Stars in Photographic Emulsions Initiated by Deuterons.

Part II. Theoretical

W. HORNING AND L. BAUMHOFF
Radiation Laboratory, Department of Physics, University of California, Berkeley, California
(Received September 27, 1948)

The theory of high energy nuclear stars depends on a theory of nuclear transparency and on a theory of nuclear evaporation. The transparency can be computed on the basis of a model proposed by R. Serber as soon as the interactions between the nucleons and the incident particle are known. The evaporation can be computed on the basis of the statistical model of the nucleus as soon as the nuclear entropy and binding energies of the evaporated particles are known. With approximate values for the above interactions, entropies, and binding energies, a probability distribution has been computed for the number of prongs per star. The results are in qualitative agreement with the observations on photographic emulsions described in Part I.

I. INTRODUCTION

It is of interest to see how far current ideas on nuclear mechanics lead toward an interpretation of nuclear stars. If the kinetic energy of the incident heavy particle is of the order of 100 Mev, then resonance phenomena and leakage through the nuclear potential barrier should be small effects. In such a case, particles may be ejected from the struck nucleus by at least two processes—direct recoil and evaporation. If a struck nucleon receives so much energy from the incident particle that it leaves the nucleus in a time short compared to a nuclear period ($\sim 10^{-21}$ sec.), then we have a recoil prong or recoil star. If, on the other hand, the energy of the incident particle is evenly distributed throughout the nucleus by many collisions of the nucleons among one another, and if particles are ejected only after many nuclear periods, then we have an evaporation star. Cosmic-ray stars appear to be of both these types.[1]

The formation of an evaporation star depends on nuclear evaporation proper, and also on nuclear transparency, which concerns the probability that an incident particle pass through a nucleus with the loss of only some fraction of its energy. Each of these parts of the theory depends on an elementary probability: transparency on the probability $Y(E, \epsilon)$ per cm path distance per Mev of energy loss that an incident particle with kinetic energy E collides with a nucleon, losing kinetic energy ϵ; evaporation on the probability $W(X, \xi)$ per second per Mev energy loss that a nucleus with excitation energy X evaporate some particle or other, losing excitation ξ.

[1] W. Heisenberg, *Cosmic Radiation* (Dover Publications, New York), p. 131.

II. THE ELEMENTARY PROBABILITIES

Unless E_0, the incident energy, is large compared to the average energy loss per collision of the incident particle ($\sim$25 Mev), the incident particle will almost certainly be unable to penetrate through the nucleus, and there will be almost zero transparency. But if E_0 is many times this critical value, things may be more complex. To get an idea of what happens, assume the stopping power of the nucleus to be like that of a Fermi gas of protons and neutrons, so that when a particle strikes it the resulting action may be analyzed into a series of $n-n$, $n-p$, and $p-p$ collisions.[2]

We represent the interaction between the incident proton or neutron and an average particle of the nucleus by $g^2e^{-r/a}/r$. A value $a^{-1}=1.19\cdot 10^{13}$ cm^{-1} is used.[3] A value of g^2 is used corresponding to an average total cross section of 0.058 barn; since total cross sections for $n-n$ and $n-p$ scattering have been measured with the results 0.034 and 0.083 barn at 90 Mev incident energy.[4] This yields $g^2/\hbar c=0.26$. The use of the above average of the $p-n$ and $n-n$ cross sections is a

[2] For the physical model here involved see R. Serber, Phys. Rev. **72**, 115 (1947).
[3] L. E. Hoisington, S. S. Share, and G. Breit, Phys. Rev. **56**, 884 (1939).
[4] Cook, McMillan, Peterson, and Sewell, Phys. Rev. **72**, 1264 (1947).

reasonable approximation whether the $p-n$ interaction is of the ordinary or exchange type. It is a better approximation for an interaction of exchange type.

The Born approximation with the above interaction gives for the differential scattering cross section

$$\sigma(E_0, \rho)d\rho = \pi m b^4 g^4 \rho d\rho / E_0(1+b^2\rho^2)^2 \quad (1)$$

where ρ is the momentum transfer, and E_0 is the energy of the incident particle in the reference system in which the struck particle is at rest, and $b=a/\hbar$. This result should be modified when applied to collisions within the nucleus, since then the struck particle is not at rest, and the exclusion principle limits possible momentum transfers. This modified cross section $\tau(E, \epsilon)$, is obtained by averaging (1) over a portion of the Fermi momentum sphere, whose radius we call p_0. That is

$$\tau(E, \epsilon)d\epsilon = d\epsilon \int \sigma(\mathbf{P}, \mathbf{p}, \epsilon)d\mathbf{p} \Big/ \frac{4}{3}\pi p_0^3. \quad (2)$$

