Then

$$\partial E(\kappa)/\partial\kappa \cong \tfrac{1}{2}[J(J+1)-E'(\kappa)] + (2J+1)(m+\tfrac{1}{2})H(d/d\kappa)(H/(G-F)) + \frac{3(m^2+m+(1/2))H[(d/d\kappa)(H/(G-F))]}{16J^2(1+(1/2J))^2(1-[H/(G-F)]^2)^2}. \quad (13)$$

The results of Eqs. (12) and (13) are to be substituted in Eq. (7).

C. The Correspondence Principle Approximation

The result of reference (6) of primary interest in the present work is the tabulation (Table I) of a "reduced energy ratio," η, as a function of a quantum number ratio λ, and κ. η and λ are defined by

$$\eta(\kappa)_{J,K}=E(\kappa)_{J,K}/J(J+1), \quad \lambda=K/[J(J+1)]^{\frac{1}{2}}.$$

K here is the limiting (prolate or oblate) symmetric top quantum number. $\eta(\kappa)$ and $\partial\eta(\kappa)/\partial\kappa$ can be taken from the table, and

$$\partial E(\kappa)/\partial\kappa = J(J+1)[\partial\eta(\kappa)/\partial\kappa]. \quad (14)$$

The derivative $\partial\eta(\kappa)/\partial\kappa$ is evaluated for constant λ, and must not be confused with the derivative $\partial\eta/\partial\lambda$ which is given, together with η, in the table.

APPENDIX

Let

$$\mathbf{H}_0 = a\mathbf{P}_a^2+b\mathbf{P}_b^2+c\mathbf{P}_c^2, \quad (15)$$

$$\mathbf{H}' = \delta c\mathbf{P}_c^2. \quad (16)$$

Then, if $E(a, b, c)$ is an eigenvalue of the unperturbed problem (15), and E' the first-order energy correction due to (16), $E'=\delta c\langle\mathbf{P}_c^2\rangle$, where the angular brackets denote an average over the (unperturbed) eigenstate corresponding to $E(a, b, c)$.

Let $E(a, b, c+\delta c)$ be the corresponding eigenvalue of the Hamiltonian,

$$\mathbf{H} = a\mathbf{P}_a^2+b\mathbf{P}_b^2+(c+\delta c)\mathbf{P}_c^2,$$

and define ϵ such that $E(a, b, c+\delta c)=E(a, b, c)+E'+\epsilon$. Then

$$\frac{\partial E(a, b, c)}{\partial c}=\lim_{\delta c\to 0}\frac{E'+\epsilon}{\delta c},$$

but $\epsilon=0(\delta c^2)$, so $\partial E(a, b, c)/\partial c=\langle\mathbf{P}_c^2\rangle$.

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On the Space Distribution of Slow Neutrons

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The behavior of the neutron density about a plane- or point-source of fast neutrons within a homogeneous slowing-down medium has been re-investigated. For the case of constant mean free path a known analytical expression for the neutron density has been reduced to a form, which is valid for slow neutrons and *for any distance* from the source. The feasibility of a numerical evaluation of the formula is demonstrated for $M=1$ (hydrogen). In particular, the asymptotic behavior at very large distances has been studied. For the more realistic example of a medium in which the mean free path decreases with decreasing energy of the neutrons, formulae are presented describing the asymptotic density and the asymptotic energy spectrum at large distances from the source.

1. INTRODUCTION

THE present paper is an extension, in two directions, of previous work[1] on the transport equation for the diffusion and slowing-down of neutrons about a point source in an infinite homogeneous medium. First, the formal solution for the case of constant mean free path has been reduced to a numerically manageable form for sufficiently slow neutrons at all distances from the source. Secondly, for the case of a mean free path that decreases as the energy of the neutrons decreases, an asymptotic formula valid at large distances from the source has been derived.

A partial result for the first case, namely the asymptotic form of the constant-mean-free-path solution at very large distances was communi-

[1] M. Verde and G. C. Wick, Phys. Rev. 71, 852 (1947), henceforward referred to as "A."

cated by the author to Dr. Marshak, who kindly included it in his general report on neutron slowing down.[2] For the convenience of the reader we shall adhere as far as possible to Marshak's notation, which is not very different from that in "A." Thorough familiarity with Parts I and III-D of Marshak's report will be assumed and those parts of our argument that have already been reproduced in the report will be treated summarily. Reference to "M" also allows us to generalize immediately our formulae to the case of an arbitrary mass number M of the slowing down element. For simplicity we shall omit the generalization for the case of a mixture, which is fairly obvious; we shall also neglect neutron capture and inelastic scattering.

The spatial distribution of neutrons about a point source can be easily derived from the distribution about a plane source, see "M" Eq. (62). We shall therefore limit ourselves to the latter case, which is slightly simpler. Our distribution function ψ will thus be a function of z, the perpendicular distance from the plane source, of u the logarithmic energy variable ($u=0$ for the primary fast neutrons of velocity v_0) and of μ the cosine of the angle between the velocity and the z axis. Integrating over the solid angle we obtain $\psi_0(z, u)$, which is the main function we wish to determine. Specifically the density of neutrons at the distance z (we shall always assume $z>0$, since the density is an even function of z) within the energy internal du is

$$v^{-1}l(u)\psi_0(z, u)du, \qquad (1)$$

where $l(u)$ is the mean free path and v is the velocity ($=v_0e^{-u/2}$).

2. THE CASE OF CONSTANT MEAN FREE PATH ($l=1$)

2.1 Outline of the Procedure

The classical method, which was independently adopted in "A" and in the various papers summarized in "M", is to reduce the transport equation to a problem of the "one-velocity" type, see "A" Eq. (2′), by means of a Laplace transformation with respect to the energy variable u, η being the new variable. In this reduced problem the mean free path is the same as before ($=1$ by definition) and the variable η substituting u appears simply as a constant parameter, that determines the scattering probability per unit solid angle: $g(\mu_0, \eta)$, μ_0 being the cosine of the scattering angle. In the general case of mass number M of the scattering atoms, this function is given by Eq. (140b) in "M" or

$$g(\mu, \eta) = \frac{\alpha}{\pi}(M^2-1+\mu^2)^{-\frac{1}{2}}[G(\mu)]^{2\eta+2}. \qquad (2)$$

Here

$$\alpha = (M+1)^2/4M;$$

and

$$G(\mu) = \frac{\mu+(\mu^2+M^2-1)^{\frac{1}{2}}}{M+1}. \qquad (3)$$

$G(\mu)$ is the ratio v/v' of the velocities after and before scattering through an angle $\arccos\mu$.[3] We shall need later the Legendre coefficients $g_n(\eta)$ of $g(\mu, \eta)$

$$g_n(\eta) = 2\pi\int_{-1}^{+1} d\mu P_n(\mu)g(\mu, \eta) \qquad (4)$$

for instance

$$g_0(\eta) = \alpha(1+\eta)^{-1}[1-e^{-q(1+\eta)}],$$
$$g_1(\eta) = \alpha(\eta+\tfrac{1}{2})^{-1}(\eta+\tfrac{3}{2})^{-1} \times[\eta+1-\tfrac{1}{2}M+(\eta+1+\tfrac{1}{2}M)e^{-q(1+\eta)}], \qquad (5)$$

where $q=2\ln[(M+1)/(M-1)]$. For $M=1$ see also Eq. (32) below. It can be seen from Eq. (2) that if $M>1$ the functions $g_n(\eta)$ are holomorphic in the whole complex plane of the η variable, and satisfy the condition

$$\lim_{n\to\infty} g_n(\eta) = 0. \qquad (6)$$

If $M=1$ these statements hold true in the half-plane $\Re(\eta)>-1$, $\Re$ meaning real part.

After the transport equation has been further reduced by means of a Fourier transformation with respect to z, y being the new variable in Marshak's notation (we use instead $k=iy$ as in

[2] R. E. Marshak, Rev. Mod. Phys. 19, 185 (1947). This paper will be referred to as "M."

[3] The general form of Eq. (2) is easily understood if one notices that the reduced transport equation describes essentially the diffusion of neutrons having a power spectrum $e^{\eta u}=(v_0/v)^{2\eta}$. It is seen that when neutrons with this spectrum are scattered through any *given* angle, the scattered neutrons have again exactly the same spectrum, but owing to the change in velocity the proportionality constant in the spectrum is changed in the ratio

$$(v/v')^{2\eta}=[G(\mu)]^{2\eta}.$$

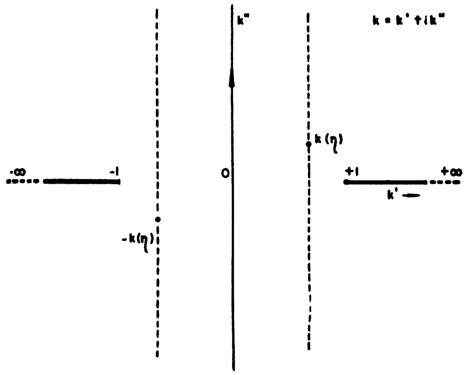

FIG. 1. The thick horizontal lines represent the "cuts" in the complex k-plane, see Eq. (13). The continuous vertical line is the integration path of Eq. (7). Two poles $k(\eta)$ and $-k(\eta)$ are indicated, the strip between the dashed lines is the convergence strip of the Fourier transform. There may be other poles outside the convergence strip.

"A") one arrives finally at the formula

$$\psi_0(z, u) = (1/2\pi i)^2 \int_{-i\infty}^{+i\infty} e^{-kz} dk \times \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\eta u} \phi_0(k, \eta) d\eta, \quad (7)$$

see "M" Eq. (142a) or "A" Eq. (39). The real constant σ must be larger than the convergence abscissa of the Laplace transform; in addition we may assume $\sigma > -1$, so that condition (6) always holds true. The function ϕ_0 must be obtained from an infinite set of linear equations, "A" Eq. (24a) or "M" Eq. (141a, b). Thus the problem consists of two parts: determination of ϕ_0 and evaluation of the integrals in (7).

We are interested in the distribution of slow neutrons, i.e., large values of u. We assume for instance $u \gtrsim 10$, corresponding to a reduction from an initial energy of more than 20 kev to 1 ev. On the other hand no restriction need be placed on z. (We shall see, however, that when z is large our formulae are valid under even *less* stringent conditions on u). We shall be able to eliminate one of the integrations in Eq. (7) by the method of residues. For the sake of clarity, we give first a general outline of the procedure to be followed, and fill in some of the details later.

