assisted in computations pertinent to the spectrum analyzer equipment itself. Finally, it must be obvious that the success of experiments involving extensive equipment depends upon painstaking tests and circuit development; without the able assistance of C. M. Bishop, the equipment used in these experiments would not have been built and operated satisfactorily.

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The Angular and Lateral Spread of Cosmic-Ray Showers*

JANE ROBERG** AND L. W. NORDHEIM
Department of Physics, Duke University, Durham, North Carolina

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The mean square angular and lateral spreads in showers have been evaluated for electrons as well as photons as functions of their energy. Analytical expressions have been obtained for high and medium energies down to the critical shower energy. The calculations have been extended by numerical methods for air down to low energies (~4 Mev). Radiative effects and ionization losses have been taken into account simultaneously for all energies, and we believe that no factors of physical significance have been omitted.

1. INTRODUCTION

THE calculation of the sidewise spread of a shower constitutes a problem of importance comparable to the evaluation of its development in depth. Particularly, the discussion of the nature of large air showers cannot be carried out without knowledge of their lateral evolution.

The first treatment of this problem has been given by Euler and Wergeland.[1] These authors discussed the mechanism of the spreading and its general features. Their numerical results, as pointed out by Bethe[2] and by the present authors[3] were, however, quite unsatisfactory and gave an extension of showers far too small. L. Landau[4] set up diffusion equations for the sidewise development in extension of the well-known Landau-Rumer[5] treatment of shower theory. However, his results are invalidated by numerical errors. The most extensive investigation was made by G. Molière.[6] Unfortunately, only an abbreviated version of his work is available. Molière uses an extension of Landau's method and carries it through to an actual evaluation of the radial density distribution in a shower. Because of the complications of the process, he is forced to neglect ionization losses for energies higher than the ionization limit and he takes low energy electrons into account according to a rather inadequate method, as he points out himself. His function will thus be subject to later revision. An evaluation of the mean square angular spread of electrons as function of energy has been given by S. Z. Belenky.[7] He starts with the Landau diffusion equations and obtains from them a set of integro-differential equations for various moments of the distribution. They are evaluated with the help of the method of Tamm

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** Now at Yale University.

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and Belenky.[8] His values differ rather markedly from those obtained here, as will be discussed later.

The presently reported investigation was carried out simultaneously and independently from Molière's and Belenky's work and is, in some respects, complementary to the previous investigations. We have not attempted to evaluate the distribution function itself, but concentrated our efforts on the determination of the mean square momenta under adequate consideration of the low energy region. We believe that no effect of significance has been omitted in our calculations, and that they can serve thus as standard against which calculations of the distribution function can be measured.

In addition to the momenta of electrons, we have also calculated the momenta of photons of all energies which differ significantly from the electron distribution. Our formulas can be considered as applicable to all materials for energies above the characteristic energy of shower theory. Calculations for lower energies have been carried through for air for reason of its particular importance for the discussion of large showers. We have extensively used the results of Richards and Nordheim,[9] quoted henceforth as R.N., and we refer to this paper for a discussion of units and cross sections.

2. GENERAL THEORY

We choose the customary units of shower theory. The unit of energy is the energy E_j, at which radiation and ionization losses are equal. The unit of length is the radiation length, that is, the distance at which one electron would lose the energy E_j, if it were subject only to ionization losses. The specific energy losses for radiation and for ionization are then

$$(dE/dt)_{\text{rad}}=E; \quad (dE/dt)_{\text{ion}}=\beta=1. \tag{2.1}$$

The values of these units for some materials are given in Table I.

TABLE I. Units of shower theory for various materials.

	Air	Water	Al	Fe	Pb
Radiation length	300 m	43 cm	9.8 cm	1.8 cm	0.51 cm
Critical energy	86 Mev	111 Mev	52 Mev	25 Mev	6.7 Mev

We denote the energy of an electron by E, the energy of a photon for easy distinction by K. We write further the function that describes the longitudinal development of a shower under neglection of its sidewise spread as

$f(E, E', t)dE'$ = number of electrons at a depth t in the interval dE' due to a primary of energy E,

$g(E, K, t)dK$ = corresponding number of photons.

According to Euler and Wergeland,[1] the only important source of deflection is the multiple Coulomb scattering of the shower electrons by the nuclei of the traversed material. This effect gives on a path dt a mean square angular deflection in one particular direction of magnitude

$$d\vartheta^2=(E_s/E)^2dt, \tag{2.2}$$

where E_s, a characteristic scattering energy, is in good approximation, for all materials[10]

$$E_s=mc^2(2\pi 137)^{\frac{1}{2}}=15 \text{ Mev}. \tag{2.3}$$

We refer all our calculations to one coordinate normal to the longitudinal direction of the shower. Due to the axial symmetry, the total mean square deviation in a radial direction will be twice the value thus calculated. Otherwise E_s has to be assumed to have a value $\sqrt{2}$ times as large as in Eq. (2.3) or 21 Mev.

The extension of the path of particles due to their deviation from straight lines can be neglected for small angular deflections, i.e., in this approximation the influence of scattering on the functions $f(E, E', t)$ and $g(E, K, t)$ does not have to be considered. This has been verified by an extensive investigation by S. Belenky.[11]

The *angular mean square deviation of electrons* of energy E at a distance t from the origin of the shower due to a primary electron of energy E_0 can then be evaluated as follows. The contribution of an electron of energy E' in an intermediate position, a distance t' backwards from the end point, to the total at t will be

$$d\vartheta^2(E, t)=(E_s/E')^2dt'f(E', E, t-t'),$$

[8] I. Tamm and S. Z. Belenky, J. Phys. U.S.S.R. **1**, 177 (1939).

[9] J. A. Richards and L. W. Nordheim, Phys. Rev. **74**, 1106 (1948).

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and the average over all possibilities then gives

$$f(E_0, E, t)\vartheta^2(E, t) = \int_0^t \int_E^{E_0} f(E_0, E', t-t') \times (E_s/E')^2 f(E', E, t')dt'dE'. \quad (2.4)$$

An exactly similar argument gives for the mean square lateral deviation

$$x^2(E, t) = \frac{1}{f(E_0, E, t)} \int_0^t \int_E^{E_0} f(E_0, E', t-t') \times (E_s/E')^2 t'^2 f(E', E, t')dt'dE'. \quad (2.5)$$

The *angular distribution of quanta* is equal to the distribution of their parent electrons. Since all photons are ultimately produced by electrons and can be considered as being destroyed by absorption, we can express the photon distribution $g(E_0, K, t_0)$ in terms of the electron distribution as follows:

$$g(E_0, K, t) = \int_0^t \int_K^{E_0} f(E_0, E, t-t') \times \sigma(E, K)e^{-\delta t'}dEdt', \quad (2.6)$$

where $\sigma(E, K)$ is the cross section for production of a photon of energy K by an electron of energy E, and δ is the total absorption coefficient for the photon. The angular mean square deviation of the photons is then obtained by averaging over the contributions of the parent electrons according to the genetic relationship Eq. (2.6), i.e.,

$$\vartheta^2(K, t) = \frac{1}{g(E_0, K, t)} \int_0^t \int_K^{E_0} f(E_0, E, t-t') \times \sigma(E, K)e^{-\delta t'}\vartheta^2(E, t-t')dEdt'. \quad (2.7)$$

