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Non-Linear Field Theories

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This is the first paper in a program concerned with the quantization of field theories which are covariant with respect to general coordinate transformations, like the general theory of relativity. All these theories share the property that the existence and form of the equations of motion is a direct consequence of the covariant character of the equations. It is hoped that in the quantization of theories of this type some of the divergences which are ordinarily encountered in quantum field theories can be avoided. The present paper lays the classical foundation for this program: It examines the formal properties of covariant field equations, derives the form of the conservation laws, the form of the equations of motion, and the properties of the canonical momentum components which can be introduced.

I. INTRODUCTION

At the present time, two great theoretical structures in physics can lay claim to containing significant parts of the "truth" which to unearth must remain the principal aim of both the experimental and the theoretical physicist. One of these structures is modern quantum physics as applied to both mechanical and field theoretical problems; the other is the general theory of relativity, which in the author's opinion represents the least imperfect "classical" (i.e., non-quantized) field theory. It is well known that at this time each one of these structures suffers from serious and apparently inherent weaknesses; Quantum physics in most of its realizations still requires a two-stage development, an underlying classical theory (a mechanical system or a set of field equations together with equations of motion) followed by a process of quantization. The result of this quantization process, when applied to a combination of fields and particles, usually leads to characteristic divergences. The general theory of relativity, on the other hand, has so far not successfully absorbed the existence of quantities possessing half-odd spin, nor can it be quantized in a satisfactory manner; as a result, the theory has been completely useless in atomic and nuclear physics.

Neither theory has as yet made any significant contribution to the problem of the constitution of the elementary particles. Moreover, while each structure undeniably contains elements of truth, their combination has so far proved unsuccessful.

The purpose of the present program is to analyze each of the two theories for its essential and, presumably, relatively permanent contributions to our present knowledge and, thus, to construct what might be called skeletonized theories. An attempt will be made to see whether such a covariant theory is at all susceptible to quantization and whether the result will be an improved theory.

Specifically, it is believed that the theory of relativity contains two great permanent achievements: (a) it is the only theory of gravitation which explains reasonably the equality of inertial and gravitational mass (the so-called principle of equivalence); (b) it is the only classical field theory in which the equations of motion of particles in the field are contained in the field equations, instead of being logical juxtapositions. That is why it is possible, in the general theory of relativity, to treat the motion of field singularities (which are used to represent particles) without having to deal with infinite interaction terms of one kind or another. It is possible that this accomplishment will also lead to a more satisfactory quantized theory, although the author has little hope that such a theory would be sufficiently powerful to attack the problem of the constitution of elementary particles.

On the other hand, there is probably no particular reason why the theory of relativity must appear in the form of Riemannian geometry, i.e., using primarily the differential covariants of a symmetric tensor of rank 2. The first step in the program sketched above consists therefore, in the setting up of a generalized classical theory and the examination of its properties. This is the scope of the present paper.

The theory to be developed is essentially non-linear, and its Hamiltonian non-quadratic. It is impossible to envisage a set of differential equations which is covariant with respect to general coordinate transformations, yet linear. While this might appear to be merely an annoying complication, the non-linear character of the general theory of relativity is crucial for the possibility of interaction between different particles. It is possible to set up linear, Lorentz-covariant field equations for a field similar to gravitational potentials, which contain equations of motion; but these turn out to represent simply Newton's first law: Each singularity is tied inexorably to a straight-line uniform motion, without any interaction between pairs of particles. It is the non-linear terms in the field equations which provide for forces, gravitational and otherwise.[1–3]

II. THE FIELD VARIABLES AND THE LAGRANGIAN

Instead of introducing a metric tensor, as is done in the general theory of relativity, we shall leave the exact nature of the field variables unspecified, denoting them merely by the symbol y_A ($A=1, \cdots, N$), where N is the number of algebraically independent components. It will be assumed that the field equations can be derived from a variational principle of the form

$$\delta I=0, \quad I=\int_V L(y_A, y_{A,\mu})d^4x. \qquad (2.1)$$

In other words, the field equations will determine values of the integral I which are stationary with respect to infinitesimal changes in the field variables; these changes are arbitrary, except that they must remain confined to the interior of the chosen four-dimensional volume V. The Lagrangian L is assumed to be an algebraic function of the field variables themselves and their first partial derivatives with respect to the coordinates. As is customary, $y_{A,\mu}$ is short for $\partial y_A/\partial x^\mu$.