It is clear, from the nature of ϕ_0 as a Laplace transform, that σ in Eq. (7) must be so large that all singularities of ϕ_0 in the η plane will be "to the left" of the integration path with respect to η. We may pull the integration path through the pole $\eta(k)$ having the largest real part $\Re(\eta)$ [4] and obtain a residue:

$$(1/2\pi i) \int \phi_0 e^{\eta u} d\eta = e^{\eta(k) u} R + \cdots \quad (8)$$

where

$$R \equiv R(k) = \lim_{\eta \to \eta(k)} [\eta - \eta(k)] \phi_0(\eta, k) = [\partial \phi_0^{-1} / \partial \eta]^{-1}_{\eta = \eta(k)}. \quad (9)$$

The remainder . . . is an integral running to the left of $\eta(k)$ but to the right of the next pole. We shall see that the difference $\delta\eta$ between the real parts of the two poles is at least of the order of $\frac{1}{2}$. Owing to the strong factor $e^{\eta u}$, the remainder is of the order $e^{-\frac{1}{2}u} \lesssim e^{-5}$ with respect to the first residue and can be neglected.[5] We are then left with the integral

$$\psi_0(z, u) = (1/2\pi i) \int_{-i\infty}^{+i\infty} \exp(-kz + u\eta(k)) R(k) dk. \quad (10)$$

We shall then consider k as a complex variable, and displace the integration path towards the right in the k plane. Considering the point k_0 where the path cuts the real axis, we shall see that we may choose k_0 to be a "saddle-point" for $\Re[-kz + u\eta(k)]$. At k_0 the exponential factor in (10) has a minimum along the real axis and a strong maximum along the path if this is chosen according to the prescription of "steepest descent." The exponent is then expanded in a Taylor series in the usual way

$$-kz + u\eta(k) = -k_0 z + u\eta_0 + \tfrac{1}{2} u \eta_0'' (k - k_0)^2 + \cdots,$$

where

$$\eta_0 = \eta(k_0), \quad \eta_0'' = \eta''(k_0).$$

One has then

$$\psi_0(z, u) \approx (2\pi \eta_0'' u)^{-\frac{1}{2}} R(k_0) \exp(-k_0 z + u\eta_0). \quad (11)$$

We notice that the minimum condition at k_0

$$-z + u\eta'(k_0) = 0 \quad (12)$$

defines k_0 as a function of z/u. Formula (11) can be used, therefore, if we can construct a table of

[4] The position of this pole in the η-plane depends on k, as we shall see.

[5] As a matter of fact, this is an overestimate of the remainder.

$\eta(k_0)$, $\eta'(k_0)$, $\eta''(k_0)$, $R(k_0)$ as functions of the real variable k_0.

In the passage from (10) to (11) we have tacitly assumed that the path of "steepest descent" is a straight path cutting across the real axis at right angles. Actually it is not straight, but it will be shown that the validity of Eq. (12) is not affected. It will also appear, that Eq. (11) goes over into the customary "age" approximation at small distances, and at very large distances into the asymptotic formula that was reproduced in "M" Eq. (175).

2.2 The Function $\phi_0(k, \eta)$

We summarize here the main results of "A," Sections 3.1 and 4.1, concerning the determination of the function ϕ_0; in addition some relationships will be derived for later use.

Let us consider ϕ_0 as a function of the complex variable k. ϕ_0 is at first defined as a Fourier transform; as such it has a meaning only within a convergence strip $k_1<\Re(k)<k_2$. It will appear later that k_1 and k_2 are functions of η. The definition of ϕ_0 may, however, be extended by analytical continuation; an expression of ϕ_0 as a continued fraction was given in "A," see also later, from which it was inferred that ϕ_0 is a one-valued analytic function of k in the whole k-plane provided this is cut along the real intervals

$$-\infty<k<-1 \quad \text{and} \quad +1<k<+\infty. \qquad (13)$$

In the k-plane ϕ_0 has only isolated poles. The situation is in general as sketched in Fig. 1.

In order to determine ϕ_0 one must consider it together with the remaining Legendre coefficients ϕ_1, ϕ_2, $\cdots$ of the angular distribution function $\phi(\mu)$, "A" Eq. (11). Within the convergence strip this is a regular function of the cosine μ, so that

$$\lim_{n=\infty}\phi_n=0. \qquad (14)$$

The sequence ϕ_0, ϕ_1, $\cdots$ satisfies an infinite set of equations

$$(2n+1)\gamma_n\phi_n-k(n+1)\phi_{n+1}-kn\phi_{n-1}=\delta_{n0}, \qquad (15)$$

the first of which is inhomogeneous ($\delta_{00}=1$), while the remaining ones form a recurrent homogeneous relation ($\delta_{n0}=0$ if $n\neq 0$). Here

$$\gamma_n=1-g_n(\eta), \qquad (16)$$

and it is assumed that η satisfies the conditions for Eq. (6) to be valid.

The analytical continuation of ϕ_0 outside the convergence strip may be effected by requiring (14) to be valid everywhere in the cut plane. Disregarding for a moment the first Eq. (15) (for $n=0$), one can see that the recurrent relation has two linearly independent solutions, since ϕ_0 and ϕ_1 may be chosen arbitrarily. Out of this double infinity of solutions, condition (14) selects one linearly independent solution. In fact it may be easily seen that the characteristic equation "A" Eq. (19) has a root ϵ such that $|\epsilon|<1$, while the other root $\epsilon'=1/\epsilon$ has $|\epsilon'|>1$. Under such circumstances, Poincaré's theorem, see "A" Sec. 3.1, assures us that there is only one solution such that

$$\lim_{n=\infty}\phi_{n+1}/\phi_n=\epsilon. \qquad (17)$$

This solution obviously satisfies Eq. (14); for all other solutions the ratio tends to ϵ', and Eq. (14) is not satisfied. The only exception occurs on the "cuts," where $|\epsilon|=|\epsilon'|=1$; (this is the reason why ϕ_0 is not unambiguously defined on the cuts).

Turning now to the remaining inhomogeneous equation, or Eq. (15) for $n=0$, two different cases may present themselves. Either the solution satisfying (17) also satisfies the equation

$$\gamma_0\phi_0-k\phi_1=0 \qquad (18)$$

in which case the inhomogeneous equation cannot be satisfied; or the left hand side of (18) is $\neq 0$, and then the arbitrary multiplicative constant in the solution can be adjusted so as to make the left hand side of Eq. (18) equal to unity, i.e., so as to satisfy the whole system (15).

This is, of course, the familiar relationship: if the homogeneous system, which is obtained by replacing δ_{n0} by 0, has a non-trivial solution, then the inhomogeneous system has none, and *vice versa*. The former case will occur only for special values of the parameter k; these have been called "eigenvalues." For each value of η, there may be several eigenvalues $k(\eta)$. When k approaches an eigenvalue $k(\eta)$, the multiplicative constant tends to infinity, so that ϕ_0 has a pole singularity at $k=k(\eta)$. On the other hand it may be easily seen from the recurrent relation that the sequence $\bar{\phi}_0$, $\bar{\phi}_1$, $\cdots$, etc. where: $\bar{\phi}_0=1$, $\cdots$,

$\bar{\phi}_n=\phi_n\phi_0^{-1}$, $\cdots$ remains finite, so that all the coefficients ϕ_n have a pole singularity of, at most, the same degree as ϕ_0. Once the eigenvalue $k(\eta)$ is known, the behavior of the sequence ϕ_n at $k=k(\eta)$ can be computed directly from the equations

$$\bar{\phi}_0=1;\quad \gamma_0-k\bar{\phi}_1=\lim\phi_0^{-1}=0;\quad \gamma_1\bar{\phi}_1-2k\bar{\phi}_2-k\bar{\phi}_0=0,\ \cdots,\ \text{etc.,} \tag{19}$$

i.e., by solving the *homogeneous* system, with the initial conditions $\bar{\phi}_0=1$, $\bar{\phi}_1=\gamma_0/k$.

The following formulae are of use in the actual evaluation of ϕ_0. From the system (15) one easily derives an expression of ϕ_0^{-1} in terms of ϕ_1/ϕ_0, then of ϕ_2/ϕ_1, $\cdots$. One arrives finally at the continued fraction[6]

$$\phi_0^{-1}=\gamma_0-\cfrac{\beta_1}{\gamma_1-\cfrac{\beta_2}{\gamma_2-\cdots-\beta_{n-1}/(\gamma_{n-1}-\lambda_n)}} \tag{20}$$

where

$$\beta_n=k^2n^2/(4n^2-1);\quad \lambda_n=[kn/(2n-1)]\phi_n/\phi_{n-1}. \tag{21}$$

The continued fraction "A" Eq. (25) is obtained on passing to the limit for $n=\infty$.[7] The customary way of evaluation consists in breaking up the fraction at the nth denominator, i.e., setting $\lambda_n=0$ in Eq. (20). One has then

$$\phi_0^{-1}\approx U_n/V_n, \tag{22}$$

where U_n and V_n are both solutions of the recurrent relation

$$w_{n+1}=\gamma_nw_n-\beta_nw_{n-1} \tag{23}$$

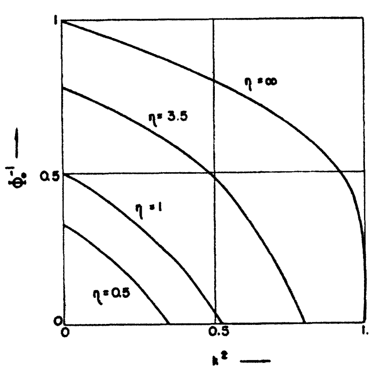

FIG. 2.

with the initial values

$$U_0=1,\quad U_1=\gamma_0,\quad V_0=0,\quad V_1=1. \tag{24}$$

The exact expression (20) can also be written

$$\phi_0^{-1}=(U_n-\lambda_nU_{n-1})/(V_n-\lambda_nV_{n-1}). \quad (25)^8$$

A more rapidly convergent expression for ϕ_0^{-1} can obviously be obtained if instead of setting $\lambda_n=0$ in Eq. (20), we replace λ_n by the limit, that follows from Eq. (17)

$$\lambda=\lim_{n=\infty}\lambda_n=k\epsilon/2=\tfrac{1}{2}(1-(1-k^2)^{\frac{1}{2}}). \tag{26}$$

The expression

$$\phi_0^{-1}\approx(U_n-\lambda U_{n-1})/(V_n-\lambda V_{n-1}) \tag{22'}$$

was in fact found to be far more satisfactory than (22) in the numerical evaluations, see next section.

An interesting limiting case arises when for some reason the coefficients $g_m(\eta)$ are negligible for $m\geq n$. Then

$$\gamma_n=\gamma_{n+1}=\cdots=1 \tag{27}$$

and one can easily show[9] that

$$\phi_n/\phi_{n-1}=Q_n(1/k)/Q_{n-1}(1/k), \tag{28}$$

where Q_n is the nth Legendre function of the second kind. This may be fed into Eqs. (21) and (20) or (25) to evaluate ϕ_0 in a finite form. An especially simple example of this kind occurs when $\eta\rightarrow\infty$ so that, see Eq. (6),

$$\gamma_0(\infty)=\gamma_1(\infty)=\cdots=1,\quad \phi_0(k,\infty)=Q_0(1/k)=\mathrm{Artanh}\,k/k \tag{29}$$

as one can easily prove.