For the calculation of the *lateral spread of photons* one has to take into account the extension of the path length of the parent electron by the photon. We can still express the lateral deviation by an average, as in Eq. (2.7),

$$x^2(K, t) = \frac{1}{g(E_0, K, t)} \int_0^t \int_K^{E_0} f(E_0, E, t-t') \times \sigma(E, K)e^{-\delta t'}x^2(E, t-t', t')dEdt', \quad (2.8)$$

but we have to replace the deviation, Eq. (2.5), by the "prolonged" scattering

$$x^2(E, t, t'') = \frac{1}{f(E_0, E, t)} \int_0^t \int_E^{E_0} f(E_0, E', t-t') \times f(E', E, t')\left(\frac{E_s}{E'}\right)^2 (t'+t'')^2 dE'dt', \quad (2.9)$$

with the difference to Eq. (2.5) that the lever arm for sidewise deflection is increased by a constant addition t''.

The above equations are exact, but unwieldy. A considerable simplification can be achieved for high energy showers. The principal contribution to the scattering of an individual shower particle will come from the last few radiation lengths of his ancestors, due to the factor $(E_s/E)^2$ in the scattering cross section. In a large shower this depth will be small compared to the total longitudinal extension of the shower. Thus, the dependence on the upper limit of the integration will disappear asymptotically in taking the average of all quantities over the longitudinal coordinate of the shower according to the definitions

$$f(E) = \int_0^\infty f(E_0, E, t)dt, \quad (2.10)$$

$$\vartheta^2(E) = \frac{1}{f(E)} \int_0^\infty \vartheta^2(E_0, E, t) f(E_0, E, t)dt, \quad (2.11)$$

and so on.

The function $f(E)$, Eq. (2.10), is called the track length distribution, or normal distribution. It has been determined for all energies in R.N.[9]

Our formulas for the scattering averaged over the longitudinal extension of the shower can now be written as

$$\vartheta^2(E) = \frac{1}{f(E)} \int_0^\infty \int_K^\infty f(E') f(E', E, t) \times (E_s/E')^2 dE'dt, \quad (2.12)$$

$$\vartheta^2(K) = \int_K^\infty f(E')\sigma(E', K)\vartheta^2(E')dE' \times \left[1 \Big/ \int_K^\infty f(E')\sigma(E', K)dE'\right], \quad (2.13)$$

$$x^2(E, t) = \frac{1}{f(E)} \int_K^\infty \int_0^\infty f(E') f(E', E, t) \times (E_s/E')^2 (t'+t)^2 dt'dE'. \quad (2.14)$$

The mean square angular scattering of a shower electron is then

$$x^2(E)=x^2(E,0). \tag{2.14a}$$

Finally we have a prolonged lateral photon scattering,

$$x^2(K,t)=\frac{\int_0^\infty\int_K^\infty f(E')\sigma(E',K)x^2(E',t'+t)e^{-\delta t'}dE'dt'}{\int_0^\infty\int_K^\infty f(E')\sigma(E',K)e^{-\delta t'}dE'dt'}. \tag{2.15}$$

The use of these longitudinally averaged values corresponds to the physical situation insofar that we will not know, in general, where an observed shower originated. Since the energy distribution in a large shower near its maximum is the same as the track length distribution, the above formulas will also be correct for the shower maximum, where there is the best chance of observing it.

It is possible and useful for later considerations to express the scattering of electrons in terms of their photon ancestors. We introduce the track length distribution of photons,

$$g(K)=\int_0^\infty g(K,t)dt, \tag{2.16}$$

and, further, the straggling function $f_0(E',E,t)$, which denotes the probability that an electron of energy E' arrives at the distance t with the reduced energy E. The angular deflection of the shower electrons can then be expressed as follows:

$$\vartheta^2(E)=\frac{\int_{E'}^\infty dK\int_E^\infty dE'\int_0^\infty dt' g(K)\sigma(K,E')[\vartheta^2(K)+\vartheta_0^2(E',E,t')]f(E',E,t')}{\int_{E'}^\infty dK\int_E^\infty dE'\int_0^\infty dt' g(K)\sigma(K,E')f(E',E,t')}. \tag{2.17}$$

Here $\vartheta_0^2(E',E,t)$ stands for the angular deflection the electron obtains as an individual from its birth at energy E' to observation with energy E at distance t'.

In order to describe the lateral deflection in an analogous manner, we have to introduce the prolonged deflection both for photons and for electrons. The expressions obtained are

$$x^2(K,t)=\frac{\int_E^\infty dE'\int_0^\infty dt' f(E')\sigma(E',K)x^2(E',t'+t)e^{-\delta t'}}{\int_E^\infty dE'\int_0^\infty dt' f(E')\sigma(E',K)e^{-\delta t'}}, \tag{2.18}$$

$$x^2(E,t)=\frac{\int_{E'}^\infty dK\int_E^\infty dE'\int_0^\infty dt' g(K)\sigma(K,E')[x^2(K,t'+t)+x_0^2(E',E,t'+t)]f(E',E,t')}{\int_{E'}^\infty dK\int_E^\infty dE'\int_0^\infty dt' g(K)\sigma(K,E')f(E',E,t')}. \tag{2.19}$$

Here again $x_0^2(E',E,t'+t)$ denotes the lateral deviation obtained by an electron as an individual when slowed down from the energy E' to E in the distance t' with the extension of the lever arm by a fixed constant t. The actual mean square deviations in the shower are again given

TABLE II. Normal energy distribution in a shower; air. $f(E)=\varphi(E)/E^2$.

E/E_i	∞	10	7	5	3	2	1.5	1
$f(E)$	$1/E^2$	0.0086	0.0166	0.030	0.073	0.145	0.23	0.42
$\varphi(E)$	1	0.86	0.81	0.75	0.65	0.58	0.52	0.42

E/E_i	0.75	0.5	0.3	0.2	0.15	0.1	0.05
$f(E)$	0.62	0.97	1.61	2.25	2.8	3.7	5.5

by $x^2(K, 0)$ and $x^2(E, 0)$. The two equations, (2.18) and (2.19), form a simultaneous system of integral equations, which can be solved by an iteration process, as will be shown in Sections 5 and 6.

The mean square angular and lateral deviations for electrons as well as photons have now been expressed in terms of the functions which describe the longitudinal development of a shower, i.e., the normal distribution $f(E)$ and the general shower functions $f(E, E', t)$ and $g(K, K', t)$. The problem reduces thus to the finding of suitable approximations for these functions that permit the evaluation of Eqs. (2.12) to (2.15) or (2.18) to (2.19). We divide the whole energy spectrum into three regions: (a) high energies, where ionization losses can be entirely neglected, (b) intermediate energies where ionization can be considered as a small correction, and (c) low energies where ionization predominates over radiation effects. The division between (b) and (c) is made at the critical energy of shower theory, where ionization and radiation losses are equal.