Here p and p' are the original and final momenta of the struck nucleon, P and P' the corresponding momenta of the incident particle, E and ϵ the incident energy and energy transfer, all in the laboratory system. It is convenient to introduce $\mathbf{q}$, the momentum of the incident particle, after collision of the two particles, in their center of mass system. By definition

$$\mathbf{q} = -\rho + \tfrac{1}{2}(\mathbf{P}-\mathbf{p}). \quad (3)$$

Conservation of energy and momentum require that

$$q^2 = \tfrac{1}{4}(\mathbf{P}-\mathbf{p})^2; \quad (4)$$

and the condition that the energy transfer be ϵ is

$$[\mathbf{q}+\tfrac{1}{2}(\mathbf{P}+\mathbf{p})]^2 = P'^2 = P^2 - 2m\epsilon. \quad (5)$$

To get the cross section $\sigma(\mathbf{P}, \mathbf{p}, \epsilon)d\epsilon$ for energy loss in range $d\epsilon$, for given $\mathbf{P}$ and $\mathbf{p}$, requires an integration around the azimuth φ', whose axis is the line of centers of the two spheres (4) and (5) in $\mathbf{q}$ space. The direction of this axis is that of $\mathbf{p}+\mathbf{P}$. From (4) and (5)

$$\rho^2 = \tfrac{1}{2}(\mathbf{P}-\mathbf{p})^2[1-\cos(\mathbf{q}, \mathbf{P}-\mathbf{p})].$$

We write

$$\theta' = \sphericalangle(\mathbf{q}, \mathbf{P}+\mathbf{p}), \quad \theta''(\mathbf{P}-\mathbf{p}, \mathbf{P}+\mathbf{p})$$
$$\cos(\mathbf{q}, \mathbf{P}-\mathbf{p}) = \cos\theta' \cos\theta'' + \sin\theta' \sin\theta'' \cos\varphi'.$$

In contrast to the more familiar case in which $p=0$, ρ here involves the azimuth. Expanding (5) according to the law of cosines gives

$$\cos\theta' = \frac{P^2-p^2-4m\epsilon}{|\mathbf{P}-\mathbf{p}|\,|\mathbf{P}+\mathbf{p}|}, \quad \sin\theta' d\theta' = \frac{4md\epsilon}{|\mathbf{P}-\mathbf{p}|\,|\mathbf{P}+\mathbf{p}|}.$$

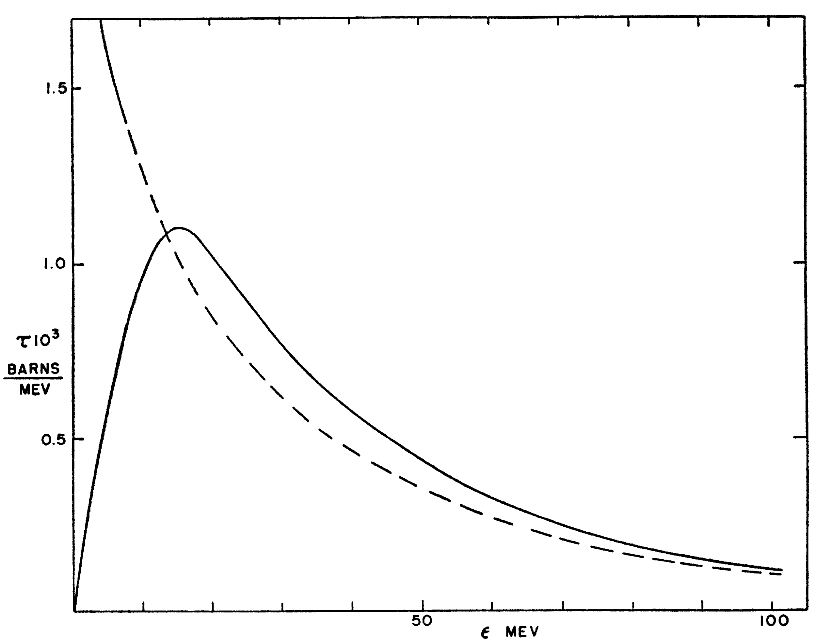

FIG. 1. Differential cross section τ for energy transfer in a collision between an incident particle of energy 100 Mev and a nucleon.

Dotted curve: struck particle free and at rest.
Solid curve: struck particle within a nucleus.

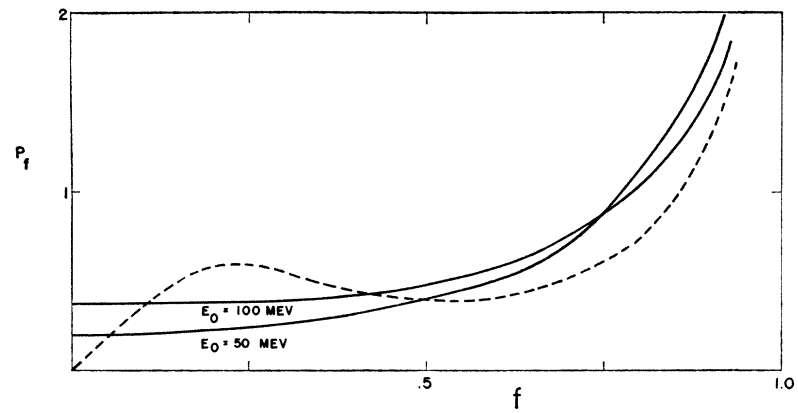

FIG. 2. $P(f)df$ is the probability that an incident nucleon with energy E_0 just after entering a nucleus of the heavy component, will have lost a fraction f of its energy in range df, just before completing its path through the nucleus.