Finally we wish to derive an expression for the derivative $\partial\phi_0/\partial k$. Deriving Eq. (14) we get

$$(2n+1)\gamma_n\partial\phi_n/\partial k-k(n+1)\partial\phi_{n+1}/\partial k-kn\partial\phi_{n-1}/\partial k=(n+1)\phi_{n+1}+n\phi_{n-1}. \tag{30}$$

Multiplying Eq. (14) by $-\partial\phi_n/\partial k$ and (30) by $+\phi_n$ and adding, one obtains

$$A_n-A_{n+1}=n\phi_{n-1}\phi_n+(n+1)\phi_n\phi_{n+1}-\delta_{n0}\partial\phi_n/\partial k,$$

[6] See I. Waller, Arkiv f. Mat. Astron. Fysik, **34A**, No. 3, 4 and 5.

[7] It will be noticed that the equation as printed in "A" contains a slip.

[8] We spare the reader the derivation of Eqs. (22) to (25), which are well-known in the theory of continued fractions. One can also arrive at them directly, setting:

$$\phi_n=w_nk^{-n}(n!)^{-1}1.3\cdots(2n-1),\quad w_n=\phi_0U_n-V_n$$

and inserting into Eq. (15).

[9] See for instance I. Waller, reference 6, especially the first paper.

where

$$A_n = nk(\phi_{n-1}\partial\phi_n/\partial k - \phi_n\partial\phi_{n-1}/\partial k)$$

(in particular: $A_0=0$). Noticing that $\lim_{n\to\infty}A_n=0$ and summing over n we have

$$\partial\phi_0/\partial k = 2\sum_{n=0}^{\infty}(n+1)\phi_n\phi_{n+1}. \quad (31)$$

2.3 Numerical Evaluation of $\phi_0(k, \eta)$

As we shall see, when k is real and within the interval $-1<k<+1$, ϕ_0 has a real pole $\eta(k)$ in the η-plane, such that $\eta>0$. This is what we call the "main pole," i.e., the pole with the largest real part, which is of paramount importance in the method of steepest descent, Eqs. (10), (11), (12). For small values of k the function $\eta(k)$ can be obtained from an expansion in powers of k^2, that will be discussed later for the general case, see Eq. (65). For the special case $M=1$ we have obtained four coefficients of the expansion, thus

$$\eta = k^2 + (11/15)k^4 + (1563/1575)k^6 + (70709/118125)k^8 + \cdots.$$

This was used for an evaluation of η, η', and η'' for $k \leq 0.5$.

When $0.5<k<1$, however, it became necessary to find the connection between η and k by a numerical investigation of the function ϕ_0 as illustrated by Fig. 2. Here ϕ_0^{-1} is plotted as a function of k^2 [10] for selected values of η, and the pole singularity of ϕ_0 appears as a zero for ϕ_0^{-1}, the position of which on the k axis varies as η varies. This gives k as a function of η, or vice versa.

The figure is qualitatively correct for any slowing down element (any M).[11] Quantitatively we have carried out the program only in the case of hydrogen ($M=1$). The calculation involves the following steps:

First, the coefficients $g_n(\eta)$ are computed for a selected value of η, by means of the relation

$$g_{n+2}(\eta) = g_n(\eta)(2\eta+1-n)/(2\eta+4+n) \quad (32)$$

and the values $g_0=1/(1+\eta)$, $g_1=1(\eta+\frac{3}{2})$. (Unfortunately a relation as simple as (32) does not exist when $M\neq 1$.)

Next a table of $\gamma_0\cdots\gamma_n$ and $\beta_1\cdots\beta_n$ is set up

TABLE I.

k	η	η'	η''	R	c (Eq. (48a))	$R(1-k)$
0.0	0	0	2	1		1
0.1	0.01007	0.2030	2.091	1.007		
0.2	0.04124	0.425_4	2.40_2	1.029		0.823
0.3	0.09670	0.695	3.06	1.073		0.752
0.4	0.183_2	1.05_7	4.3_1	1.15		0.696
0.5	0.314	1.60	6.5	1.26	0.534	0.655
0.5918*	0.5	2.48_3	12.8	1.45_6	0.539	0.594
0.64	0.637	3.22_3	18.4_4	1.61	0.541	0.58
0.7252*	1.0	5.58_7	41.1	2.01	0.543	0.553
0.8952*	3.5	38.615	737	4.98	0.546	0.522
1	∞	$\sim 2.36\eta^2$	$\sim 11.1\eta^3$	$\sim 1.18\eta$	$c_\infty \approx 0.548$	0.500

for a given k, up to a sufficiently high n, and the sequences U and V are computed by means of the homogeneous recurrent relation (23) ($n=1, 2, \cdots$) starting from the initial values (24). The expression (22′) is then found to converge satisfactorily to the value of $\phi_0^{-1}(k, \eta)$. This process is repeated for several values of k, and one of the curves in Fig. 2 is obtained.

In practice it is only necessary to evaluate ϕ_0^{-1} crudely for a few points so as to have an idea of the position of the zero-point. Then ϕ_0^{-1} is evaluated more accurately for two or three k^2-values in the neighborhood of the zero-point, which is then obtained by interpolation. It is also possible to estimate by interpolation the derivative $\partial\phi_0^{-1}/\partial k$ at the zero point; a more accurate evaluation is obtained, however, from Eq. (31) or

$$\partial\phi_0^{-1}/\partial k = -2\sum_{n=0}^{\infty}(n+1)\phi_n\phi_{n+1}/\phi_0^2. \quad (33)$$

It may be pointed out that one has (see reference 8)

$$\phi_0^{-1}\phi_n = [1\cdot 3\cdots(2n-1)/n!]k^{-n} \times (U_n - \phi_0^{-1}V_n). \quad (34)$$

At the zero-point the term in V_n vanishes. Finally

$$\partial\phi_0^{-1}/\partial k = -2\sum_{n=0}^{\infty}[1\cdot 3\cdots(2n-1)/n!]^2 \times (2n+1)k^{-2n-1}U_nU_{n+1} \quad (35)$$

involving only known quantities. It may be easily shown that the series converges like

$$\sum_n (2\lambda/k)^{2n} = \sum_n \epsilon^{2n}.$$

The rather tedious procedure we have described was actually carried out only for three η values, indicated by an asterisk in the table

[10] It is easy to see that U_n and V_n satisfying Eqs. (23) and (24) are even functions of k, so that ϕ_0, Eq. (22), is also even.

[11] In particular it is always true that the zero-point $\eta(k)$ increases from zero to $+\infty$ while k^2 varies from 0 to $+1$.

below. It then turned out that in this region η is almost exactly linear in $(1-k)^{-1}$ so that accurate interpolation is easy. This behavior corresponds to formulae (48), (48a). As a consequence it has been possible to compute without undue labor a table of the function $\eta(k)$ and its derivatives $\eta'(k)$ and $\eta''(k)$.

In order to complete the table required by the saddle-point formulae (11) and (12), one has to evaluate R, Eq. (9).

Now the function $\eta(k)$ is defined by the equation: $\phi_0^{-1}(k, \eta)=0$, so that

$$\partial\phi_0^{-1}/\partial k+\eta'(k)\partial\phi_0^{-1}/\partial\eta=0,$$

or

$$R=-\eta'(k)/(\partial\phi_0^{-1}/\partial k) \tag{36}$$

which may be computed from the value of the expression (35). The results are summarized in the Table I, which includes the data for $k<0.5$ obtained from the series expansion for η already given and the expansion for R

$$R^{-1}=1-(2/3)k^2-(17/15)k^4+(48176/23625)k^6+\cdots.$$

2.4 Asymptotic Formulae for $k(\eta)$ When $\eta\rightarrow\infty$

An inspection of the values in Table I, together with Eq. (12) shows that when we make the distance from the source z larger and larger (keeping the energy u constant), we are led to larger and larger values of η. At the same time $k(\eta)\rightarrow 1$, and $\epsilon\rightarrow 1$, so that the convergence of the sequence ϕ_n to zero becomes very slow (see Eq. (17)). This makes the convergence of the expressions (22) or (22′) unsuitable for numerical work.

Although the γ's are very close to unity when η is large, it is just their difference from unity that determines the difference $1-k(\eta)$, i.e., the quantity we wish to determine. In fact, as will appear from the following, the differences $g_n=1-\gamma_n$ cannot here be neglected up to a large value of n. This makes also the method based on the smallness of the g_n's (see Eqs. (27) and (28)) perfectly useless in the present case.

On the other hand, as we shall see, the recurrent system (15) takes for large η's a simple asymptotic form, that makes the problem easier than one might think.

In the first place, a simple asymptotic expression may be given for the coefficients $g_n(\eta)$, when $\eta\rightarrow\infty$. Going back to the scattering function g, Eq. (2), we make use of the known theorem, that if a function with a maximum is raised to a high power, the ensuing sharp maximum is of Gaussian shape. Setting $\mu=\cos\Theta$ in Eq. (3) and developing $\ln G(\Theta)$ for small Θ

$$\ln G(\Theta)=-\Theta^2/2M+(\Theta^4/24M)(1-3/M^2)+\cdots. \tag{37}$$

When this is substituted into Eq. (2) the Θ^2-term yields a Gaussian, which decays to a negligible value before the Θ^4-term makes itself felt. (If η is large, i.e.: $\eta\gg M$.) Within the same approximation one may replace the factor $(M^2-1+\mu^2)^{-\frac{1}{2}}$ simply by M^{-1}, so that finally

$$g(\cos\Theta, \eta)\approx(\alpha/\pi M)\exp(-\eta\Theta^2/M), \tag{38}$$

($\eta+1$ has been replaced by η in the exponent, for the same reasons). In evaluating the integrals (4) one may again make use of the fact that g, Eq. (38), decays to a negligible value as soon as $\Theta\gg(M/\eta)^{\frac{1}{2}}$, which is a small angle. The main contribution to the integral comes therefore from small Θ-values,[12] for which one may use the well known approximation

$$P_n(\cos\Theta)=J_0([n+\tfrac{1}{2}]\Theta) \tag{39}$$

where J_0 is the Bessel function of zero order. Inserting (38) and (39) into (4) one finds

$$g_n(\eta)=(\alpha/\eta)\exp[-M(n+\tfrac{1}{2})^2/4\eta]. \tag{40}$$

This expression has been compared with exact numerical values by Marshak, with very good agreement (see "M," footnote on page 231).

It may next be noticed that, η being large, the exponential in Eq. (40) is a slowly variable function of n in the whole region where it is not negligibly small. This suggests the introduction of a variable $\sigma=(M/4\eta)^{\frac{1}{2}}(n+\frac{1}{2})$, the step $\delta n=1$ corresponding to a very small step in σ: $\delta\sigma=(M/4\eta)^{\frac{1}{2}}$, so that g_n may be regarded as a

[12] It may be pointed out that this is not true if n becomes very large. One may show, however, that (provided η is large) the formula (40) represents $g_n(\eta)$ correctly up to n values such that $Mn^2/4\eta\gg 1$, so that the exponential becomes exceedingly small. In particular in the case of hydrogen it is possible to derive from Eq. (32) a closed expression for g_n when both n and η are large, no matter what the ratio n/η is. For $n\gg\eta$ the behavior of g_n as a function of n becomes oscillatory and differs radically from Eq. (40). These values are, however, so small that the difference does not matter.

smooth function of a practically continuous variable σ.

