This is the term which gives rise to the main part of the Lamb-Retherford line shift,[10] the anomalous magnetic moment of the electron,[11] and the anomalous hyperfine splitting of the ground state of hydrogen.[12]

The above expression L is formally simpler than the corresponding expression obtained by Schwinger, but the two are easily seen to be equivalent. In particular, the above expression does not lead to any great reduction in the labor involved in a numerical calculation of the Lamb shift. Its advantage lies rather in the ease with which it can be written down.

In conclusion, the author would like to express his thanks to the Commonwealth Fund of New York for financial support, and to Professors Schwinger and Feynman for the stimulating lectures in which they presented their respective theories.

Notes added in proof (To Section II). The argument of Section II is an over-simplification of the method of Tomonaga,[1] and is unsound. There is an error in the derivation of (3); derivatives occurring in $H(r)$ give rise to noncommutativity between $H(r)$ and field quantities at r' when r is a point on σ infinitesimally distant from r'. The argument should be amended as follows. Φ is defined only for flat surfaces $t(r)=t$, and for such surfaces (3) and (6) are correct. Ψ is defined for general surfaces by (12) and (10), and is verified to satisfy (9). For a flat surface, Φ and Ψ are then shown to be related by (7). Finally, since H_1 does not involve the derivatives in H, the argument leading to (3) can be correctly applied to prove that for general σ the state-vector $\Psi(\sigma)$ will completely describe results of observations of the system on σ.

(To Section III). A covariant perturbation theory similar to that of Section III has previously been developed by E. C G. Stueckelberg, Ann. d. Phys. **21**, 367 (1934); Nature, **153**, 143 (1944).

(To Section V). Schwinger's "effective potential" is not H_T given by (25), but is $H_T'=QH_TQ^{-1}$. Here Q is a "square-root" of $S(\infty)$ obtained by expanding $(S(\infty))^{\frac{1}{2}}$ by the binomial theorem. The physical meaning of this is that Schwinger specifies states neither by Ω nor by Ω', but by an intermediate state-vector $\Omega''=Q\Omega=Q^{-1}\Omega'$, whose definition is symmetrical between past and future. H_T' is also symmetrical between past and future. For one-particle states, H_T and H_T' are identical.

Equation (32) can most simply be obtained directly from the product expansion of $S(\infty)$.

(To Section VII). Equation (62) is incorrect. The function S_F' is well-behaved, but its fourier transform has a logarithmic dependence on frequency, which makes an expansion precisely of the form (62) impossible.

(To Section X). The term L still contains two divergent parts. One is an "infra-red catastrophe" removable by standard methods. The other is an "ultraviolet" divergence, and has to be interpreted as an additional charge-renormalization, or, better, cancelled by part of the charge-renormalization calculated in Section VIII.

[10] W. E. Lamb and R. C. Retherford, Phys. Rev. **72**, 241 (1947).
[11] P. Kusch and H. M. Foley, Phys. Rev. **74**, 250 (1948).
[12] J. E. Nafe and E. B. Nelson, Phys. Rev. **73**, 718 (1948); Aage Bohr, Phys. Rev. **73**, 1109 (1948).

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Note on a Theorem of Bloch Concerning Possible Causes of Superconductivity

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Attention is called to a theorem of Bloch, from which it is shown that even when inter-electronic interactions are taken into account, the state of lowest electronic free energy corresponds to a zero net current. This result contradicts the hypothesis that superconductivity is caused by spontaneous currents.

MANY attempts[1,2] have been made to explain superconductivity in terms of spontaneous currents, which arise because there is a special group of states of the electron gas as a whole, for which the free energy, $F=E-TS$, is lower when a finite current flows than when no current flows at all. In some of the theories, it is suggested that the current-carrying states in question may have energies which are below that of the state of zero current, while in others, it is suggested that the current-carrying states may have so high a statistical weight that their free energy is

[1] W. Heisenberg, Zeits. f. Naturforschg **32**, 65 (1948).
[2] M. Born and K. C. Cheng, Nature **161**, 1017 (1948).

a minimum, even though the energy itself may not be a minimum. It is generally realized that the Bloch one-particle wave functions always make the state of lowest free energy one of zero current, but it is hoped that the inter-electronic interactions, not taken into account completely by these wave-functions, may change this result. Bloch has, however, proved[3,4] a theorem from which one can deduce that even when these interactions are taken into account, the state of lowest free energy still corresponds to zero net current. Since this result does not appear to be widely known, and since it is not treated in much detail in the existing literature, a brief note on the subject is perhaps desirable here.