Applying the law of cosines to the identity $2\mathbf{p}=(\mathbf{P}+\mathbf{p})-(\mathbf{P}-\mathbf{p})$ gives

$$\cos\theta''=\frac{P^2-p^2}{|\mathbf{P}-\mathbf{p}|\,|\mathbf{P}+\mathbf{p}|}.$$

Substituting in (1) and performing the elementary azimuthal integration yields

$$\begin{aligned}\sigma(\mathbf{P},\mathbf{p},\epsilon)&=2\pi m^3b^4g^4(A+BC)\\&\quad\times(A+2BC+B^2[4m\epsilon]^2)^{-\frac{3}{2}}\\A&=(\mathbf{P}-\mathbf{p})^2(\mathbf{P}+\mathbf{p})^2,\quad B=\tfrac{1}{2}b^2(\mathbf{P}-\mathbf{p})^2\\C&=A+(p^2-P^2+4m\epsilon)(P^2-p^2).\end{aligned}\tag{6}$$

We now substitute from (6) in (2), after expanding σ as a power series in p/P, less than one in the experiments here of interest. The integration of (2) is to be done over the spherical shell $p<p_0$, $p'>p_0$. This last inequality may be written

$$p'^2-p^2=2m\epsilon>p_0^2-p^2$$

or

$$p>(p_0^2-2m\epsilon)^{\frac{1}{2}}.$$

If $2m\epsilon>p_0^2$, then the region of integration is instead $p<p_0$, $p>0$. The angular integrals of terms in odd powers of p/P are 0. Keeping the first three non-vanishing terms in the expansion of τ yields the result shown in Fig. 1. The elementary probability $Y(E,\epsilon)$ is simply τ times d, the numerical density of nucleons within the nucleus.

The elementary probability for evaporation has been treated by Weisskopf.[5] His Eq. (3) may be put

$$\begin{aligned}W(X,\xi)d\xi&=\gamma(\xi-V-E_b)\\&\quad\times\exp(S_B(X-\xi)-S_A(X))d\xi,\\\gamma&=\sigma_0 jm/\pi^2\hbar^3,\end{aligned}\tag{7}$$

where S_A and S_B are the entropies of the nucleus before and after the evaporation of a particle with binding energy E_b, kinetic energy $\xi-E_b$, potential barrier V, mass m, and statistical weight j; and where σ_0 is the geometrical cross section of the nucleus. On the basis of an approximate nuclear model we may write

$$S(X)\simeq\left(\frac{A}{2.2}\right)^{\frac{1}{2}}X^{\frac{1}{2}},\quad X\text{ in Mev},$$

where A is the mass number.

III. NUCLEAR TRANSPARENCY*

In the experiments of Part I, the elements of the photographic emulsion fall principally into two groups: a heavy component composed of silver and bromine, with an average mass number of 95, and a light component composed principally of carbon and oxygen. Since the hydrogen of the emulsion has too simple a structure to give rise to starts of the kinds observed, it is counted in neither component. By use of the chemical analysis listed in Part I, and values for the geometrical cross sections of the nuclei involved, we deduce that the relative probabilities that a fast incident particle hit a nucleus of the heavy component, of carbon, or of oxygen

[5] V. Weisskopf, Phys. Rev. **52**, 296 (1937).

* The problem of nuclear transparency has been treated, using a different method, by M. L. Goldberger, Phys. Rev. **74**, 1269 (1948).

are 0.76, 0.13 and 0.11. A calculation will be made for the stars from the heavy component, which are predominately evaporation stars. Some remarks will then be made about stars from the lighter elements.

As an approximation, let us ignore the scattering of the incident particle, but take account of its energy loss by the quantity $\tau(E, \epsilon)$ which expresses the probability per cm path distance per Mev energy loss, that the incident particle, with energy E, lose energy ϵ. Let $P(x, E)$ be the probability, after the incident particle has penetrated a distance x into the nucleus that it possess a kinetic energy less than E. If, at $x=x_1$, the energy E of the incident particle were known to be E_1, then the increment of P would satisfy

$$[\Delta P(x, E)]_{x=x_1}=\Delta x\int_{E_1-E}^{E_1}\tau(E_1,\epsilon)d\epsilon.$$

Since, however, the quantity E follows a probability distribution, we have in general

$$\frac{\partial P(x,E)}{\partial x}=\int_E^{E_0}\frac{\partial P(x,E')}{\partial E'}dE'\int_{E'-E}^{E'}\tau(E',\epsilon)d\epsilon. \quad (8)$$

This equation has been solved numerically, with τ as given in Fig. 1, and $E_0=100$ Mev. The result is shown by the dotted curve in Fig. 2. To make possible an analytic solution we replace τ by τ', where

$$\tau'\equiv\frac{Ae^{-b\epsilon}}{1-e^{-bE'}},\quad \int_{E'-E}^{E'}\tau(E',\epsilon)d\epsilon=By'/y$$

$$B\equiv A/b,\quad y=y(E)=e^{-bE}/(1-e^{-bE}),\quad y'=y(E').$$

The two functions τ and τ' are equivalent in a semi-quantitative way except for $\epsilon<10$ Mev, if we put $b=0.021$ (Mev)$^{-1}$ and $B=3.6\cdot10^{12}$ cm^{-1}. In view of the uncertainties in the Fermi model of the nucleus, we may regard τ' as a reasonable representation of the scattering cross section.

