# Electronic Interaction in Electrical Discharges in Gases*

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The Boltzmann transfer equation has been solved for the case of high current densities and low field strengths where electrostatic interactions may not be neglected. The solution has been examined for two different electron-molecule cross sections for momentum transfer. In one, the cross section varies inversely with electron velocity, and the distribution is found to be Maxwellian at all electron densities. In the other, the cross section is assumed independent of the electron energy. On the latter assumption, the solution varies from the Davydov distribution at low electron densities to the Maxwellian distribution at high densities. Curves have been drawn showing the transition of the distribution as the density increases. The corresponding values of average energy, drift velocity, and average velocity have been tabulated.

## I. INTRODUCTION

THIS paper deals with the effects of electrostatic interaction in low voltage discharges in gases. The field strengths are limited in order to rule out inelastic collision phenomena.

The present theory of electrical discharges, which neglects inelastic collisions and electron interaction but includes the motion of the molecules, was first developed by Davydov[1] in 1936. Davydov's electron velocity distribution function reduces to that obtained originally by Druyvesteyn[2] in 1930, and later by Morse, Allis, and Lamar,[3] in which molecular motion is neglected.

The theory which has been developed here is an extension of Davydov's work, in that it considers only elastic collisions, with the addition of electron-electron encounters. We find that below electron densities of $10^6$/cc our distribution function reduces to that of Davydov, while at very high densities, the limiting form of the distribution function is Maxwellian. These results are to be expected, and therefore the chief interest lies in our treatment of the problem. This is particularly true in view of a paper in 1939 by Haseltine[4] in which the same problem was considered but in which the analytical difficulties proved excessive.

Our good fortune in reducing this problem is largely due to a paper by Landau[5] in 1936, in which he developed an approximation to the coulomb interaction term in the Boltzmann transfer equation.[6] Because of the long range of the force, a majority of the Coulomb collisions involve a small momentum exchange, and, accordingly, Landau expands in powers of the momentum exchange.

We also justify and make use of an assumption which Haseltine found useful. When considering the momentum balance equation, there proves to be a certain range of electron and molecule densities, in which the electron-molecule collisions are the more important. Here the electrostatic interactions may be neglected insofar as momentum conservation, but not energy conservation, is concerned.

## II. ANALYTICAL FORMULATION OF THE PROBLEM

A central part of the theory of gas discharges concerns the determination of the electron velocity distribution function $f(\mathbf{r}, \mathbf{v})$, the solution of the Boltzmann transfer equation.[6] Our distribution $f$ is normalized to the electron density $n$. We consider an infinite gas with a constant impressed field $E$. The field strength for the re-

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[1] B. Davydov, Physik. Zeits. Sowjetunion **8**, 59 (1935), 9, 433 (1936).
[2] M. J. Druyvesteyn, Physica **10**, 69 (1930).
[3] P. M. Morse, W. P. Allis, and E. S. Lamar, Phys. Rev. **48**, 412 (1935).
[4] W. R. Haseltine, J. Math. and Phys. **18**, 174 (1939).
[5] E. Landau, Physik. Zeits. Sowjetunion **10**, 154 (1936).
[6] L. Boltzmann, *Vorlesungen uber Gastheorie* (Leipzig, 1896), 1, Ch. II and III.

mainder of the paper will be represented by $\gamma = eE/m$, where $e$, $m$ are the charge and mass of the electron. We assume equal ion and electron densities. In such an environment, the steady state electron distribution is the solution of

$$\gamma \cdot \Delta f = (\partial_e f/\partial t)_1 + (\partial_e f/\partial t)_M, \tag{1}$$

in which $(\partial_e f/\partial t)_1$ is the time rate of change of $f$ with respect to electron-electron collisions, $(\partial_e f/\partial t)_M$ the time rate of change of $f$ with respect to all other collisions, and $\Delta$ the velocity gradient operator.

The second term on the right of (1) includes both ions and molecules. We assume both to be in equilibrium at the absolute temperature $T$. Before evaluating $(\partial_e f/\partial t)_M$ we introduce the quantity $\lambda(v)$, defined as the mean free path for momentum transfer in terms of the differential scattering cross section, $\sigma(\theta, v)d\omega$ by the relation

$$1/\lambda(v) = N\int \sigma(\theta, v)(1-\cos\theta)d\omega, \tag{2}$$

in which $N$ is the density of scattering centers, $\theta$ is the angle, and $d\omega$ the solid angle into which the electron is scattered. The composite mean free path $\lambda$ for collisions with molecules and ions is given by