3. HIGH ENERGIES

We consider firstly the high energy region where ionization losses can be completely neglected. The track length distribution is then given[12] by

$$f(E)=1/E^2. \tag{3.1}$$

We require further the shower functions $f(E', E, t)$ for small values of the distance t and for values of the primary energy E' which is not much larger than E. The only available form which gives a good approximation for the beginning of small showers is the development of Bhabha and Heitler,[13] quoted henceforth as B.H. Their formulas, as described in appendix I, are based on an adequate approximation for high energies. The evaluation of the actual distribution functions by the Bhabha-Heitler method is complicated and can only be done approximately. It is remarkable, however, that the scattering integrals containing their functions can be integrated in closed form.

B.H. decompose the distribution function into a series

$$f(E', E, t)=\sum_0^\infty f_n(E', E, t). \tag{3.2}$$

The first term f_0 represents the probability that the primary electron E' itself has reached a depth t with an energy E. The term f_1 represents the number of second generation electrons, i.e., particles that have been created by an intermediate photon, and f_n represents electrons of the n'th generation, i.e., with n intermediate photons. The series (3.2) will obviously converge since the total number of electrons above a given energy remains finite.

Inserting Eqs. (3.1) and (3.2) into the integrals (2.12), (2.14), we obtain for the angular and prolonged lateral scattering

$$E^2\vartheta^2(E)=\int_0^\infty\int_E^\infty (dE'/E'^2)\sum_0^\infty f_n(E', E, t') \times(E_s/E')^2dt', \tag{3.3}$$

$$E^2x^2(E, t)=\int_0^\infty\int_E^\infty (dE'/E'^2)\sum_0^\infty f_n(E', E, t') \times(E_s/E')^2(t'+t)^2dt'. \tag{3.4}$$

They are composed of terms of the general form

$$(1/E^4)I_m^n(E)=\int_0^\infty\int_E^\infty (dE'/E'^4) \times f_n(E', E, t')t'^m dt'. \tag{3.5}$$

The integration gives, as shown in the appendix,

$$\begin{aligned} I_0^n&=1/(2\times12^n),\\ I_1^n&=[1/(2\times12^n)][((n+1)/2)+(n/\delta)], \\ I_2^n&=[(n+1)/(2\times12^n)]\times[[(n+2)/4]+(n/\delta)+(n/\delta^2)], \end{aligned} \tag{3.6}$$

where $\delta=7/9$ is the absorption coefficient for photons.

[12] L. W. Nordheim and M. H. Hebb, Phys. Rev. 56, 494 (1939).

[13] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. 159, 432 (1937).

The summation over n can be carried out in closed form, according to Eq. (A16) with the result that

$$I_0=\sum_0^\infty I_0^n=0.545,$$
$$I_1=\sum_0^\infty I_1^n=0.362, \tag{3.7}$$
$$I_2=\sum_0^\infty I_2^n=0.642.$$

Inserting the values into Eqs. (3.3) and (3.4), we obtain

$$\vartheta^2(E)=0.545(E_s/E)^2, \tag{3.8}$$

$$x^2(E,t)=(0.642+0.724t+0.545t^2)(E_s/E)^2. \tag{3.9}$$

The lateral scattering of electrons is obtained, of course, by taking $t=0$. The scattering integrals for photons, Eqs. (2.13) and (2.15), can now be evaluated with the help of Eqs. (3.8) and (3.9) with the result

$$\vartheta^2(K)=0.181(E_s/K)^2, \tag{3.10}$$

$$x^2(K)=(1/3)(I_2+(2/\delta)I_1+(2/\delta^2)I_0)\times(E_s/K)^2=1.126(E_s/K)^2. \tag{3.11}$$

4. INTERMEDIATE ENERGIES

Our next step will be the consideration of intermediate energies, where ionization losses have to be taken into account, but are not yet predominant. We solve this problem again by the standard integrals, Eqs. (2.12) to (2.15), but we introduce suitable modifications in the shower functions $f(E)$, $g(K)$, and $f(E',E,t)$.

The normal distribution $f(E')$ has been determined numerically for all energies by R.N.[9] There it was written as

$$f(E')=\varphi(E')/E'^2, \tag{4.1}$$

where $\varphi(E')$ is a slowly varying function of E'. It is tabulated for air in Table II. The form (4.1) is not practical here, since φ is only given numerically, and we try to find an analytical approximation. It is most essential that the latter is good in the neighborhood of the energy E for which the scattering is to be evaluated. We write

$$f(E')=(1/E'^2)+q(E)/E'^3, \tag{4.2}$$

with

$$q(E)=-E[1-\varphi(E)]. \tag{4.3}$$

This means that we take a different formula for every E value, so that we have always the correct value for $E'=E$. It would, in principle, be possible to improve the approximation by adding terms with higher negative powers in E'. However, the error in Eq. (4.3) is already less than 10 percent for all values of E', even if we take E as low as E_j.

We have, furthermore, to correct the functions $f_n(E',E,t)$ in the development, Eq. (3.2). Let us first consider the straggling function f_0. We assume now that the electron looses through ionization the energy β (or E_j) per radiation length or βt on the length t. If this loss is small compared to the loss by radiation, then we may neglect the change in radiation loss resulting from the electron energy is also reduced by ionization. We take thus as corrected straggling function

$$f_0(E',E,t)_{\text{corr}}=f_0(E',E+\beta t,t). \tag{4.4}$$

In other words, the function f_0 is to be calculated not for the final energy value, but for the energy $E''=E+\beta t$ which the electron would have had without ionization loss.

In the higher functions f_n, which represent the contribution of n'th generation particles, the ionization losses of the ancestor electrons will be relatively unimportant due to the large reduction of energy at every transformation. It will be sufficient, therefore, to consider this effect only over the last lap of the path, i.e., for the final electron after its production by a photon. We call this distance t' and decompose the function f_n as follows:

$$f_n(E',E,t)=\int_0^t dt'h_n(E',E,t',t), \tag{4.5}$$

where the function $h_n(E',E,t',t)$ gives the distribution of n'th generation electrons at t which have been created at a distance t' backwards. The function h_n can also be obtained by the Bhabha-Heitler method, as shown in the appendix. In analogy to Eq. (4.4) we take as the corrected function

$$f_n(E',E,t)_{\text{corr}}=\int_0^t dt'h_n(E',E+\beta t',t',t). \tag{4.6}$$

In order to have uniform formulas we write Eq.