Let us now turn to the study of the system (14) when η is large and k tends to a pole $k(\eta)$. As we have seen already, the poles $k(\eta)$ may also be characterized as the "eigenvalues" of the parameter k, i.e., the values for which there exists a non-trivial solution of the completely homogeneous system

$$(2n+1)\gamma_n\phi_n - k(n+1)\phi_{n+1} - kn\phi_{n-1} = 0, \quad (n=0, 1, 2, \cdots) \tag{41}$$

subject to the condition (14). The first equation (41) is now

$$\gamma_0\phi_0 = k\phi_1. \tag{42}$$

Using the difference notation

$$\delta\phi_{n+\frac{1}{2}} = \phi_{n+1} - \phi_n, \quad \delta\phi_n = \tfrac{1}{2}(\delta\phi_{n+\frac{1}{2}} + \delta\phi_{n-\frac{1}{2}}), \quad \delta^2\phi_n = \delta\phi_{n+\frac{1}{2}} - \delta\phi_{n-\frac{1}{2}}, \tag{43}$$

the remaining equations of the system may be put into the form

$$\delta\phi_{n+\frac{1}{2}} = [n/(n+1)]\delta\phi_{n-\frac{1}{2}} + [(2n+1)/k(n+1)](\gamma_n - k)\phi_n, \tag{44}$$

or alternatively,

$$\delta^2\phi_n + (n+\tfrac{1}{2})^{-1}\delta\phi_n + (2 - 2k^{-1}\gamma_n)\phi_n = 0. \tag{45}$$

The difference notation emphasizes the similarity to an eigenvalue problem of the Sturm-Liouville type, and in particular to a Schrödinger equation.[13]

We assume now that η is real and >0. It may be shown later that an eigenvalue k is then necessarily real (see Appendix A). Since the intervals $-\infty < k < -1$ and $+1 < k < +\infty$ are excluded, and since the eigenvalues occur in pairs of opposite sign (ϕ_0 being an even function of k) we may limit our search to the interval $0 < k < 1$.

A further limitation is given by the fact that a positive k must be larger than *some* at least of the γ_n's.[14] In fact, if $k < \gamma_n$ for all n's, Eq. (42) shows first that (assuming for instance $\phi_0 > 0$): $\phi_1 > \phi_0$ or: $\delta\phi_{\frac{1}{2}} > 0$; subsequently Eq. (44) shows that: $\delta\phi_{3/2} > 0$, $\cdots$, etc., so that ϕ_n increases monotonically with n, and cannot tend to zero.

Since, for large η, $\gamma_n \geq \gamma_0 \approx 1 - \alpha/\eta$, we have finally to search for an eigenvalue k in the small interval

$$1 - \alpha/\eta < k < 1 \tag{46}$$

showing that $1-k$ is at most of the order of $1/\eta$.

Again using Eq. (42) and (44) we easily see that $\delta\phi_{\frac{1}{2}}$, $\cdots$ and similarly $\delta\phi_1$, $\delta\phi_2$, $\cdots$ are at most of the order of $1/\eta$. Although the argument is not valid up to indefinitely high values of n, we may see that the statement is true even there from Eq. (17), since $1-\epsilon$ is of the order of $1/\eta$ at the most.

Thus, ϕ_n is shown to be a slowly variable function of n of the same kind as $g_n(\eta)$, and we are justified in regarding it as a continuous function of the variable σ above defined. According to (46) we get $k^{-1} = 1 + k_1/\eta$ (or $k \approx 1 - k_1/\eta$). Inserting this, together with the expression (40), into (45), and dividing by $\delta\sigma^2 = M/4\eta$, we obtain

$$\delta^2\phi/\delta\sigma^2 + \sigma^{-1}\delta\phi/\delta\sigma + 8[-k_1/M + (\alpha/M)(1 + k_1\eta^{-1})e^{-\sigma^2}]\phi = 0.$$

In the limit $\eta \to \infty$, $\delta\sigma \to 0$, the differences may be replaced with derivatives and the term $k_1\eta^{-1}$ may be neglected, so that, with

$$8\alpha/M = 2(M+1)^2/M^2,$$

the equation becomes

$$d^2\phi/d\sigma^2 + \sigma^{-1}d\phi/d\sigma + [-8k_1/M + 2M^{-2}(M+1)^2e^{-\sigma^2}]\phi = 0 \tag{47}$$

which was reproduced in "M" Eq. (169).[15] This equation, together with the boundary condition to be explained below, may be regarded as the radial Schrödinger equation for a two-dimensional particle, $-8k_1/M$ being the energy and $-2[(M+1)/M]^2e^{-\sigma^2}$ being the potential. σ is identified with the distance from the origin. For all values $M = 1, 2, \cdots$ the equation has only one bound level. The argument shows that k_1 approaches a finite value when $\eta \to \infty$. A more precise definition of k_1 is

$$k_1 = \lim_{\eta=\infty}[\eta(1-k)]. \tag{48}$$

[13] In the latter case we may compare the expression $2 - 2k^{-1}\gamma_n$ with $E - V(x)$, "2" being the energy and $2k^{-1}\gamma_n$ the potential. The eigenvalue problem then is of the type encountered when the energy level is given and a proportionality constant in the potential must be found.

[14] This is analogous to the theorem that the energy must be higher than the minimum of the potential (see reference 13).

[15] For comparison let $s = 2\sigma$. Marshak's k_1 corresponds to our k_1/M.

Now to the boundary conditions; these are

$$\phi\to 0 \text{ as } \sigma\to\infty; \quad \phi \text{ regular at } \sigma=0. \tag{49}$$

The first condition does not require any comment. The second implies

$$d\phi/d\sigma=0;$$
$$d^2\phi/d\sigma^2=\phi[4k_1/M-\{(M+1)/M\}^2] \tag{50}$$

for $\sigma=0$. That these are the correct conditions may be seen on comparing them with the differences $\delta\phi_0$ and $\delta^2\phi_0$ obtained by extrapolation from $\delta\phi_{\frac{1}{2}}$ and $\delta^2\phi_1$, as computed from Eqs. (42) and (45), again neglecting terms of higher order in $1/\eta$.

Summarizing the results obtained so far, we can say that the function $k(\eta)$ for large η can be determined approximately by solving the radial Schrödinger equation for a mass-point moving with zero angular momentum in a plane under the action of a Gaussian potential. Furthermore, one can prove that even for $M=1$, when the well has its maximum depth, the equation admits only one discrete eigenvalue. For M tending to infinity the well becomes so shallow, that it is barely sufficient for binding, k_1/M tends to a very small though finite value.

Having evaluated k_1 numerically, the behavior of k for large η's is given to a first approximation by Eq. (48). Within the same approximation we easily get an expression for R from Eqs. (33) and (36). Since ϕ_n is a slowly variable function of n, we may set:

$$\sum_n\cdots\approx(\delta\sigma)^{-1}\int\cdots d\sigma,$$

and $(n+1)\phi_n\phi_{n+1}\approx(n+\frac{1}{2})\phi_n^2$ so that:

$$\partial\phi_0^{-1}/\partial k=-[8\eta/M\phi^2(0)]\int_0^\infty\phi^2(\sigma)\sigma d\sigma;$$

moreover $k\approx 1-k_1/\eta$, $d\eta/dk\approx\eta^2/k_1$ so that finally

$$R=\eta M\phi^2(0)\left[8k_1\int_0^\infty\phi^2(\sigma)\sigma d\sigma.\right]^{-1} \tag{51}$$

Thus,

$$(R/\eta)_\infty=\lim_{\eta=\infty}(R/\eta)$$

exists and may be easily evaluated once the eigenfunction $\phi(\sigma)$ is known.

2.5 Higher Approximations

In order to study the asymptotic behavior of the neutron density it is desirable to obtain some further information about $k(\eta)$ when η is large. Let us write

$$\eta=k_1(1-k)^{-1}-c, \tag{48a}$$

where c may be regarded as a function of k or alternatively of η. We shall prove that c tends to a finite value c_∞ when $\eta\to\infty$, i.e., when $k\to 1$. We can write, therefore:

$$k\sim 1-\frac{k_1}{\eta+c_\infty}+0(\eta^{-3}). \tag{48b}$$

In order to evaluate c_∞ we might try to improve the method of the previous section, retaining terms of a higher order in $1/\eta$. In particular, we ought to correct for the errors incurred in replacing the finite differences with derivatives, errors which are, as one easily sees, of the order of $(\delta\sigma)^2=(M/4\eta)$. It is much easier, however, to arrive at this goal by following a different method, that we have developed and which was adopted by Marshak in his report. It consists in going back to the integral equation, of which the system (15) is the Legendre development (see "A" Eq. (24) or "M" (163)), and introducing directly into it the approximation (38). Let us use the same notation as in "A", where $\boldsymbol{\omega}$ was the unit vector representing the direction of motion, and μ the cosine of the angle between $\boldsymbol{\omega}$ and the z axis.

We have shown that the Legendre coefficients ϕ_n vary slowly when n varies, and are markedly different from zero over a range of n values of the order of $(4\eta/M)^{\frac{1}{2}}$. This means that the angular distribution function $\phi(\mu)$, of which the ϕ_n are the Legendre coefficients, extends over a narrow solid angle around the pole $\mu=1$, having an aperture of the order of $(M/4\eta)^{\frac{1}{2}}$. We may then use some suitable projection of the unit sphere on a plane tangent to the sphere at the pole, such that the pole is represented by the origin of a system of cartesian or polar coordinates in this plane, and such that the small area in which ϕ is different from zero is not appreciably distorted by the projection. Then arc $\cos\mu$ is approximately measured by the distance s of the representative point from the origin, and the angle

$\theta = \arc\cos(\boldsymbol{\omega}\cdot\boldsymbol{\omega}')$ is approximately measured by the distance S between the representative points of $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$. It is, however, convenient to choose the unit of length in the plane in such a way that it corresponds to an arc of $(M/4\eta)^{\frac{1}{2}}$ radians, so that the eigenfunction occupies an area in the plane of order unity.