The advantage of the function τ' is that it yields a kernel for the integral equation which is a function of E only times a function of E' only. The Laplace transform of (8) is

$$-P(0,y)+\lambda\bar{P}=\int_y^{y_0}\frac{\partial\bar{P}}{\partial y'}dy'B\frac{y'}{y}$$

$$\bar{P}=\int_0^\infty e^{-\lambda x}Pdx. \quad (9)$$

The boundary condition at $x=0$ is $P(0,y)=st(y_0-y)$, where $st\ (x)$ is a step function defined by $st(x)=0, 1$ according as $x<0$ or $x>0$. Here $y_0=y(E_0)$ where E_0 is the energy of the incident particle within the nucleus and at the start of its nuclear traverse. Multiplying (9) by y and differentiating to y yields

$$(\lambda+B)y(\partial\bar{P}/\partial y)+\lambda\bar{P}=st(y_0-y)-y\delta(y_0-y). \quad (10)$$

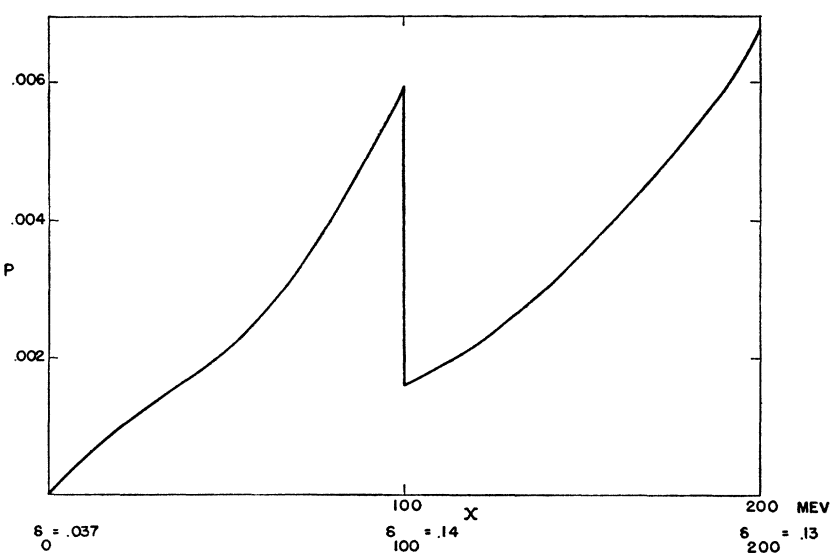

FIG. 3. $P(X)dX$ is probability of a nuclear excitation X, in range dX, when a nucleus of the heavy component is hit by a deuteron of energy 200 Mev.

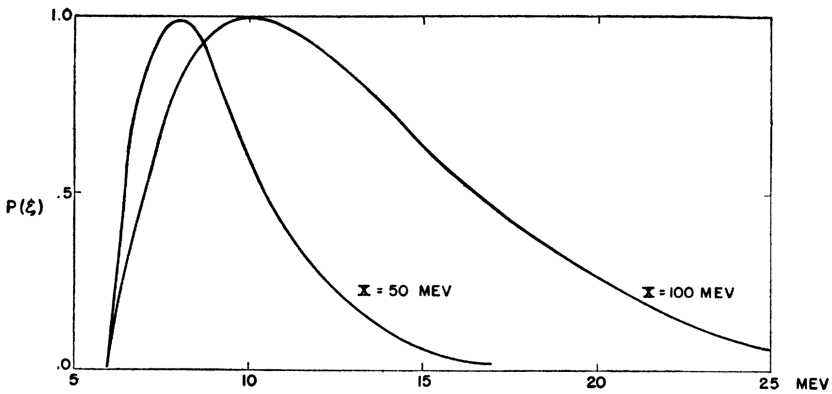

FIG. 4. $P(X, \xi)d\xi$ is proportional to the probability that a proton, evaporated from a heavy component nucleus of excitation X, reduces this excitation by an amount ξ in range $d\xi$.