(4.4) in the same form with

$$h_0(E', E, t', t) = f_0(E', E, t)\delta(t-t'). \quad (4.7)$$

Equations (2.12) and (2.14) for the angular and prolonged lateral scattering for electrons*** become now, on substitution of Eqs. (4.2), (3.2), and (4.6),

$$(\varphi(E)/E^2)\vartheta^2(E) = \int_0^\infty dt'' \int_0^{t''} dt' \times \int_{E+\beta t'}^\infty dE' \left(\frac{1}{E'^2} + \frac{q(E)}{E'^3}\right)(E_s/E')^2 \times \sum_0^\infty h_n(E', E+\beta t', t', t''), \quad (4.8)$$

$$(\varphi(E)/E^2)x^2(E, t) = \int_0^\infty dt''(t''+t)^2 \int_0^{t''} dt' \times \int_{E+\beta t'}^\infty dE'[(1/E'^2) + (q(E)/E'^3)] \times (E_s/E')^2 \sum_0^\infty h_n(E', E+\beta t', t', t''). \quad (4.9)$$

The general term of this integral is of the form

$$E^{-(s+2)}\, {}^sJ_m^n(E) = \int_0^\infty dt'' t''^m \int^{t''} dt' \times \int_{E+\beta t'}^\infty \frac{dE'}{E'^{s+2}} h_n(E', E+\beta t', t', t''), \quad (4.10)$$

where $m=0$, 1, or 2 and $s=2$ or 3. They are evaluated in the appendix. On introduction of the definition

$${}^sJ_m = \sum_{n=0}^\infty {}^sJ_m^n, \quad (4.11)$$

we obtain

$$\vartheta^2(E) = (E_s/E)^2(1/\varphi(E))[{}^2J_0(E) - (1-\varphi(E))\, {}^3J_0(E)], \quad (4.12)$$

$$x^2(E, t) = (E_s/E)^2(1/\varphi(E)) \times \{[{}^2J_2(E) - (1-\varphi(E))\, {}^3J_2(E)] + 2t[{}^2J_1(E) - (1-\varphi(E))\, {}^3J_1(E)] + t^2[{}^2J_0(E) - (1-\varphi(E))\, {}^3J_0(E)]\}. \quad (4.13)$$

The photon momenta are again obtained from Eqs. (2.13) and (2.15). Upon introduction of Eqs. (4.12) and (4.13), they take the form

$$\vartheta^2(K) = \left(E_s^2 \Big/ \int_K^\infty f(E)dE\right) \int_K^\infty (dE/E^4) \times [{}^2J_0(E) - (1-\varphi(E))\, {}^3J_0(E)], \quad (4.14)$$

$$x^2(K, t) = \left(E_s^2 \Big/ \int_K^\infty f(E)dE\right) \int_K^\infty (dE/E^4) \times \{{}^2J_2(E) - (1-\varphi(E))\, {}^3J_2(E) + (2/\delta)[{}^2J_1(E) - (1-\varphi(E))\, {}^3J_1(E)] + (2/\delta^2)[{}^2J_0(E) - (1-\varphi(E))\, {}^3J_0(E)]\}. \quad (4.15)$$

The evaluation of these integrals has been carried out numerically, except for the high energy tail (above $E > 10E_j$) where a series development has been used. The results of these calculations are contained in Table III.

It may be remarked that the methods developed here would also make possible the evaluation of the higher moments of the distribution. It also would be possible, in principle, to carry out similar calculations for other longitudinal positions than the maximum in the shower, as long as the variation of the primary distribution $f(E)$ can be neglected within the distance from the point of observation which contributes to the scattering. Such calculations have not been carried out.

5. LOW ENERGIES, ANGULAR SCATTERING

The approximations made in the preceding section are satisfactory for energies down to about the ionization limit E_j. Below this energy our functions $f_n(E', E, t)$ (Eq. (4.5)), loose their validity. Also, the cross sections for most processes become different functions of energy, and new effects, such as the Compton effect, come into play.

In order to offset these difficulties, we may here neglect the straggling of electrons, i.e., we can assume that every electron with $E < E_j$ loses its energy in a continuous fashion according to the law

$$dE/dt = E + E_j, \quad (5.1)$$

where the first term E represents the radiative losses and the second the ionization losses. This means that the path length, or range between

*** In the subsequent equations the letter s is used in two different connotations. As a subscript to the energy, i.e., E_s, it signifies the characteristic scattering energy, while as a superscript, i.e., E^s, it signifies an exponent.

two energies E' and E, is given by

$$R(E', E) = \log[(E'+E_j)/(E+E_j)], \quad (5.2)$$

or that the energy expressed as function of distance is

$$E(t) = (E'+E_j)e^{-R} - E_j.$$

The function $f_0(E', E, t)$ reduces to a function of two variables only (E' and E, or E' and t). An approximation that is more easily handled is obtained by development of the log in Eq. (5.2), i.e.,

$$R(E', E) \simeq (E'-E)/(E+E_j), \quad E' = E+(E+E_j)R. \quad (5.2a)$$

The assumption of a definite range-energy relation as in Eq. (5.2) or (5.2a) permits the direct evaluation of the scattering of an individual electron, i.e., of the functions $\vartheta_0^2(E', E, t')$ and $x_0^2(E', E, t'+t)$ in Eqs. (2.17) and (2.19).

It is then possible to solve directly the integral equations (2.12), (2.17), and (2.18), (2.19) by an iteration process similar to the method used by R.N.[9] for the determination of the energy distribution. We refer to this paper for a discussion of cross sections and for the values of a number of auxiliary functions.

The method will be best illustrated by the discussion of the actual procedure.

We decompose the electronic distribution into two groups

$$f(E) = f_0(E) + f_1(E), \quad E \leqslant E_j, \quad (5.3)$$

where f_0 is the number of electrons which have been decelerated as individuals from higher energies and f_1 is the number which have been created by photons at energies below E_j. f_0 and f_1 are numerically given by R.N.[9]

The angular scattering can be decomposed in a similar manner, i.e., in obvious notation:

$$\vartheta^2(E) = [1/f(E)][f_0\vartheta_0^2(E) + f_1\vartheta_1^2(E)]. \quad (5.4)$$

The contribution ϑ_0^2 will be the sum of the deviation the electron acquires on its path from E_j to E plus the known amount it inherited at E_j. Thus,

$$\vartheta_0^2(E) = \int_0^{R(E_j, E)} (E_s/E(t))^2 dt + \vartheta^2(E_j). \quad (5.5)$$

The first part can be evaluated as follows:

$$\int_0^R (E_s/E(t))^2 dt = \int_E^{E_j} (E_s/E')^2 (dt/dE) dE$$
$$= \int_E^{E_j} (E_s/E')^2 [dE/(E'+E_j)]$$
$$= (E_s/E_j)^2 [(E_j/E) - 1 - \log[(E+E_j)/2E]]. \quad (5.6)$$

The function $\vartheta_1^2(E)$ can be expanded in a similar manner:

$$\vartheta_1^2(E) = \left[1 \bigg/ \int_E^{E_j} h(E')dE'\right] \int_E^{E_j} dE' h(E') \times \left\{ \int_E^{E'} (E_s/E'')^2 [dE''/(E''+E_j)] + \vartheta_2^2(E') \right\}. \quad (5.7)$$

Here $h(E')dE'$ denotes the number of electrons produced in dE' by quanta, a function computed in R.N.[9] The first term on the right-hand side of Eq. (5.7) represents thus the deviation acquired on their path by electrons which have been created with energies $E' < E_j$. The second term $\vartheta_2^2(E')$ is the angular deviation inherited by these electrons from their parent quanta.