Let us first preserve the same degree of approximation as in the previous section. Writing the homogeneous integral equation corresponding to the system (41)[16] in the form

$$(1-k)\mu\phi(\mu) = -(1-\mu)\phi(\mu) + \int g\phi(\mu')d\omega' \quad (52)$$

we see that all terms are of order $1/\eta$. Consequently, we may set the factor $\mu=1$ on the left side, and $(1-\mu) = -(\arc\cos\mu)^2/2 = -Ms^2/8\eta$, on the right-hand side, remembering what we said about the units. Similarly, we have

$$d\omega = (M/4\eta)d\mathbf{s},$$

$d\mathbf{s}$ being the element of area in the plane; we also write $\phi(\mathbf{s})$ instead of $\phi(\mu)$. Remembering now Eqs. (38) and (48), Eq. (52) becomes

$$4k_1\phi(\mathbf{s}) + (M/2)s^2\phi(\mathbf{s}) - (\alpha/\pi)\int \exp(-\mathbf{S}^2/4)\phi(\mathbf{s}')d\mathbf{s}' = 0, \quad (53)$$

where the integral term on the right-hand side may be written also symbolically as: $\alpha e^{\Delta}\phi(\mathbf{s})$, Δ being the Laplacian operator in the plane. It is then clear that the equation can be considerably simplified by means of a two-dimensional Fourier transformation

$$\phi_\sigma = \mathfrak{F}(\phi) = (1/2\pi)\int \phi(\mathbf{s})\exp(i\boldsymbol{\sigma}\cdot\mathbf{s})d\mathbf{s}. \quad (54)$$

We indicate the Fourier transform with the same letter ϕ, but set the variable vector $\boldsymbol{\sigma}$ as an index to avoid confusion. Owing to the fact that $\phi(\mathbf{s})$ depends only on the modulus of $\mathbf{s}$, ϕ_σ also depends only on the modulus $\sigma = |\boldsymbol{\sigma}|$. On applying the Fourier transformation the operator e becomes simply $e^{-\sigma^2}$ whilst

$$\mathfrak{F}(s^2\phi) = -\Delta_\sigma\phi_\sigma = -(d^2/d\sigma^2 + \sigma^{-1}d/d\sigma)\phi_\sigma.$$

[16] In the same way as "A" Eq. (24) corresponds to the inhomogeneous system (15).

On making these substitutions, the equation immediately reduces to Eq. (47) of the preceding section. This is indeed not surprising, as there is the following rather trivial connection between the two methods.

A development in Legendre polynomials is essentially a development in eigensolutions of the wave equation on the sphere, whilst the Fourier transformation is a development of ϕ in eigensolutions of the wave equation in the plane. On a small polar cap on the sphere, such that the curvature may be neglected, the first development is practically equivalent to the second, this being the deeper reason for the connection (39), as is well known. The inverse of Eq. (54), or

$$\phi(s) = (1/2\pi)\int \phi_\sigma \exp(-i\boldsymbol{\sigma}\cdot\mathbf{s})d\boldsymbol{\sigma} = \int_0^\infty \phi_\sigma J_0(\sigma s)\sigma d\sigma \quad (55)$$

is, in fact, the limiting form of the Legendre development "A" Eq. (11), apart from a different normalization.

We now turn to the next higher approximation. This does not present any special difficulty, and we shall only sketch the main steps. Since it is now important to consider the distortion in the projection on the plane, we must decide upon the type of projection we want to use. We choose, for instance, the homalographic projection

$$s = (4\eta/M)^{\frac{1}{2}}2\sin(\tfrac{1}{2}\arc\cos\mu) = [8\eta(1-\mu)/M]^{\frac{1}{2}}. \quad (56)$$

The polar angle on the plane: φ remains equal to the longitude on the sphere, of course. One has then exactly

$$1-\mu = Ms^2/8\eta; \quad d\omega = (M/4\eta)sdsd\varphi. \quad (57)$$

We must use the full expression (37) where the arc distance Θ is given by

$$(4\eta/M)\Theta^2 = S^2 + (M/4\eta)\{S^4/12 - s^2s'^2/2 + ss'\cos(\varphi-\varphi')\cdot(s^2+s'^2)/4\}, \quad (58)$$

where

$$S^2 = (\mathbf{s}-\mathbf{s}')^2 = s^2 + s'^2 - 2ss'\cos(\varphi-\varphi'). \quad (59)$$

Equation (58) is derived from the cosine theorem of spherical trigonometry. Within the same approximation we may *not* replace η by $\eta+1$ in the exponent of g, as in Eq. (38), and we must also

Table II.

$M=1$	2	12	16
$\eta_1=1$	0.689	2.24	2.90
$\eta_2=11/15$	0.600	1.42	1.81

use a better approximation for the factor $(M^2-\sin^2\theta)^{-\frac{1}{2}}$. Finally we find that the integral equation may be written

$$U(\phi)+\eta^{-1}\Big\{4k_1c_\infty\phi(\mathbf{s})-(Mk_1/2)s^2\phi(\mathbf{s})$$
$$+(\alpha/64\pi M)\int\exp(-S^2/4)\phi(\mathbf{s}')fd\mathbf{s}'\Big\}=0, \quad (60)$$

where $U(\phi)$ is the left hand side of Eq. (53). Moreover, terms of order η^{-2} are neglected, and k has been expanded as in Eq. (48b). Finally, f is defined by

$$f=S^4+(M^2/2)[(s^2-s'^2)^2-S^2(s^2+s'^2)]+8(2M-1)S^2. \quad (61)$$

We can now evaluate the second-order coefficient c_∞ of the eigenvalue by means of the mean-value theorem of perturbation theory. Multiplying Eq. (60) by $\phi(\mathbf{s})$, and integrating with respect to $d\mathbf{s}$, we find as usual that the term $U(\phi)$ gives only a contribution of the order: η^{-2}.[17] Therefore, we must have

$$4c_\infty\int\phi^2(s)d\mathbf{s}=(M/2)\int s^2\phi^2(s)d\mathbf{s}$$
$$-(\alpha/64\pi k_1M)\int\exp(-S^2/4)\phi(s)$$
$$\times\phi(s')fd\mathbf{s}d\mathbf{s}', \quad (62)$$

where ϕ is now the zero-order solution.

This solves in principle the problem of computing the second order correction, i.e., the coefficient c_∞. It is also possible to apply to this formula a Fourier transformation so as to express the integrals in terms of the function ϕ_σ that can be evaluated more simply than $\phi(s)$. Numerical results for various M values will be published by Mr. Marshall.

[17] This is a consequence of the fact that ϕ differs from the zero-order solution (of Eq. (53)) only by a term of the first order, and from the fact that the kernel of Eq. (53) is symmetric.

2.6 Justification of the Method of Steepest Descent

In order to fill the gaps that were left in the derivation of the Eqs. (11) and (12), we shall now first of all collect all the available information about the poles of the function ϕ_0, i.e., the roots of the equation

$$f(k,\eta)\equiv\phi_0^{-1}(k,\eta)=0. \quad (63)$$

The integration path in the k plane, Eq. (7), passes through the origin $k=0$; this is a convenient starting point for our investigation. It was shown in "A" that when $k\to0$ any root $\eta(k)$ of Eq. (63) must tend to a root of any one amongst the equations: $\gamma_0(\eta)=0$; $\gamma_1(\eta)=0$; $\cdots$. Now $\eta=0$ is a root of the first equation, since

$$\gamma_0(0)=1-g_0(0)=1-\alpha(1-e^{-q})=0. \quad (64)$$

This indicates the existence of a function $\eta(k)$ satisfying (63) identically, such that $\eta(0)=0$. In "A" Sec. 3.2 it was indicated that this function may be expanded into a power series

$$\eta=\eta_1k^2+\eta_2k^4+\cdots. \quad (65)$$

See also "M" Eq. (142b). The first coefficients are

$$\eta_1=(3\gamma_1\gamma_0')^{-1},$$
$$\eta_2=\eta_1^2\{(4\gamma_0'/5\gamma_2)-(\gamma_1'/\gamma_1)-(\gamma_0''/2\gamma_0')\}. \quad (66)$$

The values of the γ's and their derivatives appearing in the formulae refer to $\eta=0$ and are given in "M" Eqs. (85a, b, c, d) and (147), and Table X. Some typical values are given in Table II.

The reciprocal development of k^2 in powers of η (with different notations) has been used by Waller.[6]

Let us now consider briefly the other branches of the function $\eta(k)$ in the neighborhood of $k=0$. We first examine hydrogen. In this case $\eta=0$ is the only root of $\gamma_0=0$, since $\gamma_0=1-(\eta+1)^{-1}$. The other branches of $\eta(k)$ are therefore connected with the remaining equations, $\gamma_1=0$, etc. Now for instance $\gamma_1=1-(\eta+\frac{3}{2})^{-1}$, and the root is $\eta=-\frac{1}{2}$. Assuming $\eta(0)=-\frac{1}{2}$, we get the development

$$\eta(k)=-(1/2)-(1/15)k^2+\cdots. \quad (67)$$

A numerical investigation (see "A", Fig. 1), has shown that the two solutions (65) and (67) have a very simple connection. In fact, $\eta(k)$ has two branch-points on the imaginary axis: $k=\pm ik_b$, where $k_b^2\approx 0.4$. If we start from the origin along the imaginary axis with the branch (65) and make a loop around one of the branch points, we come back to the origin with the value (67). Both values, of course, are negative but (67) is smaller than (65) which is in fact the "main pole" of the function ϕ_0 in the η plane for these values of k. Next we ought to consider branches of $\eta(k)$ such that $\eta(0)$ is a root of $\gamma_2=0$, $\gamma_3=0$, etc. These branches lie even further to the left on the η-plane and the reader will be spared further details.

Let us now turn to the general case $M>1$. The picture is now slightly more complicated since already the equation $\gamma_0=0$ has several roots besides $\eta=0$. These lie, however, considerably to the left of the imaginary axis (see "M" Eq. (22)), and will not be investigated further. The equation $\gamma_1=0$ has no simple root such as $\eta=-\frac{1}{2}$ for $M=1$. It may be proved, however, that the roots have $\Re(\eta)<-\frac{1}{2}$; the same applies to the roots of $\gamma_2=0$, etc. (see Appendix B). The connection between the branches of $\eta(k)$ associated with these roots, and the "main" branch Eq. (65), could be investigated numerically.

It may be pointed out, however, that our main purpose in studying these secondary roots was only to show that they lie sufficiently far to the left of the main pole, to justify the approximation (8). Moreover, it is not really necessary for this to be true for all k values, but only for the points of a suitable integration path in the k-plane, or indeed for the portion of the integration path that gives a significant contribution to the integral; it does not matter at all if on the remaining portions the $\eta(k)$ poles are close together and the approximation (8) fails.

In order to make this point clear, let us examine what is perhaps the most critical case, namely $M=1$, $z=0$. Because $z=0$, the saddle-point, Eq. (12), is the origin $k=0$ and the path of steepest descent is the undeformed path of Eq. (7), i.e., the imaginary axis in the k plane. Because $M=1$, the distance $\delta\eta$ between the main pole and the next starts with a value $\frac{1}{2}$, and *decreases* with increasing $|k|$, and the approximation (8) becomes increasingly worse. On the other hand, the exponential factor is, considering Eq. (65) with $\eta_1=1$:

$$e^{u\eta(k)}\approx e^{-uy^2} \tag{68}$$

if $k=iy$. Since u is large, say $u\sim 10$, this factor becomes negligible *before* the branch point is reached, i.e., before the main pole and the one next to it come close together.[18] When now the k point moves beyond the branch-point, there will be two complex conjugate η values instead of two real ones. It is clear on physical grounds, however, that these values of the Fourier variable $k=iy$ can hardly contribute to the total intensity for small values of z, because they represent Fourier components of "wave-length" short compared to the average width $\sim u^{\frac{1}{2}}$ of the z distribution.[19]

Essentially the same considerations apply when z instead of *zero* is small, i.e., such that the expansion (65) is still rapidly convergent at the saddle-point k_0. From Eqs. (12) and (65) one finds

$$z/u=\eta'(k_0)=2\eta_1k_0+\cdots,$$

or

$$k_0\sim z/2u\eta_1.$$

The condition is thus

$$k_0\ll 1,\quad\text{or}\quad z\ll 2u\eta_1\sim u.$$

It will be noticed that the approximation (8) becomes better as z increases, because the distance $\delta\eta$ when k is at the saddle-point k_0 becomes larger as k_0 increases (compares Eqs. (65) and (67)). It may also be pointed out that the case $M=1$, for which numerical examples have been given before, is really the most unfavorable case; for heavier elements the approximation is not good but excellent,[20] since the distance $\delta\eta$ is considerably larger than $\frac{1}{2}$ even at $k=0$.

