been derived by a slightly different method by W. H. Furry, Phys. Rev. **52**, 569 (1937) in an article "On fluctuation phenomena in the passage of high energy electrons through lead." In view of the different field of application, the article had escaped our notice. The parallelism of the statistical phenomena in electron multiplication in showers and in gas discharges is interesting.

the summation in the expression for $u_{0\nu}$ can be carried out with the result:

$$u_{0\nu}=\frac{q^{\nu}}{(q+1)^{\nu+1}}, \qquad (12)$$

in which

$$q=\gamma(\bar{n}-1), \qquad (13)$$

and q has a simple interpretation. It is the average number of secondary electrons liberated at the cathode if the initial electron started from the cathode. The current in a discharge will on the average increase or decrease in time depending upon whether q is greater or smaller than unity. The equation $q=1$ represents the classical condition for a sparking threshold under the assumed conditions.

After substitution of Eq. (12) into Eq. (11), we obtain

$$Q_0=\sum_{\nu=0}^{\infty}\frac{q^{\nu}}{(q+1)^{\nu+1}}Q_0{}^{\nu}=\frac{1}{q+1-qQ_0}.$$

We can solve for Q_0 and find the two values $Q_0=1$ and $Q_0=1/q$. The corresponding breakdown probabilities are, according to Eq. (10): $P_0=0$ and $P_0=1-1/q$. From physical considerations it is clear that we must take the first solution in the case $q<1$, and the second in the case $q>1$. Mathematically this follows rigorously from a more complicated derivation, not given here,** which unambiguously gives the result

$$\begin{aligned}P_0&=0 \text{ for } q<1,\\ P_0&=1-1/q \text{ for } q>1.\end{aligned} \qquad (14)$$

** The steps in this derivation are as follows. The probability $w(a, b)$ that a primary electrons, starting from the cathode, are succeeded by b secondary electrons can be calculated to be,

$$w(a, b)=\binom{a+b-1}{b}(q+1)^{-a-b}q^{b}.$$

Q_0 can be given in the following form,

$$Q_0=\sum_{a,b\cdots z=0}^{\infty} w(1, a)w(a, b)\cdots w(z, 0)$$

(infinite number of parameters a, $b\cdots z$), which can be evaluated and gives the result:

$$Q_0=(1/2q)(q+1-((q-1)^2))^{\frac{1}{2}}.$$

The cases $q<1$ and $q>1$ are distinguished by the value of the square root

$q<1$, $((q-1)^2)^{\frac{1}{2}}=1-q$ and therefore $Q_0=1$,
$q>1$, $((q-1)^2)^{\frac{1}{2}}=q-1$ and therefore $Q_0=1/q$,

which immediately leads to Eq. (14).

This solution, in contrast to the one by Hertz,[2] is simple and explicit in form. It is seen that P_0 is appreciably less than unity in a region extending far beyond the threshold $q=1$. However, P_0 as a function of the potential difference V between the electrodes is determined only if one knows q as a function of V. Since in general q increases rapidly with V in the region above the breakdown potential, P_0 may rise from zero to a value very close to 1 in a rather narrow voltage region in contrast to values of q.

Suppose now that the initial electron is liberated at a point x in the gas. In analogy to Eqs. (10), (11), (12), and (13) we have

$$P_x=1-Q_x, \qquad (10a)$$

$$Q_x=\sum_{\nu=0}^{\infty}u_{x\nu}Q_0{}^{\nu}, \qquad (11a)$$

$$u_{x\nu}=[q_x{}^{\nu}/(q_x+1)^{\nu+1}], \qquad (12a)$$

in which

$$q_x=\gamma(\bar{n}_x-1). \qquad (13a)$$

Substitution of Eq. (12a) into Eq. (11a) gives:

$$Q_x=\sum_{\nu=0}^{\infty}\frac{q_x{}^{\nu}}{(q_x+1)^{\nu+1}}Q_0{}^{\nu}.$$

For $q<1$, $Q_0=1$, so that $Q_x=1$.
For $q>1$, $Q_0=1/q$. In that case we find:

$$Q_x=\sum_{\nu=0}^{\infty}\frac{q_x{}^{\nu}}{(q_x+1)^{\nu+1}}q^{-\nu}=\left(q_x+1-\frac{q_x}{q}\right)^{-1}.$$

Substitution of the two values of Q_x into Eq. (10a) gives as a final result:

$$\begin{aligned}P_x&=0 \text{ for } q<1,\\ P_x&=((1-1/q)/(1-1/q+1/q_x)) \text{ for } q>1\end{aligned} \qquad (14a)$$

in which Eq. (14) is included as a special case.