We note that by definition

$$f_1(E) = [1/(dE/dt)] \int_E^{E_j} h(E')dE' = [1/(E+E_j)] \int_E^{E_j} h(E')dE'. \quad (5.8)$$

The first term of Eq. (5.7) can then be brought into the easily integrable form

$$\int_E^{E_j} dE' h(E') \int_E^{E_j} (E_s/E'')^2 [dE''/(E''+E_j)]$$
$$= \int_E^{E'} (E_s/E'')^2 [dE''/(E''+E_j)] \int_{E''}^{E_j} h(E')dE'$$
$$= \int_E^{E_j} (E_s/E'')^2 f_1(E'') dE''.$$

The evaluation of $\vartheta_2^2(E')$ requires the knowledge of the scattering of the parent quanta, which

TABLE III. Mean square angular and lateral spread of shower electrons and photons as function of energy.

E/E_j or K/E_j	$x^2(E)$	$X^2(E)$	$x^2(K)$	$X^2(K)$	$\vartheta^2(E)$	$\theta^2(E)$	$\vartheta^2(K)$	$\theta^2(K)$
∞	$0.642(E_s/E)^2$	$0.214(E_s/E)^2$	$1.13(E_s/K)^2$	$0.38(E_s/K)^2$	$0.545(E_s/E)^2$	$0.182(E_s/E)^2$	$0.181(E_s/K)^2$	$0.060(E_s/K)^2$
10	0.49	0.18	0.95	0.32	0.47	0.16	0.16	0.054
7	0.46	0.17	0.88	0.285	0.44	0.15	0.15	0.050
5	0.43	0.16	0.80	0.260	0.42	0.14	0.14	0.047
3	0.40	0.140	0.71	0.220	0.38	0.13	0.12	0.040
2	0.33	0.125	0.62	0.195	0.34	0.11	0.11	0.036
1.5	0.30	0.105	0.55	0.175	0.31	0.095	0.095	0.034
1	$0.25(E_s/E_j)^2$	$0.085(E_s/E_j)^2$	$0.46(E_s/E_j)^2$	$0.155(E_s/E_j)^2$	$0.27(E_s/E_j)^2$	$0.085(E_s/E_j)^2$	$0.085(E_s/E_j)^2$	$0.029(E_s/E_j)^2$
0.75	0.28	0.120	0.64	0.255	0.34	0.135	0.110	0.043
0.5	0.39	0.165	0.80	0.42	0.69	0.22	0.145	0.062
0.4	0.46	0.190	1.30	0.53	0.97	0.28	0.170	0.076
0.3	0.60	0.235	1.75	0.72	1.40	0.35	0.23	0.100
0.2	0.86	0.31	2.55	1.02	2.40	0.57	0.38	0.145
0.15	1.13	0.48	3.4	1.30	3.4	0.80	0.50	0.185
0.10	1.6	0.46	4.7	1.8	6.7	1.23	0.68	0.26
0.05	2.8	0.64	7.4	2.7	12.8	2.3	1.25	0.40

in turn is determined by

$$\vartheta^2(K)=\frac{\int_K^\infty f(E)\sigma(E,K)\vartheta^2(E)dE}{\int_K^\infty f(E)\sigma(E,K)dE}. \quad (5.9)$$

The contribution to $\vartheta^2(E)$ from ϑ_2^2 must evidently be small, since the angular deviations of photons are small and $\vartheta_2^2(E)$ refers in addition to photons of energies larger than E_j. We can carry out the simultaneous determination of $\vartheta^2(K)$ and of $\vartheta_2^2(E)$ by an iteration process, i.e., we evaluate $\vartheta^2(K)$ with a $\vartheta^2(E)$ not yet containing the $\vartheta_2^2(E)$ terms, and then use this function to determine $\vartheta_2^2(E)$. The evaluation of Eq. (5.9) could then be corrected again and so on. It was found, however, by estimates based on this procedure that the contribution from $\vartheta_2^2(E')$ is entirely negligible.

The evaluation of Eq. (5.9) was carried out with the cross sections given by R.N. The results of the calculations are contained in Table III.

6. LOW ENERGIES, LATERAL SCATTERING

The lateral scattering of electrons and photons can be treated by the method of the preceding section, with the added complication that the "prolongation" of the lever arm by subsequently added path lengths has to be taken into account. We will deal throughout with the "prolonged" scattering expressions $x^2(E,t)$, $x^2(K,t)$ from which the actual scattering is obtained by taking $t=0$.

We write the expression for the prolonged photon scattering as follows (cf. Eq. (2.15)):

$$x^2(K,t)=\frac{\int_K^\infty f(E)\sigma(E,K)\bar{x}^2(E,t)dE}{\int_K^\infty f(E)\sigma(E,K)dE}, \quad (6.1)$$

where we introduced the "averaged" prolonged electron scattering,

$$\bar{x}^2(E,t)=\int_0^\infty x^2(E,t'+t)\delta e^{-\delta t'}dt'. \quad (6.2)$$

We decompose $x^2(E,t)$ again as in the preceding section:

$$x^2(E,t)=[1/f(E)][f_0x_0^2(E,t)+f_1x_1^2(E,t)], \quad (6.3)$$

where the first term contains the contribution of electrons decelerated from E_j as individuals and the second term those produced with energies below E_j. For the first part one finds

$$x_0^2(E,t)=\int_E^{E_j}[(R(E',E)+t)^2/(E'+E_j)]\times(E_s/E')^2dE'+x^2(E_j,R(E',E)+t), \quad (6.4)$$

where $R(E',E)$ is the range of the electrons between the energies E' and E (cf. Eq. (5.2)). It is clear that $x_0^2(E,t)$ is a quadratic function in t. It follows then from Eq. (6.2) that

$$\bar{x}_0^2(E,t)=(E_s/E_j)^2[a(E)+2b(E)t+c(E)t^2], \quad (6.5)$$

where a, b, c, are functions of E that can be obtained from Eqs. (6.2) and (6.4).

For the actual computation we have used the approximation, Eq. (5.2a). We find, with $R_0=(E_0-E)/(E+E_j)$, for the prolonged scattering of an electron between the energies E_0 and E:

$$
\begin{aligned}
x_0^2(E_0, E, t) & \\
&= \int_E^{E_0} [(R(E',E)+t)^2/(E'+E_j)](E_s/E')^2 dE' \\
&= \int_0^{R_0} (R+t)^2 (E_s/E(R))^2 dR \\
&\simeq E_s^2 \int_0^{R_0} \left[\frac{R+t}{E+(E+E_j)R}\right]^2 dR \\
&= (E_s/E_j)^2 [E_j/(E+E_j)]^2 \\
&\quad \times \{[1-(E^2/E_0^2)-2(E/E_0)\log(E_0/E)] \\
&\quad +2t[\log(E_0/E)-(1-(E/E_0))] \\
&\quad +t^2(1-(E/E_0))(1+(E_j/E))\}. \qquad (6.6)
\end{aligned}
$$

As the next step we evaluate a first approximation to the photon momentum $x_0^2(K,t)$ from Eq. (6.1), using the previously found expression (6.5) for $E<E_j$ for the electron scattering at $E<E_j$ in place of the not yet determined full expression (6.3). For $E>E_j$, of course, the results from Section 4 have been used. We obtain again a quadratic function in t,

$$
\begin{aligned}
x_0^2(K,t) &= (E_s/E_j)^2 \\
&\quad \times [A(K)+2B(K)t+C(K)t^2], \qquad (6.7)
\end{aligned}
$$

where A, B, C, are numerically obtained functions of the photon energy K. It turned out that these functions do not have to be corrected again, and the results of this calculation for $t=0$ are given in Table III as our final results.