Let us now consider what happens when z becomes $\sim u$ or larger, so that the saddle-point k_0 approaches unity. In the first place, we must prove that the integration path can be pulled to

[18] Considering only the quadratic term in Eq. (65) involves, of course, an error. The true value of $\eta(k)$ at the branch point $k^2=-0.4$ is about 0.34. This makes $e^{u\eta}\propto e^{-3.4}=0.033$ instead of $e^{-4}=0.02$.

[19] From a mathematical standpoint one would have to show that the poles $\eta(k)$ with the largest real part move further to the left as y increases.

[20] We refrained from extending our computations to this case, because we understand that work on the heavier elements is in progress elsewhere.

the right in the k-plane so as to pass through the saddle-point; to this end we first choose the abscissa σ in Eq. (7) very large. We know that when $\Re(\eta)\rightarrow+\infty$, the function ϕ_0 tends to the expression (29) for all k points outside the cuts. This is a regular analytic function in the k plane, so that all the singularities $k(\eta)$ must move toward the cuts when $\Re(\eta)\rightarrow+\infty$. Thus, when σ is sufficiently large we may deform the integration path in the k plane so as to pass through any desired point k_0 on the real axis; a suitable path, for instance, is a straight line parallel to the imaginary axis: $k=k_0+iy$ with, say, $|y|\leq 1$. From $k=k_0\pm i$ we then move to infinity on straight lines at an angle $\vartheta<\pi/2$ with the real axis. We then show that only a small portion of the central vertical path is important. On this central portion we may evaluate the η-integral by means of Eq. (8); in fact, although Eqs. (65) and (67) cannot be used when $|k|\sim k_0$ is comparable to unity, the qualitative statement that the secondary pole (67) is considerably to the left of the main pole remains true. We then expand the exponent $u\eta(k)-zk$ in powers of $k-k_0=iy$, as stated before

$$u\eta-zk=u\eta_0-zk_0-\tfrac{1}{2}u\eta_0''y^2 -(i/6)u\eta_0'''y^3+\cdots. \quad (69)$$

An estimate of the various terms can be made remembering Eq. (48a). The derivatives of the slowly variable quantity c may be neglected, if we are interested merely in orders of magnitude. Thus

$$\eta_0'\sim k_1(1-k_0)^{-2};\quad \eta_0''\sim 2k_1(1-k_0)^{-3};\quad\cdots. \quad (70)$$

The positive sign of η_0'' shows that the exponential of the expression (69) has indeed a maximum at $y=0$. We show, furthermore, that the exponential may be represented by a Gaussian. The condition (12) for k_0 in connection with (70) shows that $1-k_0$ is of order unity when z is comparable with u, and when $z\gg u$

$$1-k_0\approx(uk_1/z)^{\frac{1}{2}}\ll 1. \quad (71)$$

Now from (70) we have:

$$\eta_0''\approx 2\eta_0'(1-k_0)^{-1}=2z/u(1-k_0);$$

hence the quadratic term in (69) is

$$-\tfrac{1}{2}u\eta_0''y^2=-z(1-k_0)^{-1}y^2=-(y/y_0)^2, \quad (72)$$

where $y_0=[(1-k_0)/z]^{\frac{1}{2}}\ll 1$ because $z\gtrsim u\gg 1$. If only the quadratic term is retained, the exponential becomes small when $y>y_0$. Now since $\eta_0'''\approx 3\eta_0''(1-k_0)^{-1}$, the cubic term is of the order of $y/(1-k_0)$ with respect to the quadratic term; for instance, if $y=y_0$ the cubic term is

$$y_0(1-k_0)^{-1}(y/y_0)^3=y_0(1-k_0)^{-1} =[z(1-k_0)]^{-\frac{1}{2}}\ll 1. \quad (73)$$

In fact, if $z\sim u$, $1-k_0\sim 1$ and the inequality holds true. If on the other hand $z\gg u$, we may use (71) so that

$$[z(1-k_0)]^{-\frac{1}{2}}=(k_1uz)^{-\frac{1}{4}}\ll 1. \quad (74)$$

The same argument applies *a fortiori* to the higher order terms, which are of the order $[y/(1-k_0)]^n$, $n\geq 2$, with respect to the quadratic term. This proves conclusively that the gaussian approximation applies to the vertical straight part of the path. As regards the inclined parts of the path, we know from previous considerations that $\eta(k)$ moves to the left as k moves toward infinity so that $e^{u\eta(k)}$ becomes even smaller. At the same time the factor e^{-zk} tends to zero. We have thus reached a sufficient justification for the steepest descent expression (11), even though a closer investigation of upper limits might be desirable for purposes of mathematical rigor.

A few remarks remain to be made. The path we have used is not strictly speaking the path of steepest descent; as usual, it was sufficient to use a path sufficiently close to the path of steepest descent to achieve the result. It may be seen from the formula (48a), neglecting the variation of c, that the path of steepest descent in the limit of large z, or k_0 close to unity, is a small circle of equation: $(1-k')^2+k''^2=uk_1/z=(1-k_0)^2$ if $k=k'+ik''$, i.e., a circle having the center at $k=1$ and passing through the saddle-point. The significant contributions to the integral come from a small angular fraction of this circle, of the order of y_0 divided by the radius, i.e., $y_0/(1-k_0)\ll 1$ according to (73). This is why it was immaterial to replace the circle by a straight line.[21]

[21] In this limiting case of z very large, one might also present the procedure as an asymptotic expansion with respect to z, i.e., perform the integration with respect to k first by pulling the integration path in the k plane and taking the residue at $k(\eta)$, this being given by Eq. (48a). The result is, however, the same.

Another remark of interest is that when z is very large, we need no longer request $u \gg 1$, but only the weaker condition

$$uz \gg 1. \tag{75}$$

In fact, this is seen to ensure the validity of (74) and therefore of the Gaussian approximation. At the same time it is no longer necessary for u to be large in order to enable us to retain only the contribution from the main pole in Eq. (8). In fact, $\eta(k)$ at the main pole is now large, indeed of the order

$$\eta(k_0) \sim k_1(1-k_0)^{-1} \sim (k_1 z/u)^{\frac{1}{2}} \tag{76}$$

while the secondary poles are negative. Therefore, the distance $\delta\eta$ between the main pole and the one next to it is at least of the order of (76), and the factor expressing the ratio of the residues or $e^{-u\delta\eta}$ has an exponent of the order

$$u\delta\eta \gtrsim u\eta(k_0) \sim (k_1 uz)^{\frac{1}{2}}.$$

The condition for the secondary poles to be negligible is then precisely the same condition (76), ensuring the validity of the method of steepest descent.

Summarizing the conditions under which Eq. (11), (12) are expected to hold, we find

$$u \gg 1, \quad z \text{ arbitrary} \tag{77}$$

or

$$z \gg 1, \quad uz \gg 1. \tag{78}$$

2.7 Special Limiting Cases and Numerical Results for Hydrogen

If $z \ll u$ the expression (11) takes a simple form, owing to the fact that the saddle-point k_0 is close to the origin and one may neglect all the higher powers of k in the series expansions of η and R in powers of k^2. For instance,

$$R^{-1} = \gamma_0'(0) + \cdots,$$

as one can see from Eqs. (9) and (20) neglecting terms $\sim k_0^2$. Following "M" Eq. (85a) we have therefore: $R^{-1} = \xi$. Similarly we set:

$$\eta(k) = \eta_1 k^2 + \cdots,$$

and the saddle-point k_0 is given by: $z/u = \eta'(k_0) = 2\eta_1 k_0$, etc. Finally one finds

$$\psi_0(z, u) = \xi^{-1}(4\pi\eta_1 u)^{-\frac{1}{2}} \exp(-z^2/4\eta_1 u) \tag{79}$$

which coincides with the "age" approximation, "M" Eq. (145); see also "M" Eq. (85c) and our Eq. (66).

Turning now to the other extreme case: $z \gg u$, we may use Eq. (71) for the saddle-point. This follows from Eq. (48a) when we neglect the variation of c with k. Correspondingly $\eta(k_0)$ is obtained from (48a) with $c = c_\infty$, so that, using Eq. (71):

$$\eta_0 = \eta(k_0) = (k_1 z/u)^{\frac{1}{2}} - c_\infty. \tag{80}$$

Computing η_0'' in a similar way, one has finally

$$\psi_0 \sim A(zu)^{-\frac{1}{4}} \exp(-z + 2(k_1 uz)^{\frac{1}{2}} - c_\infty u),$$
$$A = (4\pi)^{-\frac{1}{2}} k_1^{\frac{1}{4}} (R/\eta)_\infty. \tag{81}$$

The constants in this formula are determined by the Eqs. (47), (51), and (62). Apart from a slightly different notation and a minor slip, Eq. (81) corresponds to "M" Eq. (175).

The range of validity of the asymptotic formula (81) depends on the accuracy with which Eq. (48a) represents $\eta(k)$ when c is regarded as a constant c_∞. If the error is of the order of $1-k$. which corresponds to the error $\sim\eta^{-3}$ in Eq. (48b), then, as one easily sees, we must expect a correction term in the exponent of Eq. (81) of the order $u^{\frac{3}{2}} z^{-\frac{1}{2}}$, which would be negligible only if $z \gg u^3$. More generally we may regard the three terms in the exponent of Eq. (81) as the

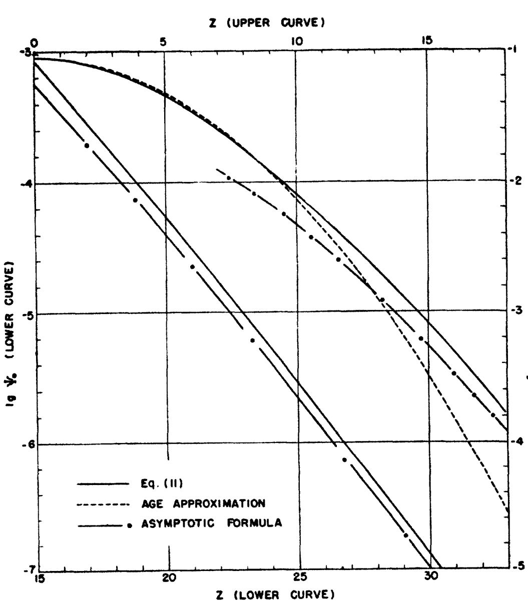

FIG. 3. The logarithm of the neutron density ψ_0 as a function of the distance z for $u = 10$ and $M = 1$.

first terms of an expansion in reciprocal powers of η_0, i.e., in powers of $(u/z)^{\frac{1}{2}}$. This will converge quite slowly, except when z is enormously large. There is not much point in computing further coefficients of the expansion.