Let the Hamiltonian be denoted by

$$\mathcal{H}=\sum_n [V(\mathbf{X}_n)-\hbar^2\nabla_n^2/2m]+\tfrac{1}{2}\sum_{m\neq n}V(\mathbf{X}_{mn}).$$

$V(\mathbf{X}_n)$ represents the potential of the nth electron in the field of the ion lattice, while $V(\mathbf{X}_{mn})$ represents the potential energy of inter-electronic interaction (coulomb).

Let us observe that the total current, $\mathbf{j}$, is related to the total electronic momentum, $\mathbf{P}$, by $\mathbf{j}=e\mathbf{P}/m$.

In the first part of the proof, we assume temporarily that the lowest state carries a current, in order to show that one is thereby led to a contradiction. Suppose that the exact wave-function is $\psi(\mathbf{X}_1, \mathbf{X}_2, \cdots \mathbf{X}_n \cdots)$. Let us consider a slightly different wave function,

$$\phi=\exp[(i\delta\mathbf{P}/\hbar\sum_n \mathbf{X}_n)]\psi$$

where $\delta\mathbf{P}$ is very small. This corresponds to a state in which each electron has been given an additional momentum, $\delta\mathbf{P}$. If $\mathbf{P}_0$ is the total electronic momentum in the state, ψ, then their total momentum in the state, ϕ, is

$$\mathbf{P}=\mathbf{P}_0+N\delta\mathbf{P},$$

where N is the total number of electrons. (Note that if ψ is totally anti-symmetric in space and spin coordinates of all the particles, then ϕ must possess the same antisymmetry, because the exponential factor is a totally symmetric function of the particle coordinates.)

[3] Appendix to L. Brillouin, J. de phys. et rad. 4, 334 (1933).
[4] H. G. Smith and J. O. Wilhelm, Rev. Mod. Phys. 7, 266 (1935).

Let us now compute the mean electronic energy for the wave function, ϕ. It is clear that the potential energy is exactly the same as when the wave function is ψ. If T_0 represents the mean kinetic energy in the state, ψ, one obtains, however,

$$T=-\hbar^2/2m\sum_n\left(\int\phi^*\nabla_n^2\phi dX_1dX_2\cdots dX_n\cdots\right)$$

$$=T_0+\frac{N(\mathbf{P}_0\cdot\delta\mathbf{P})}{m}+\frac{N(\delta P)^2}{2m}$$

$$=T_0+m\mathbf{j}_0\cdot\delta\mathbf{j}/e^2+m(\delta j)^2/2Ne^2.$$

One can choose $\delta\mathbf{P}$ opposite in sign to $\mathbf{P}_0$, so that $\mathbf{P}_0\cdot\delta\mathbf{P}$ is negative. If $\delta\mathbf{P}$ is small enough, the term involving $(\delta P)^2$ can be neglected. One concludes that for the state ϕ, the kinetic energy and therefore the total energy, is less than for the state ψ. Although the function, ϕ, is not necessarily a solution of Schrödinger's equation, it is a well-known theorem that the energy of the lowest state must be lower than the mean energy calculated with any other wave function. We have thus proved that unless $\mathbf{j}_0=0$, the state, ψ, cannot be the lowest.

One can conclude immediately from the above that if superconductivity is due to spontaneous currents, then it must disappear at absolute zero, where the system occupies the lowest state. It is possible to go farther, however, and to show that states of finite current cannot be thermodynamically the most stable even at non-zero temperatures. To do this, we show that for each solution of Schrodinger's equation with a non-vanishing current, there exists another solution with a lower current and with a lower energy.