It remains to compute the contribution $x_1^2(E,t)$ of the electrons which have been produced with energies $E'<E_j$. It can be written as

$$
\begin{aligned}
x_1^2(E,t) &= \int_E^{E_j} dE' \int_{E'}^{\infty} dK g(K)\sigma(K,E') \\
&\quad \times [x_0^2(K, R(E',E)+t)+x_0^2(E',E,t)] \\
&\quad \times 1 \Big/ \int_E^{E_j} dE' \int_{E'}^{E} g(K)\sigma(K,E')dK. \qquad (6.8)
\end{aligned}
$$

TABLE IV. Comparison of shower spreads at high energies obtained by various authors (notations as in Table III).

	$(E/E_s)^2x^2(E)$	$(E/E_s)^2x^2(K)$	$(E/E_s)^2\vartheta^2(E)$	$(E/E_s)^2\vartheta^2(K)$
Euler-Wergeland	0.074		0.33	
Molière	0.835	1.314	0.6	0.2
Belenky	0.94			
Janossy	0.724		0.570	
Roberg-Nordheim	0.642	1.13	0.545	0.181

Here $g(K)$ is the energy distribution of photons, $\sigma(K,E')$ the production cross section for electrons, for which we have to add pair production and Compton effect, and $x_0^2(E',E,t)$ the contribution to the scattering of the electron after it has been created, as determined in Eq. (6.6). All functions in Eq. (6.8) are then known.

In order to simplify the evaluation of Eq. (6.8), we have assumed that every electron E' is created in the average by a quantum of definite energy K. For Compton electrons of not too low energies we can assume without much error that they have the same energy as their parent photons. The average energy K_{Av} of the photon which gives the correct scattering inheritance to a pair electron E' is defined by the relation

$$
x^2(K_{\text{Av}}) = \frac{\int_{E'}^{\infty} g(K)\sigma_p(K,E')x^2(K)dK}{\int_{E'}^{\infty} g(K)\sigma_p(K,E')dK},
$$

where σ_p is the differential cross section for pair production. If we assume that $x^2(K_{\text{Av}})$ is proportional to K^{-n} and $g(K)$ to K^{-m}, we find, since $\sigma_p(K,E')\sim K^{-1}$,

$$
\begin{aligned}
(K_{\text{Av}})^{-n} &= \int_E^{\infty} K^{-(m+n+1)}dK \Big/ \int_0^{\infty} K^{-(m+1)}dK \\
&= [m/(m+n)]E^{-n},
\end{aligned}
$$

or

$$
K_{\text{Av}} = [1+(n/m)]^{1/n}E = \alpha E. \qquad (6.9)
$$

The value of α is fairly insensitive to the exponents. At very high energies, we would have $n=2$, $m=2$, and $\alpha=\sqrt{2}$. At very low energies $n\sim 1$, $m\sim 1$, $\alpha=2$. We choose as a suitable average value $\alpha=\sqrt{3}=1.73$. The evaluation of Eq. (6.8) reduces then to single integrals of the general form

$$\int_{\alpha E}^{E_j} h_p(\alpha E')x_0^2(\alpha E', R)dE', \quad (6.10)$$

where $h_p(E')$ is the number of pair electrons produced per interval dE', a function determined by R.N.[9], and x_0^2 is to be taken from Eq. (6.7).

The process could be repeated, i.e., one could go back with the new value of $x_1^2(E)$ into Eq. (6.3) and then re-evaluate the photon scattering from Eq. (6.1). However, as mentioned before, this turned out to be unnecessary. We also have left out of our calculations the knock-on electrons and post-Compton photons, since their contribution to the mean square deviations could be estimated to be negligible.

Our calculations have been extended down to energies of $0.05E_j$ or $\sim$4 Mev, the lower limit to which the auxiliary functions $f(E)$, $g(K)$ have been given by R.N.[9] For the final results see Table III.

7. DISCUSSION

Our results are collected in Table III which gives the mean square deviations of electrons and photons as function of their energy. The symbols have the following meaning:

$x^2(E)$ = mean square lateral spread of electrons of energy E,
$x^2(K)$ = mean square lateral spread of photons of energy K,
$\vartheta^2(E)$ = mean square angular spread of electrons of energy E,
$\vartheta^2(K)$ = mean square angular spread of photons of energy K.

The capitalized letters denote the same quantities averaged over all energies in a shower larger than a given lower limit, for instance,

$$X^2(E) = \int_E^\infty x^2(E')f(E')dE' \times 1 \Big/ \int_E^\infty f(E')dE'. \quad (7.1)$$

$X^2(E)$ gives thus the total spread of a shower as measured by an arrangement that records only particles above a given energy E.

The units of energy and length are those of shower theory (cf. Table I). Angles are measured in radians. The critical scattering energy E_s (cf. Eq. (2.3)) has to be taken as 21 Mev for evaluation of radial deviations, and equal to $21/\sqrt{2}$ Mev = 15 Mev for deviations in one particular direction normal to the shower axis.

The mean square deviations are obtained from Table III by multiplying the figures in the upper half ($E/E_j \geqslant 1$) by $(E_s'/E)^2$, in the lower half ($E/E_j \leqslant 1$) by $(E_s/E_j)^2$.

Table III should be used in conjunction with Table I of R.N.,[9] which gives the corresponding energy distribution functions in a large shower. The energy distribution for electrons is also reproduced in Table II, Section 4, of this paper.

The first row in Table III represents asymptotic values for high energies under complete neglection of ionization losses. They have been obtained (cf. Section 3) without any further neglection from the shower theory in the form of Bhabha and Heitler.[13] The values down to $E/E_j = 1$ have been obtained from the formulas of Section 4, using asymptotic cross sections for high energies. The results are valid as far as these cross sections are a satisfactory approximation. The low energy part has been calculated for air. The results should, however, have at least qualitative validity for other materials.