A solution of the homogeneous equation corresponding to (10) is

$$P_1 = y^{(-\lambda/\lambda+B)}.$$

A solution of the inhomogeneous equation is

$$P_2 = \frac{1}{\lambda}, \quad y < y_0;$$

$$P_2 = \left(\frac{1}{\lambda} - \frac{1}{\lambda+B}\right)\left(\frac{y_0}{y}\right)^{(\lambda/\lambda+B)}, \quad y < y_0.$$

The general solution is $\bar{P} = P_2 + CP_1$. In (9) put $y = y_{0+}$ and substitute for $\bar{P}$. The result is

$$\lambda\left[\frac{1}{\lambda} - \frac{1}{\lambda+B} + C\left(\frac{1}{y_{0+}}\right)^{(\lambda/\lambda+B)}\right] = \frac{B}{\lambda+B}$$

from which $C = 0$. The solution satisfying boundary conditions, for $y > y_0$ is thus

$$\bar{P} = \frac{B}{\lambda(\lambda+B)}\left(\frac{y_0}{y}\right)^{(\lambda/\lambda+B)}$$

The inversion theorem for the Laplace transform yields

$$P = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda x}\frac{B}{\lambda(\lambda+B)}\left(\frac{y_0}{y}\right)^{(\lambda/\lambda+B)} d\lambda.$$

The contour may be deformed into two infinitesimal circuits, one around the pole at $\lambda = 0$, the other around the essential singularity at $\lambda = -B$. The contribution from the pole is simply 1.

To get information on nuclear excitation, we need to evaluate P at $x = x_0$, the thickness of the nucleus, and then average over impact parameters. If R is the nuclear radius and r the impact parameter, then

$$x_0 = 2(R^2 - r^2)^{\frac{1}{2}}, \quad 4rdr = -x_0 dx_0.$$

The desired average is then

$$P(E, R) = \frac{1}{\pi R^2}\frac{1}{4i}\oint\left(\frac{2R}{\lambda}e^{2R\lambda} - \frac{e^{2R\lambda}}{\lambda^2} + \frac{1}{\lambda^2}\right) \times \frac{B}{\lambda(\lambda+B)}\left(\frac{y_0}{y}\right)^{(\lambda/\lambda+B)} d\lambda$$

$$\equiv (p_1 + p_2 + p_3)/\pi R^2.$$

To evaluate the integral p_1, put $\lambda = -B + t$. Then

$$p_1 = \frac{2RB}{4i}e^{-2RB}\frac{y_0}{y}\oint\frac{\exp(2Rt - Bt^{-1}\log(y_0/y))dt}{t(t-B)^2}.$$

Put $t = (a/2R)^{\frac{1}{2}}\tau$ and $a = B\log(y/y_0)$, greater than 0 in the range of y of interest, and get

$$p_1 = \frac{2RB}{4i}e^{-2RB}\left(\frac{y_0}{y}\right)\oint\frac{\exp((2aR)^{\frac{1}{2}}(\tau + \tau^{-1}))d\tau}{\tau[(a/2R)^{\frac{1}{2}}\tau - B]^2}.$$

If $(a/2R)^{\frac{1}{2}} < B$, then expand the denominator in a binominal series, obtaining

$$p_1 = \frac{2RB}{4i}e^{-2RB}\left(\frac{y_0}{y}\right)\oint\frac{\exp((2aR)^{\frac{1}{2}}(\tau + \tau^{-1}))d\tau}{B^2} \times\left[\frac{1}{\tau} + 2\left\{\frac{1}{B}\left(\frac{a}{2R}\right)^{\frac{1}{2}}\right\} + \frac{2\cdot 3}{1\cdot 2}\left\{\frac{1}{B}\left(\frac{a}{2R}\right)^{\frac{1}{2}}\right\}^2\frac{1}{\tau} + \cdots\right].$$

Each term in this series involves a modified

Bessel function, according to the standard formula

$$I_n(z)=\frac{1}{2\pi i}\oint\frac{\exp\left(\frac{z}{2}(\tau+\tau^{-1})\right)d\tau}{\tau^{n+1}}.$$

The series, which converges with moderate rapidity if $(a/2R)^{\frac{1}{2}}<B$ may now be used to get the numerical value of p_1. If $(a/2R)^{\frac{1}{2}}>B$, the similar series obtained by using the alternate binominal expansion of the denominator is used. The evaluation of p_2 is similar to that of p_1. Evaluating p_3 involves only elementary functions.

Let $\mathcal{P}(f)df$ be the probability that the incident particle lose a fraction f of its incident energy E_0 in range df, during a nuclear traverse. Then

$$\mathcal{P}(f)=E_0\frac{\partial}{\partial E_0}P(E_0-E, R).$$

In Fig. 2, $\mathcal{P}(f)$ is plotted for mass number 95, and for incident energies of 100 and 50 Mev. When the kinetic energy of the incident particle becomes about $E_F=22$ Mev, the depth of the Fermi potential well, it can then no longer emerge from the nucleus. Thus the continuous curve $\mathcal{P}(f)$ should be corrected by making it 0 from $f=(E_0-E_F)/E_0$ to $f=1$; and by adding to it a δ-function at $f=1$, with a coefficient equal to the area under $\mathcal{P}(f)$ between $f=(E_0-E_F)/E_0$ and $f=1$.