In the particular case of hydrogen, however, the numerical data reveal a remarkable constancy of c, i.e., c approaches c_∞ sooner than one would expect in general, and as a result the asymptotic formula (81) is better than one might expect.[22]

The distribution function has been computed, as an example, for hydrogen for $u=10$, using the data of Table I. The result is plotted in Fig. 3, where the "age" approximation and the asymptotic formula are included also for comparison. The age approximation fails, as one might expect, when z becomes comparable to u, so that k_0 is no longer $\ll 1$.

3. THE CASE OF VARIABLE MEAN FREE PATH

In most practically important cases the mean free path cannot be considered constant. For many moderators, though not for all, it will decrease with decreasing velocity of the neutrons. At large distances from the source, an equilibrium between the primary and the slow neutrons is then readily established, so that the slow neutron density is determined to a large extent by the law for the primary neutrons, i.e., $r^{-2}e^{-r/l(0)}$ in the case of a point source. We shall see, however, that the asymptotic formula for the slow neutrons differs from the above-mentioned law in that it has a smaller (in general a fractional) power of the reciprocal distance before the exponential.

3.1 The Asymptotic Behavior of $\psi_0(z, u)$

Our problem is to find an asymptotic formula for the neutron density $\psi_0(z, u)$ when $z\rightarrow\infty$, while u is kept fixed at some value. It may be pointed out that the range of u values in which we are interested is finite, say: $0<u<\ln(E_0/kT)$, E_0 being the initial energy. We consider again the case of a plane primary source. Again ψ_0 may be expressed in terms of its Fourier-Laplace transform ϕ_0, Eq. (7), but ϕ_0 is given by a different set of equations.

We must, in fact, go back to "A" Eq. (2) or "M" Eq. (63) and take into account the dependence of l on the energy. Going through the same transformations as those leading to "A" Eq. (24) one finds

$$\phi(\mu)-k\mu\lambda(\mu)-\int g(\omega\cdot\omega', \eta)\phi(\mu')d\omega'=1/4\pi, \quad (82)$$

where λ is the Fourier-Laplace transform of $l(u)\psi$; the connection between λ and ϕ is thus: $\lambda=\mathcal{L}\{l(u)\mathcal{L}^{-1}\phi\}$ if $\mathcal{L}$ means Laplace transform. We have to obtain ϕ from Eq. (82), and then insert $\phi_0=\int\phi d\omega$ into Eq. (7).

In order to study the asymptotic behavior of ψ_0 we may try again to pull the integration path in the k-plane, see Fig. 1, toward the right until we are stopped by a singularity; the position and character of this singularity determines the asymptotic behavior. Now we shall see that the analysis of the singularity in the k-plane when $\Re(\eta)$ is large can be made on similar lines as in Section 2.4. In order to make use of this fact we first prove that:

(a) When $\Re(\eta)>\sigma$ (this being a suitable constant) the convergence abscissae in the k-plane are independent of η[23] and precisely the convergence strip is

$$-1<\Re(k)<+1$$

if we choose the mean free path of the primary neutrons $l(0)$ as the unit of length.

(b) The convergence strip is determined by the fixed singular points $k=\pm1$, and the character of the singularity when $k\rightarrow\pm1$ is independent of η as long as $\Re(\eta)>\sigma$.

For the sake of simplicity we shall prove these statements under the assumption

$$l(u)=l_1+l_2e^{-\epsilon u}, \quad (83)$$

when l_1, l_2, ϵ are positive constants. As we said, we set

$$l(0)\equiv l_1+l_2=1. \quad (84)$$

The proof can be easily generalized to a formula containing a sum of many exponentials.

With Eq. (83) one easily expresses the connection between λ and ϕ as follows:

$$\lambda(\mu, \eta)=l_1\phi(\mu, \eta)+l_2\phi(\mu, \eta+\epsilon). \quad (85)$$

[22] This statement depends rather critically on the value of k_1 chosen in computing c from Eq. (48a). The eigenvalue k_1 is known with limited accuracy, so that the constancy of c as exhibited in the last column of Table I may be fortuitious. This, of course, does not affect the usefulness of Eq. (48a) as an interpolation formula.

[23] In sharp contrast with the constant mean free path case.

Inserting into Eq. (82) we get

$$[1-kl_1\mu]\phi(\mu,\eta)-\int g(\omega\cdot\omega',\eta)\phi(\mu')d\omega' = kl_2\mu\phi(\mu,\eta+\epsilon)+1/4\pi. \quad (86)$$

Regarding for a moment the right-hand side of this equation as given, we notice that the left-hand side coincides with that of the integral equation "A" Eq. (24) except that k is substituted by kl_1. Hence (86) can be solved for $\phi(\mu,\eta)$ if kl_1 is different from any of the "eigenvalues" $k(\eta)$ that we studied before. Now let us choose σ in such a way that when $\Re(\eta)>\sigma$ the eigenvalue $k(\eta)$ with the smallest real part satisfies

$$l_1<\Re\{k(\eta)\}. \quad (87)$$

This is always possible since $k(\eta)\rightarrow 1>l_1$ when $\Re(\eta)\rightarrow+\infty$. Then clearly if: $-1<\Re(k)<+1$ (as we shall assume from now on), kl_1 in Eq. (86) *can not* be equal to an eigenvalue. This proves, then, that when $\Re(\eta)>\sigma$ one can use Eq. (86) to find $\phi(\mu,\eta)$ if $\phi(\mu,\eta+\epsilon)$ is known. If the latter is regular, $\phi(\mu,\eta)$ is also regular.

Now we know that when $\Re(\eta)\rightarrow+\infty$, $\phi(\mu,\eta)$ becomes the Laplace transform of the primary neutrons for which the convergence strip is $-1<\Re(k)<+1$. Thus, for a very large η, ϕ is regular within this strip. Then, according to the above statement, we can move in steps $\delta\eta=\epsilon$ toward smaller values of η and prove that ϕ remains regular, as long as $\Re(\eta)>\sigma$. On the other hand, the convergence strip of $\phi(\mu,\eta)$ cannot be wider than that for $\phi(\mu,+\infty)$,[24] which becomes singular when $k\rightarrow\pm 1$. The same is expected, therefore, of $\phi(\mu,\eta)$ as long as $\Re(\eta)>\sigma$. This completes the proof of statements (a) and (b).

We are now going to investigate the properties of the Fourier-Laplace transform when η is very large. We shall find that as $k\rightarrow 1$, ϕ behaves like $(1-k)^{\rho-2}$ where ρ is a constant <2. The above argument shows then that the behavior of ϕ is of the same type also for values of η that are not large, subject only to the condition $\Re(\eta)>\sigma$.

We can now drop the special assumption (83), since the discussion is equally simple in the general case, provided $l(u)$ is a decreasing function of u. Referring to the form (86) of the integral equation, we notice that the connection between ϕ and λ, in the case of large η, takes a simple form, owing to the fact that the main contribution to the Laplace integral comes from very small values of u. Therefore, omitting for simplicity the remaining variables,

$$\lambda(\eta)=\int_0^\infty e^{-\eta u}l(u)\psi(u)du = \int_0^\infty e^{-\eta u}[l(0)+ul'(0)+\cdots]\psi(u)du, \quad (88)$$

so that, remembering Eq. (84) and setting $l'(0)/l(0)=-\gamma$, we find

$$\lambda(\eta)=\phi(\eta)+\gamma\partial\phi/\partial\eta+\cdots. \quad (89)$$

We must expect the successive terms in this development to be of decreasing order of magnitude, for large η, as can also be verified *a posteriori* on the final result. This shows, at least, that the approximation is consistent.

Since in the case of large η we are dealing mainly with neutrons that have suffered only small energy losses, the results obtained for constant m.f.p. should offer some guidance. Therefore, we surmise that we shall have to study the function ϕ for small values of $\vartheta=\arccos\mu$, and set $\mu=1-\vartheta^2/2$, then introducing approximation (38) for the scattering function and the variables $s=(M/4\eta)^{-\frac{1}{2}}\vartheta$. With this change of variable the partial derivative in Eq. (89) suffers the transformation: $\partial/\partial\eta\rightarrow\partial/\partial\eta+(s/2\eta)\partial/\partial s$. Taking this into account and introducing Eq. (89) into the integral equation, we find

$$[(1-k)+(kMs^2/8\eta)]\phi - \gamma k\mu[\partial\phi/\partial\eta+(s/2\eta)\partial\phi/\partial s] - (\alpha/4\pi\eta)\int\exp(-S^2/4)\phi(s')ds'=1/4\pi. \quad (90)$$

In this equation we neglect terms of higher order than η^{-1}, and thus replace $\gamma k\mu$ with γk before the square bracket. Finally, after multiplying with ηk^{-1} and remembering the abbreviation introduced after Eq. (53), the equation becomes

$$h\eta\phi-\eta\partial\phi/\partial\eta-(s/2)\partial\phi/\partial s+(M/8\gamma)s^2\phi - (\alpha/k\gamma)e^{\Delta_s}\phi=\eta/4\pi k\gamma;\quad h=(1-k)/k\gamma. \quad (91)$$

[24] If it were, the distribution of the slow neutrons would decay faster than that of the primary ones with which they are in equilibrium, an obvious impossibility.

In order to solve this equation we consider the eigenvalue problem

$$-(s/2)dw/ds+(M/8\gamma)s^2w -(\alpha/k\gamma)e^{\Delta_s}w=(\rho-1)w \quad (92)$$

which is similar to Eq. (53) and can be treated in a similar manner; $\rho-1$ corresponds to k_1 in Eq. (53) and is the variable parameter. The eigenvalues of ρ form a discrete sequence ρ, ρ_1, ρ_2, $\cdots$ (see later). These eigenvalues depend on k, but when k is close to unity we may take them equal to their limiting values for $k=1$, i.e., set $\alpha/k\gamma=\alpha/\gamma$ in Eq. (92). We call ρ the lowest eigenvalue, which as we shall see is <2, when k is close to unity. Call w, w_1, w_2, $\cdots$ the corresponding eigenfunctions and develop ϕ:

$$\phi(\eta, s)=\sum_n a_n(\eta)w_n(s). \quad (93)$$

Inserting into Eq. (91) and using the orthogonality property of the eigenfunctions w_n:

$$\int w_n w_m d\mathbf{s}=\delta_{nm},$$

we find

$$(h\eta+\rho_n-1)a_n-\eta da_n/d\eta=(\eta/k\gamma)A_n, \quad (94)$$

where

$$4\pi A_n=\int w_n d\mathbf{s}. \quad (95)$$

The solution of Eq. (94) contains an integration constant that may be determined from the condition

$$a_n(+\infty)=A_n(1-k)^{-1}. \quad (96)$$

This follows from Eq. (86), remembering that $g\rightarrow 0$ when $\eta\rightarrow+\infty$. Therefore,

$$\phi(\mu, +\infty)=[4\pi(1-k\mu)]^{-1}. \quad (97)$$

Multiplying by $w_n(s)$ and integrating over $d\mathbf{s}$, and setting

$$\mu=1-\vartheta^2/2\approx 1-(M/8\eta)s^2\approx 1,$$

we get Eq. (96).