It is to be noted, of course, that the forward progress of a shower will be stopped for energies such that the root mean square angular deflection of electrons becomes of order unity. Our calculations are actually based on the assumption that the scattering angles are small. $(\vartheta^2(E))^{\frac{1}{2}}$ reaches the value 0.5 for air at $E/E_j = 0.2$ or $E = 16$ Mev, and for lead at $E/E_j = 3.7$ or $E = 25$ Mev and the value unity for air at $E/E_j = 0.05$ or $E = 4$ Mev, and for lead at $E/E_j = 1.8$ or $E = 12$ Mev. We can take the latter values as a rough measure of the energy below which electrons do not show any preference for the forward direction. Using the results of R.N., we find that for air about $\frac{1}{6}$ of the electron track length is contained in this low energy, non-directional part while this fraction increases to 80 percent for lead.†

The root mean square lateral deviation of electrons with $\vartheta^2(E) \sim 1$ can be considered as a rough measure of the total radius of a large shower. This radius is for normal air about 120 m, for

† The position of the shower maximum will not be affected much, since it depends mainly on the particles of high energy, compare S. Belenky, reference 11.

lead about 0.45 cm. The root mean square deviation for all shower electrons above these energies is less, namely, ~60 m for air or 0.3 cm for lead. Spreads of this order of magnitude are sufficient to explain observations on large air showers; compare f.i. the investigation by Cocconi[14] and collaborators.

The general behavior of the spread as function of energy is similar to that found by Euler and Wergeland[1] and by Molière,[6] and it gives, as shown by these authors, a qualitative explanation of the general features of showers. For example, photographs of showers penetrating several lead plates often show quite clearly that electrons with larger angular spread are more easily absorbed. Furthermore, electrons emerging from lead with a pronounced forward direction must have energies at least of order 20 to 30 Mev. The root mean square radial deviation of all electrons of this and higher energies is only about 1.5 mm. Thus, lead showers seem to diverge virtually from single points.

An interesting feature of our results is that the angular deviations of photons are smaller than those of electrons of the same energy while their radial deviation is larger. The reason for this behavior is that a photon inherits its angular deflection from a higher energy electron parent. For the lateral deflection this effect is overcompensated by the comparatively long mean free path of photons.

During the last years, calculations on the spread of showers have been given by various authors, and it seems to be of importance to compare their results with ours and to assess their accuracy.

Table IV gives the results for the mean square deviation at high energies under total neglection of ionization losses.

The original Euler-Wergeland[1] theory contains a serious underestimate of the spread. The results of the other authors are in rough agreement, the differences being due to the use of different forms of the shower theory and different approximations for the cross sections. Molière uses simplified cross sections and his method, based on Landau's[4] general equations, cannot be judged completely from the short available abstracts.

Belenky's[7] value has been obtained after correction of a factor in his equations which made the mean square deviation four times too large. He uses the Tamm-Belenky[8] formalism to evaluate correct diffusion equations.

Janossy,[15] in his recent book, evaluates our Eqs. (2.12) and (2.14) in an elegant and simple way, using the full asymptotic cross sections. His values have to be considered as the best ones today. His method, unfortunately, does not seem to yield itself to the inclusion of ionization losses. Our values are obtained from a rigorous evaluation of the Bhabha-Heitler theory, the difference to Janossy being due to the basic approximations underlying their formulas.

Data on $x^2(E)$ for lower energies under inclusion of ionization losses have been given by Belenky and Janossy. Belenky's curve, as judged from a very small graph, gives approximately the correct reduction due to ionization at the critical energy. It is, however, considerably too flat, i.e., it gives too little reduction at higher energies and too small a spread at lower energies. The sharp upturn of our values for very low energies is the result of the comparatively long range of low energy photons, an effect that is neglected in the Tamm-Belenky formalism. Janossy gives plausibility arguments for the rough interpolation formula

$$x^2(E)=0.724(E_s/(E+E_j))^2. \qquad (7.2)$$

This formula gives a reduction somewhat too high at energies in the neighborhood of E_j and also fails to give the upturn at very low energies.

The only serious attempt to obtain the full density distribution averaged over all electron energies as function of distance from the shower axis is due to Molière.[6] He neglects ionization losses for all energies above the critical energy E_j, while our Table III shows that this effect reduces $x^2(E)$ by a factor ~2.5 at E_j. The number of electrons in the neighborhood of E_j is also reduced by a similar factor. The contribution of electrons with $E \gtrsim E_j$ is thus very considerably overestimated by Molière. On the other hand, he considers in the low energy region only those electrons that have been slowed down from higher energies (our contribution f_1, compare

[14] G. Cocconi, A. Loverdo, and V. Tongiorgi, Phys. Rev. **70**, 846 (1946).

[15] L. Janossy, *Cosmic Rays* (Oxford University Press, London, 1948).

TABLE V. Values of $K_m(E)$ for $s=2$ and 3; $m=0, 1, 2$.

$\beta b/2E$	$^2K_0(E)$	$^2K_1(E)$	$^2K_2(E)$	$^3K_0(E)$	$^3K_1(E)$	$^3K_2(E)$
1/2	0.370	0.151	0.0679	0.315	0.109	0.0415
2/5	0.421	0.187	0.105	0.363	0.143	0.0607
1/3	0.463	0.222	0.134	0.404	0.175	0.0810
1/4	0.531	0.282	0.173	0.471	0.234	0.122
1/7	0.656	0.438	0.308	0.602	0.375	0.235
1/10	0.730	0.551	0.330	0.66	0.384	
1/20	0.835	0.69	0.55	0.792	0.619	
1/50	0.93	0.86	0.81	0.91	0.83	0.705
0	1	1	1	1	1	1

Section 5). This underestimates the number as well as the individual deviations in the low energy range. It would seem thus that a more accurate calculation should give a somewhat stronger compression of the shower core coupled with a larger sidewise dispersion from the lower energy region. The outmost part of the shower is, according to Molière, due to single scattering through comparatively large angles†† which gives asymptotically for large distances r from the shower axis a decrease of density as r^{-3}. This latter result cannot be strictly correct since it would result in a divergent mean square deviation and it is also physically clear that finally there must be an exponential cut-off.

It is thus difficult to judge how good an approximation Molière's density function represents. In order to obtain a rough criterium, we have calculated the mean square radial moment of Molière's distribution, cutting it off at about two radiation lengths, from where on he has a pure $1/r^2$ law. The result is an $X^2(E)=1.14(E_s/E_j)^2$, while our value for a lower energy cut-off of 4 Mev has the numerical factor 0.64. Molière thus overestimates somewhat the extension of showers and his curve has more qualitative than quantitative significance. The theoretical interpretation of experimental data on large showers is, however, inherently of very rough character. Use of Molière's results will thus probably not lead to a seriously distorted picture.