Since $q_x<q$ for all values of x it follows from comparison of Eqs. (14) and (14a) that $P_x<P$ for all values of $q>1$.

The calculations can easily be extended to cases in which more than one initial electron is liberated at the same moment. For example in the case that N electrons are liberated at point x in the gas the breakdown probability is equal to $1-Q_x{}^{N}$. All the curves for the breakdown probability as a function of q have the feature in common that they are zero for $q<1$.

The breakdown potential can be defined as the potential difference between the electrodes for which $q=1$. It follows from Eqs. (14) and (14a) that below the breakdown potential $(q<1)P=0$, which means that the series of avalanches discontinues sooner or later. It must be emphasized, however, that this only holds in the case of a gas at low pressure. At higher pressure, where space charge formation is an essential element in the development of the spark,[5] the breakdown probability may be quite different, and there may be no well defined breakdown potential below which the breakdown probability is exactly zero. An alternative definition of the breakdown potential for this case is discussed by Loeb.[3]

IV. EXPERIMENTAL DETERMINATION OF BREAKDOWN PROBABILITY

There are several experimental methods by which the breakdown probability can be determined. One of these is to observe the time lag between the application of a certain voltage between the electrodes and the occurrence of a spark, while the cathode is irradiated continuously with ultraviolet light giving a known number of initiating electrons from the cathode per second. Experiments and calculations by Zuber,[6] V. Laue,[7] and Schade,[4] prove that the time lag consists of two parts. The first part gives a contribution subject to statistical fluctuations and is caused by the fact that breakdown depends first on the liberation of an initial electron by the ultraviolet light, which may occur with some delay, and second on the probability that this initial electron leads to breakdown. The second part of the time lag gives approximately a constant contribution and is due to the fact that the current has to build up from zero to a finite value, the magnitude of which is more or less arbitrarily determined by the experimental observability. This formative time lag is not appreciably influenced by the statistical fluctuations in the number of charged particles present, because these fluctuations are only important in the very early stage of the development of the spark.[4] The formative time is of course strongly dependent on the voltage between the electrodes. It should be remarked at this point that the breakdown probability, as calculated in Section III, gives the probability that the series of avalanches goes on indefinitely, whereas the result of a measurement depends on the probability that the series of avalanches is not interrupted during a finite time interval, fixed by the measurement. However, if this time interval is large compared to the time it takes for the positive ions to cross the gap, the results of Section III will give the right value for the breakdown probability to a very good approximation.

An effective method in observing the sparking time lag employs the Kerr cell optical shutter. This method has been used by Wilson[8] to observe sparking in air at atmospheric pressure. The results of measurements of this type give the probability that a spark occurs within a fixed time interval after the application of a voltage between the electrodes. This probability is called by Wilson *sparking probability*, but should be clearly distinguished from the *breakdown probability* as defined in this paper. For the relation between both probabilities the following expression can be derived.

$$\begin{aligned} R(V,T) &= 1-\exp[-nP_0(T-T_0)] \text{ for } T>T_0 \\ &= 0 \text{ for } T<T_0, \end{aligned} \tag{15}$$

in which $R(V,T)$ = probability that a spark occurs within a time interval T after application of a voltage V between the electrodes, n = average number of electrons liberated per second at the cathode by an external source of radiation, P_0 = breakdown probability for each liberated electron, for a gas at low pressure given by Eq. (14), and T_0 = formative time lag. P_0 and T_0 are both functions of V.

With help of Eq. (15) the theory of breakdown probability in a gas at low pressure, as presented in this paper, can be checked experimentally. The only experiments of the type described above have been performed in air at atmospheric

[5] L. B. Loeb and J. M. Meek, J. App. Phys. 11, 438, 459 (1940). L. B. Loeb and J. M. Meek, *The Mechanism of the Electric Spark* (Stanford University Press, California, 1941). L. B. Loeb, *Fundamental Processes of Electrical Discharge in Gases* (John Wiley & Sons, Inc., New York, 1919). H. Raether, Zeits. f. Physik. 112, 464 (1939).