The field equations which result from the variation of the field variables in the interior shall be designated by L^A,

$$L^A \equiv (\partial L/\partial y_A)-(\partial L/\partial y_{A,\rho})_{,\rho}=0. \qquad (2.2)$$

More precisely, the symbol L^A will be used for the left-hand side of these equations, no matter whether they are satisfied or not.

In this paper it will be assumed that the transformation law of the field variables is linear and homogeneous in the field variables themselves and that it depends algebraically on the first derivatives of the new coordinates with respect to the old ones. With respect to *infinitesimal* coordinate transformations, the field variables will transform according to a law having the form

$$\bar{\delta} y_A = F_{A\mu}{}^{B\nu}\xi^\mu{}_{,\nu} y_B - y_{A,\mu}\xi^\mu. \qquad (2.3)$$

The four functions ξ^μ represent the infinitesimal changes of the coordinate values of a fixed world point. The $F_{A\mu}{}^{B\nu}$ are numbers, independent of both the choice of coordinate system and the coordinate values themselves, but characteristic for the type of field variables representing the field. Finally, the $\bar{\delta}y_A$ are the changes produced in the field variables y_A *as functions of their arguments* because of the infinitesimal coordinate transformations. Because the transformed y_A are not compared with the original values at the same world point, but with the original values at that world point which possesses the same coordinate values prior to the transformation, Eq. (2.3) contains a "transport" term, the second term on the right-hand side. Incidentally, in this equation, as well as throughout the paper, the summation convention is being applied both to the Greek indices which are associated with the coordinates and run from 1 to 4 and to the capital indices which run from 1 to N.

The transformation law (2.3) is, of course, not the most general law which may be encountered in geometrical objects, but it does include all types of tensors and tensor densities and also spinors. Being an infinitesimal transformation law, it is subject to the requirement that the commutator should again be an operator of the same type. This condition is represented by the

[1] Einstein, Infeld, and Hoffman, Annals of Mathematics **39**, 65 (1938).
[2] A. Einstein and L. Infeld, Annals of Mathematics **41**, 455 (1940).
[3] L. Infeld and P. R. Wallace, Phys. Rev. **57**, 797 (1940).

identity

$$F_{A\mu}{}^{C\nu}F_{C\rho}{}^{B\sigma}-F_{A\rho}{}^{C\sigma}F_{C\mu}{}^{B\nu}\equiv\delta_\rho{}^\nu F_{A\mu}{}^{B\sigma}-\delta_\mu{}^\sigma F_{A\rho}{}^{B\nu}. \quad (2.4)$$

The field equations (2.2) must be covariant, i.e., if they are satisfied in one coordinate system, they must also be satisfied automatically in any other coordinate system. They would certainly be covariant if the integral I were an invariant, for in that case its extremization would also be an invariant operation. However, this condition is too strong, as it cannot even be satisfied in the general theory of relativity. There it is possible to introduce either a Lagrangian of which the integral is an invariant but which contains explicitly second derivatives of the field variables, or one which contains only first derivatives but which has no invariant integral. If the Lagrangian, in the face of an infinitesimal coordinate transformation, adds a divergence, then that condition is sufficient (though possibly not necessary) to assure covariant field equations. We shall, therefore, require that

$$\bar{\delta}L=Q^\mu{}_{,\mu}, \quad (2.5)$$

where the four expressions Q^μ are some functions of the ξ^μ and their derivatives (including those of higher order). In other words, it will be assumed that the infinitesimal change in the integral I can be expressed by means of a surface integral

$$\bar{\delta}I=\oint_S Q^\mu n_\mu dS, \quad (2.6)$$

without being affected by the values of the ξ^μ in the interior. The four quantities n_μ represent the components of a "unit normal vector" which is introduced so that Gauss' theorem can be formulated. This is possible even without the introduction of a metric.

By considering the transformation of a variation of the Lagrangian, $\bar{\delta}(\delta L)$, it can easily be shown that the field equations, because of Eq. (2.5) or its equivalent (2.6), satisfy the transformation law

$$\bar{\delta}L^B=-F_{A\mu}{}^{B\nu}\xi^\mu{}_{,\nu}L^A-(L^B\xi^\mu)_{,\mu}, \quad (2.7)$$

where the constants $F_{A\mu}{}^{B\nu}$ are identical with those introduced in Eq. (2.3).