$$1/\lambda = 1/\lambda_m + 1/\lambda_i, \tag{3}$$

in which $\lambda_m$ is the mean free path for molecules and $\lambda_i$ is that for ions. The approximate calculation of $\lambda_i$ is generally made using the Debye-Huckel[7] shielding radius $\rho$ as the upper limit of the collision parameter.[8] The approximate value of $\lambda_i$ is then

$$1/\lambda_i = (n\pi e^4/\bar{\epsilon}_r^2)\ln(\rho\bar{\epsilon}_r/e^2), \tag{4}$$

where $\rho = (\bar{\epsilon}_r/12\pi ne^2)^{\frac{1}{2}}$, $\bar{\epsilon}_r$ is the average relative electron-ion energy, and $n$ is the electron or ion density. If we assume the electron-molecule cross section to be of the order of $10^{-15}$ cm$^2$, the ratio $\lambda_i/\lambda_m$ for one-volt electrons is

$$\lambda_i/\lambda_m = N/n(10)^{-3}, \tag{5}$$

where $N$ is the molecular density. Relation (5) gives us an accurate estimate of the role played by the ions in the discharge.

The form of $(\partial_e f/\partial t)_M$ is derived by Chapman and Cowling.[9] If we expand $f(\mathbf{v})$ in the form

$$f(\mathbf{v}) = f_0(v) + v_z f_1(v), \tag{6}$$

with the field in the $z$ direction, and write $s$ for $v^2$, their expression is

$$\left(\frac{\partial_e f_0}{\partial t}\right)_M = \frac{\delta}{\beta s^{\frac{1}{2}}}\frac{d}{ds}\frac{s^2}{\lambda(s)}\left(\frac{df_0}{ds}+\beta f_0\right),$$
$$\left(\frac{\partial_e v_z f_1}{\partial t}\right)_M = -v_z\frac{s^{\frac{1}{2}}f_1}{\lambda(s)}, \tag{7}$$

where $\delta = 2m/M$ is twice the ratio of electronic to molecular mass and $\beta = m/2kT$, $k$ being Boltzmann's constant.

The calculation of $(\partial_e f/\partial t)_1$, as stated in the introduction, was indicated by Landau.[5] He expanded this term in powers of the momentum exchanged on collision, retaining terms of the second order, as the zeroth and first-order terms vanish. The result was that $(\partial_e f/\partial t)_1$ could be expressed as the velocity divergence of a vector $\mathbf{J}$,

$$(\partial_e f/\partial t)_1 = \nabla \cdot \mathbf{J}. \tag{8}$$

The vector $\mathbf{J}$ is most conveniently written using the summation convention over repeated indices. Letting $i = x, y, z$,

$$J_i(\mathbf{v}) = \mathfrak{L}\int_{v' \text{ space}}\left[f'(\mathbf{v}')\frac{\partial f(\mathbf{v})}{\partial v_j} - f(\mathbf{v})\frac{\partial f'(\mathbf{v}')}{\partial v_j'}\right] \times (u^2\delta_{ij} - u_i u_j)/(u^3)d\mathbf{v}', \tag{9}$$

where $\delta_{ij}$ is the Kronecker delta and $\mathfrak{L} = 2\pi e^4/m^2 \ln\bar{\epsilon}\rho/e^2$. The vector u is the relative velocity $\mathbf{v}' - \mathbf{v}$ of the scattering (primed) eleetron relative to the incident (unprimed) electron; $\bar{\epsilon}$ is the relative electron energy $\frac{1}{2}mu^2$.

In order to carry out the integration in (9) after substituting (6), we found suitable the coordinate system devised by Hylleraas[10] for the helium ion in which $v$ serves as polar axis for $v'$.

The manner of performing the angular integration is shown in Appendix I. As a result,

[7] P. Debye and E. Huckel, Physik. Zeits. **24**, 185 (1923).
[8] B. Davydov, Physik. Zeits. Sowjetunion **12**, 269 (1937).
[9] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1939), pp. 348 *et seq.*
[10] E. A. Hylleraas, Zeits. f. Physik **54**, 347 (1929).