To get the corresponding result for deuteron bombardment, we introduce $\bar{r}=1.7\cdot10^{-13}$ cm, the average of the projected distance between the two deuteron particles on a plane perpendicular to the incident beam. This value is easily computed with an approximate deuteron wave function of form $e^{-\alpha r}/r$. If we assume the two deuteron particles always have this average projected spacing, then when one of these particles hits the nucleus, the other must do so also, unless the first hits the nucleus within a projected distance $\bar{r}$ of the edge. On the average, when one of the particles hits the nucleus with impact parameter such that the second has a non-zero chance of not hitting, the first will hit at distance $\bar{r}/2$ from the edge; then the chance of the second not hitting will be $\sim\frac{1}{3}$, if $\bar{r}<R$, R the nuclear radius. The probability of one particle hitting the nucleus and the other not hitting is thus

$$\frac{2}{3}\frac{R^2-(R-\bar{r})^2}{R^2}$$

which for a nucleus of the heavy component is 0.30. Let $P_1(E_0, \epsilon)$ be the distribution function for energy transfer ϵ when only one particle of the deuteron hits the nucleus, E_0 being the incident energy of such a particle. If $P_2(E_0, \epsilon)$ is the corresponding function when both particles of the deuteron hit, then

$$P_2(E_0, \epsilon)=\int_0^{\epsilon}P_1(E_0, \epsilon_1)P_1(E_0, \epsilon-\epsilon_1)d\epsilon_1.$$

And then the corresponding distribution, given simply that a deuteron collides with the nucleus, will be $P_d=0.3P_1+0.7P_2$. A plot of P_d is given

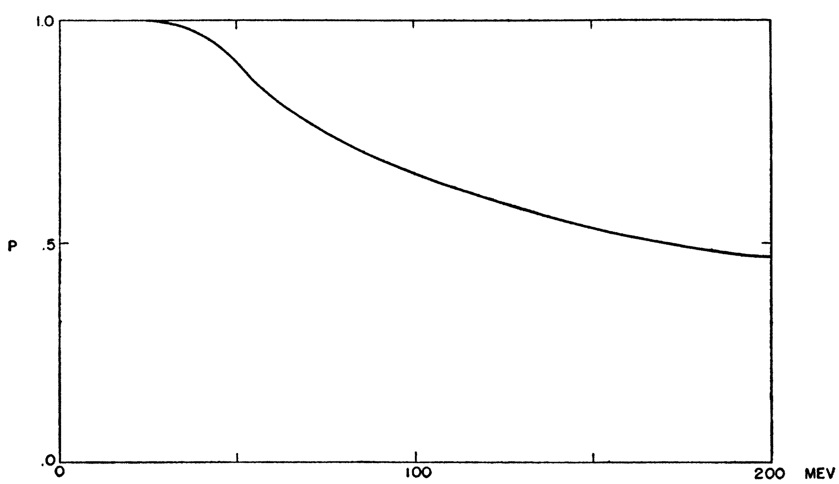

FIG. 5. $P_p(X)$ is probability that an ion evaporated from a nucleus of the heavy component be a proton, when the nucleus is at excitation X.

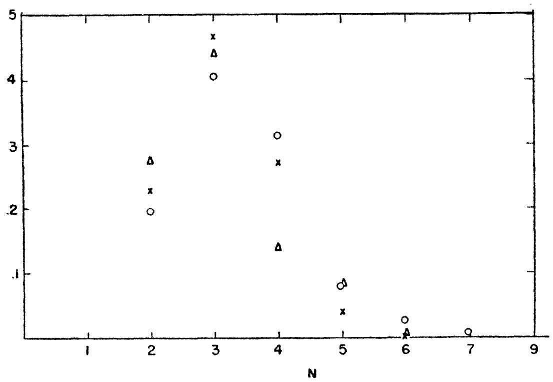

FIG. 6. The ordinate is the probability that a nuclear star contain N prongs formed by the paths of charged particles. The crosses give the prong distribution in the photographic emulsion as observed in the experiments of Part I. The triangles give the distribution observed in cloud chamber filled with air and alcohol vapor, with a neutron beam of 95 Mev. The circles give the theoretical distribution for stars from the heavy component of the emulsion produced by a deuteron beam of 200 Mev.

in Fig. 3. The rather striking discontinuity in this curve arises from the strongly discontinuous mass distribution within the deuteron.

IV. NUCLEAR EVAPORATION

To apply formula (7) to the heavy component of the photographic emulsion ($A=95$), binding energies for the neutron proton, alpha particle and deuteron are needed. According to the Weizsäcker formula,[6] they are

$$E_n=8.0,\quad E_p=8.0,\quad E_\alpha=4.0,\quad E_d=13.3 \text{ Mev.}$$

The Coulomb barrier for the proton and deuteron is taken to be $V=6.2$, for the alpha particle $2V$. The least change in the excitation energy X of a nucleus when it evaporates a particle is, in the approximation for large X, the sum of the binding energy and Coulomb energy for that particle, or the "threshold energy" for the particle. This threshold energy for a given type of evaporated particle varies when the parent nucleus, as a result of evaporation, travels down the Heisenberg valley. It is reasonable to suppose that the excited nucleus will usually not wander more than about three steps from the bottom of the valley, in which case these variations in the thresholds will be a secondary, though perhaps not entirely negligible effect, at least for a heavy nucleus. In what follows, these variations will be neglected. The formula (7) yields a finite probability for the evaporation of particles even heavier than the alpha particle. But even for 200 Mev excitation of the heavy component, these probabilities are small and will be neglected.