III. IDENTITIES

Consider again the transformation law (2.5), which specifies that in the event of an infinitesimal coordinate transformation the change in the Lagrangian is a divergence. That change can also be represented in the form of a *variation* induced by the infinitesimal transformation:

$$\bar{\delta}L=L^A\bar{\delta}y_A+((\partial L/\partial y_{A,\rho})\bar{\delta}y_A)_{,\rho}. \quad (3.1)$$

Substitution of Eqs. (2.3) and (2.5) results in

$$(Q^\rho-(\partial L/\partial y_{A,\rho})\bar{\delta}y_A)_{,\rho}=L^A(F_{A\mu}{}^{B\nu}\xi^\mu{}_{,\nu}y_B-y_{A,\mu}\xi^\mu). \quad (3.2)$$

The right-hand side will be a divergence only if the left-hand sides of the field equations satisfy the four identities

$$(F_{A\mu}{}^{B\nu}y_BL^A)_{,\nu}+y_{A,\mu}L^A\equiv 0. \quad (3.3)$$

In the general theory of relativity these identities are known as the contracted Bianchi identities. They hold no matter whether the field equations are satisfied or not.

The expressions L^A contain the field variables themselves and also their first and second derivatives. The second derivatives occur only linearly, and their coefficients can be represented in the form

$$\begin{aligned}L^A&=L^{AB\rho\sigma}y_{B,\rho\sigma}+\cdots,\\ L^{AB\rho\sigma}&=-\tfrac{1}{2}[(\partial^2L/\partial y_{A,\rho}\partial y_{B,\sigma})+(\partial^2L/\partial y_{A,\sigma}\partial y_{B,\rho})].\end{aligned} \quad (3.4)$$

When the L^A are substituted into the identities (3.3), the terms containing second derivatives will, in turn, lead to terms containing third derivatives, and those must cancel each other, irrespective of other terms,

$$F_{A\mu}{}^{B\rho}y_BL^{AC\sigma\tau}y_{C,\rho\sigma\tau}\equiv 0. \quad (3.5)$$

It follows that the coefficients $L^{AB\sigma\tau}$ must satisfy the identities

$$(F_{A\mu}{}^{B\rho}L^{AC\sigma\tau}+F_{A\mu}{}^{B\sigma}L^{AC\tau\rho}+F_{A\mu}{}^{B\tau}L^{AC\rho\sigma})y_B\equiv 0. \quad (3.6)$$

For what follows, $4N$ of these identities are of special interest, those in which ρ, σ, and τ all equal 4,

$$F_{A\mu}{}^{B4}\Lambda^{AC}y_B \equiv 0,$$
$$\Lambda^{AC} = L^{AC44} = -\partial^2 L/\partial y_{A,4}\partial y_{C,4}. \tag{3.7}$$

Before concluding this section, it will be well also to formulate the conservation laws which are satisfied in a covariant theory. It is well known that in the presence of a Lagrangian, and provided the field equations are satisfied, there exist 16 quantities $t_\iota{}^\kappa$, functions of the field variables and their first derivatives only, which satisfy four divergence relationships:

$$t_\iota{}^\rho{}_{,\rho} = 0,$$
$$t_\iota{}^\kappa = \delta_\iota{}^\kappa L - y_{A,\iota}(\delta L/\delta y_{A,\kappa}). \tag{3.8}$$

If now the presence of matter in the field is represented by continuous and differentiable right-hand sides of the field equations,

$$L^A = P^A, \tag{3.9}$$

then the right-hand side of Eq. (3.8) no longer vanishes, but one obtains instead

$$t_\iota{}^\rho{}_{,\rho} = y_{A,\iota}P^A. \tag{3.10}$$

Because of the existence of the differential identities (3.3), this right-hand side can also be given the form of a divergence. Naturally, the field equations (3.9) can be satisfied only if the right-hand sides satisfy the same relationships which are satisfied identically by the left-hand sides. Therefore, Eq. (3.10) may be written in the form

$$T_{\iota,\rho}{}^\rho = 0,$$
$$T_\iota{}^\kappa = \delta_\iota{}^\kappa L - y_{A,\iota}(\partial L/\partial y_{A,\kappa}) + F_{A\iota}{}^{B\kappa}P^A y_B. \tag{3.11}$$

Only in this "strong" form are the conservation laws useful in the consideration of situations in which matter is represented by continuous expressions P^A, or else by discrete singularities of the field variables.