$\mathbf{J}$ may be written in the form

$$\mathbf{J}(\mathbf{v})=\mathfrak{L}'\mathbf{v}\int_0^\infty (5U^{00}+5v_zU^{01}+2v_zU^{10}+2v^2U^{11}+f_1f_1')v'dv'+\mathbf{k}\mathfrak{L}'\int_0^\infty [v^2U^{10}+v^2v_zU^{11}+5(f_0'f_1-f_0f_1')-3v_zf_1'f_1]v'dv', \quad (10)$$

where $\mathbf{k}$ is a unit vector in the $z$ direction, $\mathfrak{L}'=(8\pi)/(15)\mathfrak{L}$, and

$$U^{kl}=f_k'\frac{\partial f_l}{v\partial v}-f_l\frac{\partial f_k'}{v'\partial v'}.$$

The calculation of the velocity divergence of $\mathbf{J}(\mathbf{v})$ is most easily carried out if $\mathbf{J}$ is expressed as a vector in spherical coordinates. The isotropic and anisotropic terms of $(\partial_e f/\partial t)_1$ are separated in the form

$$(\partial_e f/\partial t)_1=\nabla\cdot\mathbf{J}=(\partial_e f_0/\partial t)_1+(\partial_e v_z f_1/\partial t)_1. \quad (11)$$

The results of the differentiation as performed in Appendix II are

$$\left(\frac{\partial_e f_0}{\partial t}\right)_1=\frac{10}{s^{\frac{1}{2}}}\frac{d}{ds}s^{\frac{1}{2}}(Af_0'+Bf_0)+\frac{2}{s^{\frac{1}{2}}}\frac{d}{ds}s^{5/2}(Cf_1'+Df_1),$$

$$\left(\frac{\partial_e v_z f_1}{\partial t}\right)_1=v_z\left\{\frac{10}{s^{\frac{1}{2}}}\frac{d}{ds}s^{5/2}(Af_1'+Bf_1)+\frac{6}{s^{\frac{1}{2}}}\frac{d}{ds}s^{5/2}(Cf_0'+Df_0)-5(2Cf_0'+Df_0+Bf_1)\right\}, \quad (12)$$

where the primes now refer to differentiation with respect to $s\equiv v^2$. We define the four coefficients $A$, $B$, $C$, and $D$ in the following way:

$$A=\mathfrak{L}'\int_0^\infty f_0ds, \quad B=\mathfrak{L}'f_0(0),$$

$$C=\mathfrak{L}'\int_0^\infty f_1ds, \quad D=\mathfrak{L}'f_1(0).$$

We now have analytic expressions for (1) and so can assemble the Boltzmann equation. It will immediately be observed in (7) and (12) that both terms on the right of (1) can be expressed as sums of isotropic and anisotropic terms. These represent the first two terms of an expansion in Legendre polynomials following the form of the expansion of $f(\mathbf{v})$ in (6). The left-hand side of (1) separates in similar fashion, giving rise to the following pair of simultaneous integro-differential equations:

$$\left(\frac{\partial_e f_0}{\partial t}\right)\equiv\frac{10}{s^{\frac{1}{2}}}\frac{d}{ds}s^{\frac{1}{2}}(Af_0'+Bf_0)+\frac{2}{s^{\frac{1}{2}}}\frac{d}{ds}s^{5/2}(Cf_1'+Df_1)+\frac{\delta}{\beta s^{\frac{1}{2}}}\frac{d}{ds}\frac{s^2}{\lambda(s)}(f_0'+\beta f_0)=\frac{2}{3}\frac{\gamma}{s^{\frac{1}{2}}}\frac{d}{ds}s^{\frac{1}{2}}f_1, \quad (13a)$$

$$\left(\frac{\partial_e f_1}{\partial t}\right)\equiv\frac{10}{s^{\frac{1}{2}}}\frac{d}{ds}s^{5/2}(Af_1'+Bf_1)+\frac{6}{s^{\frac{1}{2}}}\frac{d}{ds}s^{5/2}(Cf_0'+Df_0)-5[2Cf_0'+Df_0+(B+s^{\frac{1}{2}}/5\lambda(s))f_1]=2\gamma f_0'. \quad (13b)$$

Equation (13a) is seen to be exact. This is necessary for the conservation of particle density, since the transport of mass, written as

$$n\Delta m=\int_{v\text{ space}} m(\partial_e f/\partial t)d\mathbf{v},$$

must vanish.[11] The integrated Eq. (13a) is the energy balance equation. This is shown by computing the energy transport

$$n\Delta\epsilon=\int_{v\text{ space}}\epsilon\left(\frac{\partial_e f}{\partial t}\right)d\mathbf{v},$$

making use of both sides of (1). Because of symmetry only the isotropic parts of (1) survive the integration, so we need only consider (13a). From

[11] See reference 9, pp. 47 *et seq.*

TABLE I. Mean values calculated from (23) at different electron densities $n$.