The solution of (94) is then

$$a_n(\eta)=(A_n/k\gamma)e^{h\eta}\eta^{\rho_n-1}\int_\eta^{+\infty} e^{-hx}x^{1-\rho_n}dx. \quad (98)$$

Examine now the behavior of this expression when $k\rightarrow 1$, t.i.: $h\rightarrow 0$, remembering that we are interested in the singularity of ϕ at this point. Now we can see that for an eigenvalue that satisfies $\rho>2$, the expression (98) remains finite when $h\rightarrow 0$, while if $\rho<2$ we can write

$$\int_\eta^{+\infty} e^{-hx}x^{1-\rho}dx=h^{\rho-2}\int_{h\eta}^{+\infty} e^{-y}y^{1-\rho}dy \approx(1-\rho)!h^{\rho-2}. \quad (99)$$

Hence the highest singularity will arise from the lowest eigenvalue. We consider, therefore, only the first term in the expansion (93) and drop henceforward the index n. Noticing that as $k\rightarrow 1$, $h\approx(1-k)/\gamma$, we have finally the behavior in this limit

$$\phi\sim A\gamma^{2-\rho}(1-\rho)!w(s)\eta^{\rho-1}e^{\eta(1-k)/\gamma}(1-k)^{\rho-2}. \quad (100)$$

For us the most important part of this formula is the last term or: $(1-k)^{\rho-2}$, that gives, as we have said, the singular behavior of ϕ also for finite values of η. This solves, therefore, the problem of the asymptotic behavior for large distances, for according to a well-known Tauberian theorem it is permissible, in computing the asymptotic behavior of the inverse transform, to consider all factors as constant, except the singular term of the type indicated, thus obtaining a dependence on z of the type (compare Eq. (7))

$$\cdots(1/2\pi i)\int_{-i\infty}^{+i\infty} e^{-kz}(1-k)^{\rho-2}dk =\text{const.}z^{1-\rho}e^{-z}. \quad (101)$$

For a *point* source, remembering "M" Eq. (62), we get the asymptotic dependence

$$\psi_0\sim r^{-\rho}e^{-r}, \quad (102)$$

r being the distance from the source measured in mean free paths of the primary neutrons. In order to use this law we have to solve the eigenvalue problem Eq. (92) with $k=1$. The procedure that was applied to Eq. (53) can be used here; namely, we perform a Fourier transformation (Eq. (54)) and notice that

$$\mathfrak{F}(sdw/ds)=-(\nabla_\sigma\cdot\boldsymbol{\sigma})\mathfrak{F}(w) =-(2+\sigma d/d\sigma)\mathfrak{F}(w). \quad (103)$$

Finally we set

$$U(\sigma)=e^{-\gamma\sigma^2/M}\mathfrak{F}(w) \quad (104)$$

and find

$$\{d^2/d\sigma^2+\sigma^{-1}d/d\sigma+(4\gamma/M)(2\rho-3)+2[(M+1)/M]^2e^{-\sigma^2}-(2\gamma/M)^2\sigma^2\}U(\sigma)=0 \quad (105)$$

that differs from Eq. (47) in having a different potential and a different meaning for the "energy" term. Clearly the presence of the quadratic term in the "potential" makes the eigenvalue spectrum discrete. When γ tends to zero, the lowest eigenvalue must tend to the discrete eigenvalue of problem (47), and the higher eigenvalues must crowd together, reproducing the continuous spectrum. If γ is sufficiently small the second eigenvalue must be close to the zero line, i.e., ρ must be close to $\frac{3}{2}$.

A very rough evaluation of the lowest eigenvalue has been made by the Ritz method, using the eigenfunction $e^{-\lambda\sigma^2/2}$ with λ the variable parameter. The minimum condition is

$$\alpha(1+\lambda)^{-2}+\gamma^2\lambda^{-2}/2M=M/8. \quad (106)$$

The γ term is often small, so that it is permissible to develop λ in powers of γ. As an example we find for carbon, $M=12$, the result: $\rho=1.5-0.42\gamma^{-1}+0.078_5\gamma+\cdots$. Assuming an energy dependence of the m.f.p.: $l(u)=2.75+2E$, E being the energy in Mev, the data in Table III have been obtained.

TABLE III. Variable mean free path; carbon.

Initial energy of neutrons in Mev:	0	0.5	1	2
$\gamma=0$		0.27	0.42	0.59
$\rho=-\infty$		−0.03	0.53	0.84

3.2 Remarks on the Energy Spectrum at Large Distances

Using the η-dependence of Eq. (100) and inverting the Laplace transformation, it would be easy to derive also the dependence of the neutron density on u, i.e., the spectrum. The formula, however, would hold only for very small u, or energies very near the primary one, being derived from an expression valid only for very large η.

A greater interest attaches the spectrum for large u, or in the slow neutron region. This can be derived (compare "A" Section 4.2) from the position of the first singularity that one meets as η moves to the left in the complex plane.

We must now drop the assumption $\Re(\eta)>\sigma$, that was made in Section 3.1. It is then easy to see that the singularity in the η plane must occur when kl_1 in Eq. (86) becomes equal to the eigenvalue $k(\eta)$ with the largest real part, i.e., the eigenvalue given by Eq. (65), or: $k(\eta)\approx(\eta/\eta_1)^{\frac{1}{2}}$, when η is small and by (48b) if η is large.

We write therefore

$$kl(\infty)=k(\eta)l(0) \quad (107)$$

introducing explicitly the value of the mean free paths for zero velocity: $l(\infty)$ and for the initial velocity: $l(0)$.

In order to find the spectrum, we now first keep k fixed and invert the Laplace transform (compare "A" (22a) and "A" (40)) finding an expression of the type

$$(1/2\pi i)\int_{\beta-i\infty}^{\beta+i\infty}\phi(\eta)e^{\eta u}d\eta=F(k)e^{\eta_k u}+\cdots, \quad (108)$$

where η_k is the solution of Eq. (107) and $F(k)$ is the residue of ϕ at η_k, and the terms omitted correspond to singularities further to the left in the complex η-plane. Unfortunately, it is not easy to determine $F(k)$ accurately, except by the kind of numerical work described in "A" Section 5. We know, however, that as $k\rightarrow1$, $F(k)$ behaves like $(1-k)^{\rho-2}$. For large r that is all we need, and we finally find that for large u the energy spectrum at large distances, i.e., the dependence of ψ_0 on u, is given by

$$\psi_0\sim e^{\eta_1 u} \quad (109)$$

where η_1 is the solution of

$$l_1\equiv l(\infty)=k(\eta)l(0). \quad (110)$$

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APPENDIX A

Eigenvalues $k(\eta)$ when η is real and >0. We first prove that in this case: $|g_n(\eta)|<1$ for all n. Equation (2) may be written

$$g(\mu,\eta)=(\alpha/\pi)G^{2\eta+1}dG/d\mu. \quad (111)$$

One has then from Eq. (4)

$$|g_n|<2\pi\int_{-1}^{+1}|g|d\mu=2\alpha\int G^{2\eta+1}dG$$

$$=\alpha(\eta+1)^{-1}\{1-e^{-q-q\eta}\}.$$

In the interval: $0<\eta<+\infty$ the maximum of the right hand side occurs at $\eta=0$ so that

$$|g_n|<\alpha(1-e^{-q})=1. \tag{112}$$

Multiplying now Eq. (41) by ϕ_n^* and summing over n we get

$$\sum_{n=0}^{\infty}(2n+1)\gamma_n|\phi_n|^2$$

$$=k\sum_{n=0}^{\infty}(n+1)(\phi_{n+1}\phi_n^*+\phi_n\phi_{n+1}^*). \tag{113}$$

Owing to (112): $\gamma_n\equiv 1-g_n>0$, therefore k is the ratio of two real forms one of which is essentially positive and must be real, as stated in the text.

Needless to say, this proof assumes that the sums in (113) are convergent; according to Eq. (17) this will always be the case unless $|\epsilon|=1$, i.e., k real and $>+1$ or <-1.

It may be pointed out, however, that the latter exceptional values of k may also be regarded as eigenvalues, and more precisely as forming the continuous spectrum of eigenvalues of Eq. (41). Without going into details, we may point out that the question is entirely analogous to that about the continuous eigenvalues of a differential equation such as, for instance, Eq. (47). As is well known all positive values of the "energy" parameter $-8k_1/M$ are regarded as eigenvalues, although the corresponding eigensolutions do not satisfy the ordinary condition of quadratic integrability, but only a weaker condition. Entirely similar considerations apply here; and, needless to say, in the limit in which our system may be approximated by Eq. (47) the continuous eigenvalues of the latter equation, i.e., those with $k_1<0$, go over, according to Eq. (48b), into the continuum of real values of k with $k>+1$.

APPENDIX B

Roots of $\gamma_1=0$, $\gamma_2=0$, *etc.* It will suffice to prove that $|g_n|<1$ for $n=1, 2, \cdots$ when $\Re(\eta)\geq -\frac{1}{2}$. From Eq. (111) we have, remembering that $G\leq 1$,

$$|g|\leq(\alpha/\pi)dG/d\mu.$$

Therefore, Eq. (4) yields

$$|g_n(\eta)|<2\alpha\int_{-1}^{+1}|P_n(\mu)|(dG/d\mu)d\mu. \tag{114}$$

For $n=1$ the integral is quite simple; one finds

$$|g_1|<(M+1)/2M\leq 1 \tag{115}$$

as we want. For $n>1$ we consider $M=1$ first. It is easily found that the roots of $\gamma_2=0$ are $-1\pm i2^{-\frac{1}{2}}$, those of $\gamma_3=0$ are $-(3/2)\pm i(3/2)^{\frac{1}{2}}$, etc., i.e., all these roots have $\Re(\eta)<-\frac{1}{2}$.

If $M>1$ we apply Schwartz's inequality to Eq. (114) and find

$$|g_n|^2<(2\alpha)^2\int P_n{}^2d\mu\int(dG/d\mu)^2d\mu.$$

Now

$$\int(dG/d\mu)^2d\mu=\int(dG/d\mu)dG<(M^2-1)^{-\frac{1}{2}}\int GdG$$

$$=(2\alpha)^{-1}(M^2-1)^{-\frac{1}{2}}$$

by a simple majoration. Finally

$$|g_n|^2<2\alpha(M^2-1)^{-\frac{1}{2}}2/(2n+1) \tag{116}$$

which is <1 for $M=2, 3, \cdots, n=2, 3, \cdots$.