IV. EQUATIONS OF MOTION

The equations of motion are obtained by the method initiated by Einstein and collaborators.[1,2] Because of the identities derived in Section III, any singularities present in the field are subject to certain restrictions which represent both the conservation of mass (or its equivalent) and the equations of motion. Just as in those papers, it is impossible to formulate these conditions precisely without taking recourse to an *approximation method*. This is because the motions of the singularities cannot be completely determined unless such effects as spontaneous polarization and spontaneous emission of radiation by the singularities are specifically excluded. This exclusion is accomplished by assuming that all motions are "slow" in the sense that differentiation of a field variable with respect to x^4 (the time) reduces the order of magnitude at every stage of the approximation method.

It shall be assumed that solutions of the field equations are to be obtained in the form of a power series expansion with respect to some parameter ϵ, which might, for instance, represent the order of magnitude of the material velocities involved, (v/c):

$$y_A = \overset{0}{y}_A + \epsilon \overset{1}{y}_A + \epsilon^2 \overset{2}{y}_A + \cdots. \tag{4.1}$$

Moreover, the zeroth approximation shall be the "trivial" solution, a rigorous solution of the field equations in which all field variables are constants. (In the theory of relativity, this trivial solution is represented by the flat Minkowski metric.) The first approximation must then satisfy the following linear, homogeneous equations:

$$\overset{0}{L}{}^{ABrs}\overset{1}{y}_{B,rs} = 0. \tag{4.2}$$

In this approximation, there appear no time derivatives, because of the assumption that these are of a higher order of magnitude. The $\overset{0}{L}{}^{ABrs}$ are the coefficients (3.4), and the indices r and s are coordinate indices running from 1 to 3, in accordance with the usual notation in the literature. Because of the condition of "slow motion," this approximation will not contain radiation, but rather solutions corresponding to material particles, with as yet undetermined motions.

Generally, these solutions will not be defined throughout space. There will be singularities, probably of the $(1/r)$ type; more precisely, at each instant of time (x^4) there will be certain three-dimensional domains in which the field equations have no bounded solutions. However, only such solutions will be considered in which each one of these singular regions can be surrounded by a closed surface S (in three-dimensional space, S is two dimensional) on which the field equations are satisfied. It is this condition of separateness of the singular regions which makes possible the formulation of additional requirements on the solutions outside.

If we proceed to the next approximation, the equations will have the form

$$\overset{0}{L}{}^{ABrs}\overset{2}{y}_{B,rs}=-\overset{2}{L}{}^{A}(\overset{0}{y}+\epsilon\overset{1}{y}). \tag{4.3}$$

The right-hand sides will be the L^A of the second order, formed from the first-order solutions (the $\overset{1}{L}{}^A$ vanish, of course). They will contain terms which are linear in the first and second time derivatives of the first approximation, and quadratic in the purely spatial derivatives of the first approximation.

Because of Eqs. (3.6), the left-hand sides of the second-order equations (4.3) satisfy identically the four relationships

$$F_{A\mu}{}^{Ct}\overset{0}{y}_C(\overset{0}{L}{}^{ABrs}\overset{2}{y}_{B,rs})_{,t}\equiv 0, \tag{4.4}$$

irrespective of the choice of the second-order field variables. In particular, it is possible to satisfy the field equations outside the singular regions and to continue the second-order variables throughout the interior of the singular regions with arbitrary continuous and three times differentiable functions. Then the conditions (4.4) will be satisfied throughout (three-dimensional) space and will permit the application of Gauss' theorem to a three-dimensional domain which includes a singular region but which is bounded by a closed surface on which the field equations are satisfied. On that surface, on which Eqs. (4.3) are to be satisfied, we have then

$$\begin{aligned}0&\equiv\oint_S F_{A\mu}{}^{Ct}\overset{0}{y}_C\overset{0}{L}{}^{ABrs}\overset{2}{y}_{B,rs}n_t dS\\&=-\oint_S F_{A\mu}{}^{Ct}\overset{0}{y}_C\overset{2}{L}{}^{A}(\overset{0}{y}+\epsilon\overset{1}{y})n_t dS,\end{aligned} \tag{4.5}$$

which represent for that singularity inside S the three equations of motion and the law of the conservation of mass (μ takes all four values $1\cdots 4$). It can be shown easily that these conditions for the first approximation are empty unless S encloses a singular region. The number of conditions which can be obtained equals, therefore, four times the number of separate singular regions.