[6] K. Zuber, Ann. d. Physik 76, 231 (1925).

[7] U. V. Laue, Ann. d. Physik 76, 261 (1925).

[8] R. R. Wilson, Phys. Rev. 50, 1082 (1936).

pressure, and therefore have no bearing on the theory valid for a gas at low pressure. The results of these experiments indeed seem to indicate that at high pressure the curve showing P_0 as a function of q is markedly different from the type of curve expressed by Eq. (14), especially at the foot of the curve. This is not surprising since the processes of spark formation in the two cases—high and low pressure—are quite different.

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Electron Velocity Distribution Function in High Frequency Alternating Fields Including Electronic Interactions

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The Boltzmann transport equation has been solved for the electronic velocity distribution function in a high frequency gas discharge. The distribution is examined as a function of the electron density. The conductivity is computed for two electron-molecule cross sections, one, in which the cross section is inversely proportional to the electron velocity and two, in which the cross section is independent of electron energy. The results show the extension of Margenau's distribution to high densities.

THIS paper is an extension of the work of Margenau[1] in which the Boltzmann Equation

$$(e/m)\mathbf{E}\cdot\nabla f=(\partial_e f/\partial t) \tag{1}$$

was solved for high frequency fields $\mathbf{E}_z=E\cos\omega t$ for the case of low current densities. In it we will apply the results of a former paper[2] (hereinafter referred to as I) in which the Coulomb interaction term $(\partial_e f/\partial t)_1$ of the Boltzmann Equation was derived. Following the notation in I, the distribution function $f(\mathbf{v})$, normalized to the electron density n, will be expanded in the form

$$f(\mathbf{v})=f_0(v)+v_z(f_1(v)\cos\omega t+g_1(v)\sin\omega t). \tag{2}$$

The approximate collision term $(\partial_e f_0/\partial t)_1$ is given by

$$\left(\frac{\partial_e f_0}{\partial t}\right)_1=\frac{10}{s^{\frac{1}{2}}}\frac{ds^{\frac{3}{2}}}{ds}(Af_0'+Bf_0), \tag{3}$$

in which $s=v^2$, $A=\mathcal{L}'\int_0^\infty f_0 ds$, $B=\mathcal{L}'f_0(0)$, $\mathcal{L}'=16\pi^2/15\cdot e^4/m^2\ln\bar{\epsilon}\rho/e^2$, $\rho=(\bar{\epsilon}_1/12\pi ne^2)^{\frac{1}{2}}$, e and m are the charge and mass of the electron; the average energy of electrons relative to electrons is $\bar{\epsilon}$, that relative to ions is $\bar{\epsilon}_1$.

We use the same electron-molecule and electron-ion collision term as in I, namely

$$\left(\frac{\partial_e f_0}{\partial t}\right)_M=\frac{\delta}{\beta s^{\frac{1}{2}}}\frac{d}{ds}\frac{s^2}{\lambda}(f_0'+\beta f_0), \tag{4}$$

where $\beta=m/2kT$, $\delta=2m/M$, T is the gas temperature, and M is the molecular mass, so that our total $\partial_e f_0/\partial t$ for electrons, molecules, and ions is the sum of (3) and (4).

We will make the same simplifications as in I in which the Coulomb interaction was neglected in the momentum balance equation. We can then use Eq. (23) in the paper by Margenau and Hartman,[3]

$$\frac{\partial_e f_0}{\partial t}=\frac{\gamma}{6v^2}\frac{d}{dv}(v^2 f_1),\quad \frac{v}{\lambda}f_1+\omega g_1=-\frac{\gamma}{v}f_0'(v),$$
$$g_1=(\omega\lambda/v)f_1. \tag{5}$$

[1] H. Margenau, Phys. Rev. 69, 508 (1946).
[2] J. H. Cahn, Phys. Rev. **75**, 293 (1949).
[3] H. Margenau and L. M. Hartman, Phys. Rev. **73**, 309 (1948).