Gardner and Peterson[7] have observed stars with two, three, and more visible prongs, each prong consisting of the track of an evaporated ion; they have measured the relative frequencies of these types of stars. To compute these frequencies, we introduce a function $P(\nu, X)$, the probability that the excited nucleus will have emitted ν or more prongs when, as a result of evaporation, its excitation has fallen to a value X from an original value $X_0>X$. Let $q(X)$ be the probability per unit drop in excitation that the nucleus emit an ion. Then P will satisfy the equation

$$\frac{\partial P(\nu+1, X)}{\partial X}=[P(\nu+1, X)-P(\nu, X)]q(X).$$

Or, putting

$$Q=\int_0^X q(X)dX$$

$$\frac{\partial P(\nu+1, Q)}{\partial Q}=P(\nu+1, Q)-P(\nu, Q). \qquad (11)$$

The use of a continuous function $q(X)$, to describe evaporation is justified only if the excitation is many times the threshold energy for evaporation of a particle. This condition is quite well satisfied when the excitation is of the order of 200 Mev. We write

$$(1/q(X))=\xi_i+(p_n/p_i)\xi_n \qquad (12)$$

where ξ_i is the average drop in excitation per evaporated ion, at excitation X, ξ_n the average drop per evaporated neutron; p_n and p_i are the relative probabilities for neutron and ion evaporation at excitation X. The quantities on the right of (12) are easily calculated from (7).

To solve (11), we note that $P(0, Q)=1$. Hence

$$e^{-Q}P(1, Q)=e^{-Q}+C_1.$$

The constant of integration C_1 is fixed by re-

[6] H. A. Bethe, Rev. Mod. Phys. 8, 165 (1936).

[7] E. Gardner and V. Peterson, Phys. Rev. 75, 364 (1949).

quiring that $P(1, X)=0$ when $X=X_1\equiv X_0-T_p$ where T_p is the least of the ion thresholds, and X_0 the original excitation. That is $C_1=-e^{-Q_1}$, $Q_1\equiv Q(X_1)$. Similarly

$$e^{-Q}P(2, Q)=e^{-Q}-C_1Q+C_2,$$

$$C_2=-e^{-Q_2}+C_1Q_2,$$

$$Q_2=Q(X_0-2T_p)$$

$$e^{-Q}P(3, Q)=e^{-Q}+C_1\frac{Q^2}{2}-C_2Q+C_3,$$

$$C_3=-e^{-Q_3}-C_1\frac{Q_3^2}{2}+C_2Q_3,$$

$$Q_3=Q(X_0-3T_p).$$

. . . .

The probability of evaporating just ν prongs is

$$p_\nu=P(\nu, Q_T)-P(\nu+1, Q_T), \quad Q_T=Q(T_p). \quad (13)$$

The mean prong number $\bar{\nu}=\Sigma\nu p_\nu$ turns out to be 4.0 at $X=200$ Mev.

A more accurate method of finding $\bar{\nu}$ involves the use of the function $P(t, X)$, which is the probability at time t, after the nucleus is struck, that it have excitation X or less. Let

$$W(X, \xi)=W_n+W_p+W_\alpha+W_d$$

be the sum of the function (7) for the four types of evaporated particles. Then

$$\frac{\partial P(t, X)}{\partial t}=\int_X^{X_0}\frac{\partial P(t, X')}{\partial X'}dX'\int_{X'-X}^{X'}W(X', \xi)d\xi.$$

The complicated form of the kernel makes difficult an analytic solution. Differentiating this equation yields

$$\frac{\partial p(t, X)}{\partial t}=-p(t, X)\int_0^X W(X, \xi)d\xi$$

$$+\int_X^{X_0}p(t, X')dX'W(X', X'-X),$$

$$p(t, X)=\frac{\partial}{\partial X}P(t, X). \quad (14)$$

Multiply (14) by $e^{-\lambda t}$ and integrate between $t=0$ and $t=\infty$. Write

$$\bar{p}(\lambda, X)=\int_0^\infty p(t, X)e^{-\lambda t}dt,$$

$$-p(0, X)+\lambda\bar{p}(\lambda, X)=\int_0^\infty\frac{\partial p(t, X)}{\partial t}e^{-\lambda t}dt.$$

Then to solve the resulting equation numerically break the interval from 0 to X_0 into a number of equal parts $(0, X_n)$, $(X_n, X_{n-1})\cdots(X_1, X_0)$; and take $\bar{p}(\lambda, X)$ as an unknown constant in each such sub-interval except (X_1, X_0), in which take $\bar{p}(\lambda, X)=1$. Then the integral equation breaks up into n algebraic equations which are easy to solve successively, since only those values of the kernel $W(X', X'-X)$ enter into the equation for which $X'>X$. The principal labor is in tabulating the function $W(X', X'-X)$. The method yields $\bar{p}(\lambda, X)$ as a histogram in X.