APPENDIX

The Bhabha-Heitler formula and evaluation of the scattering integrals. Bhabha and Heitler[13] (abbreviated as B.H.) use for the probability of emission of a quantum K by an electron E the formula

$$\sigma(E, K)dKdl=\log 2(dK/K)dl. \quad \text{(A1)}$$

Their unit of length is thus connected with the one used here through

$$l \log 2=t. \quad \text{(A2)}$$

They write for the probability of pair production in dl

$$\sigma(K)dl=\alpha dl. \quad \text{(A3)}$$

Their value for the absorption coefficient of photons is then

$$\alpha=(7/9)\log 2=\delta \log 2. \quad \text{(A4)}$$

B.H. express their energies by a logarithmic variable

$$y=\log(E'/E), \quad \text{(A5)}$$

where E' is an initial and E a final energy. The total number of electrons plus positrons produced by a primary E' having an energy $\geqslant E$ at a depth l is then expressed in a series

$$F(E', E, l)=F(y, l)=F_0+\sum_{n=1}^{\infty} F_n. \quad \text{(A6)}$$

F_0 is the probability that the primary electron has arrived with an energy $\geqslant E$. It is given by the "straggling function" $W(l, y)$ of B.H. (Eq. (6) or Eq. (34) of reference 12).

$$F_0(y, l)=\int_0^y (e^{-\eta}\eta^{l-1}/\Gamma(l))d\eta. \quad \text{(A7)}$$

The terms F_n give the numbers of electrons at l with energy $\geqslant E$ which had been produced from the primary with n intermediate quanta. According to B.H.'s formula (reference 13, top of p. 442), it can be written as

$$F_n(y, l)=(2\alpha \log 2)^n e^{-\alpha l}\int_0^l dl' \times\int_0^{l-l'} \frac{dl'' e^{\alpha(l'+l'')}l''^n(l-l'-l'')^{n-1}}{(n-1)!(n-1)!} \times\int_0^y \frac{(y-y')^{n-1}}{(n-1)!}F_0(y', l'+l''+n)dy'. \quad \text{(A8)}$$

In comparing this expression with the one in B.H., it is to be noted that our F_n is twice the B.H. f_n which represents the number of electrons of one charge only. Further, the formula quoted above (B.H., p. 442) represents f_{n+1}. Tracing the deviation of this formula back to B.H., Eq. (21), and the developments on p. 440, one verifies that the variable l' in Eq. (A8) denotes the distance an electron of the n'th generation has traveled as an individual after it had been created by the immediate ancestor photon. In order to express this dependence explicitly, we write

$$F_n(y, l)=\int_0^l H_n(y, l', l)dl', \quad n\geqslant 1. \quad \text{(A9)}$$

For $n=0$, i.e., the straggling of an individual electron, l' is, of course, equal to the total path l, and we write

$$H_0(y, l', l)=F_0(y, l')\delta(l-l'). \quad \text{(A10)}$$

The scattering integrals. In Section 4 we had introduced the differential distribution function of electrons

$$f_n(E', E, t)=\partial F_n(E', E, t)/\partial E.$$

†† In the evaluation of the mean square as in this paper the single scattering is included through proper choice of the scattering constant E_s.

In order to take into account ionization losses, we approximated the f_n by Eq. (4.6)

$$f_n(E', E, t)_{\rm corr} = \int_0^t h_n(E', E+\beta t', t', t)dt'.$$

Here $h_n(E', E'', t', t)$ is the probability under neglection of radiation losses that an electron of the n'th generation arrived at t with the energy E'' after having been created the distance t' backwards. The functions h_n are given by

$$h_n(E', E'') = \partial H_n/\partial E'' = (1/E'')(\partial H_n/\partial y). \quad \text{(A11)}$$

The identification $E = E'' - \beta t$ gives then the modification of the distribution by ionization losses.

The mean square scattering deviation requires the evaluation of the integrals

$$E^{-(s+2)}\,{}^sJ_m^n = \int_0^\infty t^m dt \int_0^t dt' \times \int_{E+\beta t}^\infty (dE'/E'^{s+2}) h_n(E', E+\beta t', t', t), \quad \text{(A12)}$$

where m is 0, 1 or 2, s is 2 or 3, and n goes from 0 to ∞. Upon introduction of the unit length, Eq. (A2), we obtain

$${}^sJ_m^n = (\log 2)^{m+1} \int_0^\infty l^m dl \int_0^l dl' \times \int_0^\infty [e^{-(s+1)y}(\partial H_n/\partial y_n)dy/(1+(\beta l'/E)\log 2)^{s+2}], \quad \text{(A13)}$$

where the H_n are given by Eqs. (A7) to (A10). The expressions (A13) are sixfold integrals. They can be evaluated by one of the so-called "simple calculations" by suitable changes of the order of integration and introduction of new variables.††† The results are as follows:

$${}^sJ_m^0 = (m!/2^{m+1}) b\, {}^sK_m, \quad (m \text{ any value}),$$

$${}^sJ_0^n = [b^{n+1}/2[(s+1)(s+2)]^n]\, {}^sK_0, \quad n = 1, 2, \cdots,$$

$${}^sJ_1^n = \frac{b^{n+1}}{2[(s+1)(s+2)]^n} \left\{ n\left(\frac{1}{\delta} + \frac{b}{2}\right) {}^sK_0 + \frac{b}{2}\, {}^sK_1 \right\},$$

$${}^sJ_2^n = \frac{b^{n+1}}{2[(s+1)(s+2)]^n} \times \left\{ \left(\frac{n(n+1)}{\delta^2} + \frac{n^2 b}{\delta^2} + \frac{n(n+1)}{4} b \right) {}^sK_0 + bn\left(\frac{1}{\delta} + \frac{1}{2}\right) {}^sK_1 + \frac{b}{2}\, {}^sK_2 \right\}, \quad \text{(A14)}$$

where

$${}^sK_m(E) = \frac{1}{m!} \int_0^\infty \frac{e^{-x} x^m dx}{[1+(x\beta b/2E)]^{s+2}}, \quad \text{(A15)}$$
$$b = \log 4/\log(s+2).$$

All the $K(E)$ go to unity in the limit $E \to \infty$. The ${}^2J_m^n$ reduce for $\beta = 0$ to the I_m^n given in Eq. (3.6). The integrals ${}^sK_m$ can be evaluated for large E by expansion of the denominator which leads to a semiconvergent series. They can also be expressed in terms of the exponential integral and thus be computed. Values are given in Table V.

The summations over n can be carried out in closed form with the help of the summation formulas

$$\sum_0^\infty z^n = 1/(1-z); \quad \sum_1^\infty n z^n = z/(1-z)^2;$$
$$\sum_1^\infty n^2 z^n = z(z+1)/(1-z)^3. \quad \text{(A16)}$$

We denote the sums so obtained by

$${}^sJ_m = \sum_{n=0}^\infty {}^sJ_m^n. \quad \text{(A17)}$$

The expressions of these quantities in terms of the integrals K follow.

For $s = 2$:
$$\begin{aligned} {}^2J_0 &= 0.545\, {}^2K_0, \\ {}^2J_1 &= 0.273\, {}^2K_1 + 0.0885\, {}^2K_0, \\ {}^2J_2 &= 0.273\, {}^2K_2 + 0.0885\, {}^2K_1 + 0.281\, {}^2K_0. \end{aligned}$$
For $s = 3$: (A18)
$$\begin{aligned} {}^3J_0 &= 0.450\, {}^3K_0, \\ {}^3J_1 &= 0.194\, {}^3K_1 + 0.0348\, {}^3K_0, \\ {}^3J_2 &= 0.167\, {}^3K_2 + 0.030\, {}^3K_1 + 0.100\, {}^3K_0. \end{aligned}$$

††† See J. Roberg, Ph.D. Thesis, Duke University, 1942, for all details of this and subsequent calculations.