Let us denote by $\psi_i(\mathbf{X}_1, \mathbf{X}_2,\cdots\mathbf{X}_n\cdots)$, the complete set of orthonormal exact solutions of Schrödinger's equations, possessing energies, E_i. We then define an alternative set of functions,

$$\phi_i=\exp[(i\delta\mathbf{P}/\hbar)\cdot\sum_n\mathbf{X}_n]\psi_i,$$

which are also clearly orthonormal, but which are not, in general, exact solutions of Schrödinger's equation. The Hamiltonian matrix can be expressed in terms of the functions, ϕ_i, as follows

$$H_{ij}=\int((\phi_i^*H\phi_j)dX_1dX_2\cdots dX_n\cdots)$$

$$=\left(E_i+\frac{N(\delta P)^2}{2m}\right)\delta_{ij}$$

$$+(\hbar/im)\delta\mathbf{P}\cdot\int(\psi_i^*\sum_n\nabla_n\psi_j)dX_1\cdots dX_n.$$

If $\mathbf{j}_i$ represents the mean current in the state ψ_i, the diagonal elements are

$$H_{ii}=E_i+N(\delta P)^2/2m+\mathbf{j}_i\cdot\delta\mathbf{P}/e$$

while the off-diagonal elements are

$$H_{ij}=(\hbar\delta\mathbf{P}/i)\cdot\int(\psi_i^*\sum_n\nabla_n\psi_j)dX_1dX_2\cdots dX_n\cdots.$$

If $\delta\mathbf{P}$ is small, H_{ij} is a small perturbation, and one can by standard perturbation procedures, obtain a solution for the wave-functions and energy levels. The off-diagonal elements of $\mathcal{H}$ contribute to the energy only in second order. Thus, one obtains for the energy of the state for which ϕ_i is the zeroth approximation

$$W_i=E_i+\mathbf{j}_i\cdot\delta\mathbf{P}/e+\text{terms of order }(\delta P)^2.$$

By choosing $\delta\mathbf{P}$ opposite in sign to $\mathbf{j}_i$ and very small, one obtains, for each ψ_i, another state having an energy lower than that of ψ_i. Thus, if we assume that there is a group of states carrying some current, $\mathbf{j}_i$, they cannot have a minimum free energy, because there is always another group of states with the same statistical weight, but with a lower energy, hence with a lower free energy.

It has sometimes been suggested that superconductivity is always associated with circulating currents, such as those in a superconducting ring. Since the total current, $\mathbf{j}_i$, vanishes here, it is clear that the above results do not apply. By noting that the current for this case is proportional to the total electronic angular momentum, however, one can give a similar treatment, and one obtains again the result that the state of lowest free energy corresponds to zero net current.* Even when the magnetic field caused by the electronic motion is taken into account, the same result is obtained. The proof of the theorem requires, however, that the coupling between electron spin and orbital magnetic fields be neglected. It would seem safe to conclude that the spontaneous current hypothesis cannot explain superconductivity, unless one wishes to show that the energy stabilizing the state of finite current is due to coupling between electron spin and orbital magnetic fields.

It would seem preferable to state that if superconductivity is caused by interactions between electrons, it is probably due to a somewhat localized tendency for electrons of the same velocity to move together as a unit, which is held together in some way by the inter-electronic forces. In order to stop such a group of electrons, it would be necessary to scatter all of them at once. Such a process would be enormously less probable than one in which electrons are scattered individually by lattice vibrations, or other irregularities in the lattice. The formation of such localized groups of co-moving particles would liberate energy, thus making the superconducting state the most stable one. Yet, it would still remain true that a superconducting state which was carrying a large current would have a higher energy than one which carried no current; the current carrying state would then be very long-lived because of the small probability of scattering. It is possible to regard the theories of Heisenberg and Born from this viewpoint, provided that one interprets the interaction energy as associated with a local ordering of velocities, rather than with the stabilization of spontaneous currents. The author is also planning to publish a theory shortly in which another example of this type of explanation is studied in detail, through the means of treating inter-electronic interactions in terms of plasma oscillations.

* In carrying out this treatment, one multiplies the wave function by $\exp[(im/\hbar)\sum_i\phi_i]$, where m is an integer. Since m must be at least unity, the smallest change of angular momentum obtainable in this way is $N\hbar$. In order to obtain an arbitrarily small change of current, however, one can consider a superconducting ring of radius, r, made of a very thin wire. The total change of current density is then $\rho\hbar/mr$, where ρ is the electron density. By going to very large r, one can obtain arbitrarily small changes in current. Hence, for a large enough ring, one can certainly apply the same argument as with non-circulating currents. Since the limiting current for a superconductor is known to be independent of the ring diameter, it is clear that one can, in this way, show that spontaneous currents are not the cause of superconductivity in such a ring.