If $N(t)$ is the expected number of ions evaporated between the instant the nucleus is hit and time t

$$\frac{dN(t)}{dt}=\int_0^{X_0}p(t, X')dX'\int_0^{X'}W_i(X', \xi)d\xi$$

where $W_i(X_1\xi)=W_p+W_\alpha+W_d$. Then

$$\lambda\bar{N}(\lambda)=\int_0^{X_0}\bar{p}(\lambda, X')dX'\int_0^{X'}W_i(X', \xi)d\xi$$

which yields $\lambda\bar{N}(\lambda)$ as the result of a quadrature. But

$$\bar{\nu}=\lim_{\lambda\to 0}\lambda\bar{N}(\lambda).$$

When $X_0=200$ Mev and the interval $(0, X_0)$ is broken into 30 parts, the numerical method just described yields

$$\bar{\nu}=4.4.$$

The distribution (13) thus yielded a mean prong number about 10 percent too small. The distribution will hence be improved by multiplying by a scale factor and renormalizing so as to get a mean 4.4.

To get a result to compare with experiment, the distribution (13) must now be evaluated for $X_0=150$, 100 and 50 Mev as well as 200 Mev. The results for various X_0's are then weighted according to the graph of Fig. 3.

We may crudely represent the insensitivity of the emulsion to high energy protons by assuming that it detects all protons with energies below 15 Mev, but none with energies above this value. To subtract from the computed prong number distribution the effect of unobserved protons, the energy distribution of the evaporated protons is needed, shown for a special case in Fig. 4. For this is needed also the proportion of all evaporated ions which are protons, shown in Fig. 5. After a correction for this emulsion insensitivity to protons, the calculated prong number distribution for the heavy component, at 190 Mev bombarding energy, becomes as shown by the circles in Fig. 6.

It is doubtful whether a simple statistical theory can be applied to the evaporation of ions from a highly excited nucleus as light as carbon or oxygen. Fortunately, information on this process has been obtained by cloud-chamber measurements.[8] Of 499 stars produced by the 90 Mev neutron beam in a chamber filled with water vapor and alcohol, the numbers with 2, 3, $\cdots$ prongs were as follows:

2 prongs	267 stars
3 prongs	159 stars
4 prongs	46 stars
5 prongs	25 stars
6 prongs	2 stars

Of 106 two pronged stars, observed in detail in the chamber, 66 percent were observed to possess at least one prong with an energy in excess of 15 Mev. Multiplying the entry 267 by $\frac{1}{3}$ yields a corrected prong distribution for the cloud chamber stars as shown by the circles in Fig. 6.

As the deuteron bombarding energy varies from 35 to 190 Mev, the prong distribution in the photographic emulsion remains the same, within the limit of experimental error. The mean of the theoretical distribution, for the heavy component, varies from about 1.1 at 35 Mev bombarding energy to 3.3 at 190 Mev. Since at 35 Mev a struck nucleus of the heavy component is almost sure to evaporate neutrons only; the cross section for production of a visible star from a nucleus of the heavy component varies from about $0.1\sigma_0$ at 35 Mev to over $0.9\sigma_0$ at 190 Mev, where σ_0 is the geometrical cross section. Due to the ease with which carbon and oxygen nuclei may break up into alpha-particles, the cross section for visible star production from these lighter nuclei may be expected, in contrast, to change very slowly with bombarding energy.

The stars observed by Gardner and Peterson must then arise almost entirely from the lighter component of the emulsion at the lower bombarding energies, and predominantly from the heavy component at the higher energies. If we make the plausible assumption that the prong number distribution for the light component is almost unchanging throughout the range of bombarding energies used, then the observed and calculated distributions are in good qualitative agreement.

[8] Cloud-Chamber Group, Radiation Laboratory, University of California private communication.

V. ANGULAR DISTRIBUTION

The angular distribution of star prongs observed in Part I is predominantly in the forward direction. We may assume that particles evaporated from a nucleus emerge with an angular distribution spherically symmetric with respect to the nucleus. But the nucleus itself may be moving. In the case of the heavy component, the mean recoil velocity of the center of the excited nucleus, just after it is struck, will be small compared to the mean velocity of the evaporated ions. But in the case of the light component, it will be of the same order or exceed the mean velocity of the ions evaporated from it. The observed asymmetry is of the order to be expected on the assumption that it is due primarily to this recoil of the lighter nuclei.

VI. ACKNOWLEDGMENTS

We wish to thank Professor Robert Serber for suggesting and guiding this problem.

This paper is based on work done under the auspices of the United States Atomic Energy Commission in connection with the Radiation Laboratory, University of California, Berkeley, California.