V. UNIQUENESS OF SOLUTIONS, MOMENTA

In this section it will be shown first that if the field variables and their first derivatives are given on a three-dimensional hypersurface in space-time (e.g., throughout three-space at a specified time), the continuation of the solution beyond that initial hypersurface is not unique. As a corollary, it will then be shown that if canonical momenta are introduced in that formalism, not all the time derivatives of the field variables can be expressed in terms of these momenta. Both of these results can be predicted qualitatively from the covariance of the field equations. Suppose that one solution were known which satisfies the initial conditions on the hypersurface. Then one could always carry out a coordinate transformation such that the new coordinates coincide on the surface with the old ones, up to second derivatives, but not elsewhere. Then the solutions would be transformed into *formally* different solutions satisfying the same initial conditions. *Physically*, one would be inclined to call such equivalent solutions "the same solution;" in that sense, the solutions are presumably uniquely determined by the initial conditions.

It was pointed out previously that the field equations are linear with respect to the second derivatives of the field variables. If we choose as the hypersurface one with x^4 constant, then the y_A as well as the $y_{A,4}$ are given on that hypersurface. The continuation would be unique if the field equations could be solved with respect to the terms containing $y_{A,44}$. This, however, is not possible. Consider the coefficients of these second time derivatives. They are the quantities L^{AB44}. If the matrix

$$\|\Lambda^{AB}\|=\|L^{AB44}\| \tag{5.1}$$

were regular, then this solution could be accomplished. Actually, there are four different sets of N quantities which are zero eigenvectors of that matrix, as shown in Eq. (3.7). Conversely, there exist four linear combinations of field equations which are free of second time derivatives and which, therefore, represent restrictions on the choice of initial conditions on the hypersurface. They are

$$y_B F_{A\mu}{}^{B4}L^A=0. \tag{5.2}$$

It follows that four linear combinations of the second time derivatives remain completely arbitrary on each space-like hypersurface. Naturally, the singling out of some particular coordinate as the "time" is completely arbitrary, and the term "time" was used only to conform to physical intuition.

The ambiguous character of the continuation of the solution with given initial conditions leads to pecularities of the canonical momenta which are analogous to those encountered in quantum electrodynamics. As is customary, the derivatives of L with respect to $y_{A,4}$ are designated as the momenta,

$$\partial L/\partial y_{A,4}=\pi^A. \tag{5.3}$$

These N equations cannot be solved with respect to the $y_{A,4}$, but, on the contrary, there exist four relationships between the π^A and y_A, $y_{A,s}$. First, Eqs. (5.3) can be solved with respect to $y_{A,4}$ only if the determinant of the partial derivatives

$$(\partial\pi^A(y_B, y_{B,s}, y_{B,4})/\partial y_{C,4}) = (\partial^2 L/\partial y_{A,4}\partial y_{C,4}) = -\Lambda^{AC} \tag{5.4}$$

does not vanish. But Λ^{AC} is a singular matrix, and its determinant is zero. It follows that an attempt to solve with respect to the $y_{A,4}$ will result in the establishment of four relationships which do not contain any (first-order) time derivatives. In view of the $4N$ identities

$$y_B F_{A\mu}{}^{B4}(\partial\pi^A/\partial y_{C,4})\equiv 0, \tag{5.5}$$

these four relationships can be obtained by straightforward integration. They are

$$y_B F_{A\mu}{}^{B4}\pi^A - K_\mu(y_C, y_{C,s})\equiv 0, \tag{5.6}$$

where the K_μ are functions introduced by the integration, but actually determined in any theory.

VII. CONCLUSION

The relationships set up in the last Section will give rise to the usual difficulties in quantization, since the N momenta are not algebraically independent of each other. Four of the Hamiltonian equations will turn out to be empty, as a result. When the field variables y_A and π^A are reinterpreted as operators, it will not be possible to interpret Eqs. (5.6) as linear relationships satisfied by the operators π^A; such an assumption would be incompatible with the commutation relations. Rather, they will have to be interpreted as initial conditions which are imposed on the state vector and which are preserved automatically in the course of time. At each instant, the infinitesimal contact transformation leading from the state at t to the state at $(t+dt)$ will contain four arbitrary functions of the spatial coordinates. These arbitrary functions are usually eliminated by the setting up of so-called coordinate conditions, analogous to the gauge condition of electrodynamics; but it is also possible to retain this arbitrariness in the formalism and set up a quantized theory in which the Hamiltonian is determined only up to four arbitrary functions. It is proposed to examine these problems in a future paper.