| $n$ | $\bar{v}$ (cm/sec.) | $\bar{\epsilon}$ (electron volts) | $\bar{v}_z$ (cm/sec.) |
|---|---|---|---|
| $10^8$ | $3.1(10)^7$ | 0.32 | $2.5(10)^6$ |
| $10^{10}$ | $2.7(10)^7$ | 0.22 | $3.4(10)^5$ |
| $10^{12}$ | $1.1(10)^7$ | 0.072 | $7.5(10)^5$ |

the left hand side of (13a) we find on performing a partial integration that

$$n\Delta\epsilon = -\int_0^\infty (\tfrac{1}{2}ms)\Big[10(Af_0'+Bf_0) + 2s(Cf_1'+Df_1)+\frac{\delta s^{\frac{1}{2}}}{\beta\lambda(s)}(f_0'+\beta f_0)\Big](2\pi s^{\frac{1}{2}}ds),$$

while from the right hand side of (13a),

$$n\Delta\epsilon = -\int_0^\infty (\tfrac{1}{2}ms)\left(\frac{2\gamma f_1}{3}\right)(2\pi s^{\frac{1}{2}}ds),$$

where of course $\epsilon=\frac{1}{2}ms$. Assuming detailed balancing, the integrands of both expressions must be equal, which leads to the equation

$$10(Af_0'+Bf_0)+2s(Cf_1'+Df_1) + \frac{\delta s^{\frac{1}{2}}}{\beta\lambda(s)}(f_0'+\beta f_0) = \tfrac{2}{3}\gamma f_1. \quad (14)$$

Equation (14) will be seen to be the first integral of (13a) with the constant of integration set equal to zero to suppress singular solutions of $f$.

Momentum balance is assured by (13b). The rates at which momentum and energy are added to the electron gas by the applied field are easily computed from the left hand sides of (13b) and (13a) to be $eEn$ and $eE\bar{v}_z n$, respectively.

## III. SOLUTION OF THE TRANSFER EQUATION

Equations (13b) and (14) in the variable $v$ take the form

$$(10A+\delta v/\beta\lambda(v))f_0'+2v(10B+\delta v/\lambda(v))f_0 + 2v^2(Cf_1'+2vDf_1)=(4/3)\gamma v f_1, \quad (15)$$

$$\tfrac{3}{2}Cf_0''+[3Dv+(C-\gamma)/v]f_0'+10Df_0 + 5/2Af_1''+(5Bv+10A/v)f_1' + (20B-v/\lambda(v))f_1=0. \quad (16)$$

The simplifications we have found necessary, but which will yield results of interest, are twofold. Haseltine[4] assumed that in the momentum balance Eq. (16) the electron-electron impacts could be neglected relative to the electron-molecule collisions. In the energy balance Eq. (15) the situation differs materially in that the electron loses a large fraction of its energy when colliding with another electron, while in a collision with a molecule the electron loses a fraction $\delta=2m/M$ of its energy. If we compare the coefficient of $f_0$ in (15) with that of $f_1$ in (16), the term in $B$ will be seen to have a very much greater importance in (15) than in (16), verifying Haseltine's assumption. It will be noted, however, that at sufficiently high electron densities the electron-electron interaction terms in (16) must be retained. Consequently, we neglect the interaction terms in (16), obtaining with Haseltine[4] and Morse, Allis, and Lamar,[3]

$$\gamma f_0'+[v^2/\lambda(v)]f_1=0. \quad (17)$$

The second simplification comes in neglecting terms in $f_1^2$ relative to $f_0^2$ in (15), which then reads

$$(10A+\delta v/\beta\lambda(v))f_0' + 2v(10B+\delta v/\lambda(v))f_0=4/3\gamma v f_1. \quad (18)$$

The elimination of $f_1$ in (18) by (17) leads to a simple first-order equation in $f_0$ whose solution is

$$f_0=N_0\exp\left\{-\int_0^v \frac{(10B\lambda/\delta)\beta v+\beta v^2}{(4/3)((\beta\gamma\lambda)^2/\delta)+(10\beta A\lambda/\delta)\beta v+\beta v^2}(2\beta v dv)\right\}, \quad (19)$$

where $N_0$ is the normalization constant.

Our solution (19) is quite general in that the dependence of $\lambda$ on $v$ has not been specified. For precise investigations of particular gas discharges it is often desirable to use the empirical electron-gas cross sections, which (19) permits. Equation (19) has a drawback in that it is a non-linear integral equation, because of the presence of the normalization constant in $B$ and the integral form of $A$.

We have investigated (19) for two assumed cross sections. In case (a) the cross section is inversely proportional to $v$, and in (b) the cross section is independent of electron energy.

In case (a) the time $\tau=\lambda/v$ between collisions is a constant. Then (19) reduces to

$$f_0=N_0\exp\left\{-\frac{(10B\tau/\delta)+1}{(4/3)(\beta(\gamma\tau)^2/\delta)+(10\beta A\tau/\delta)+1}\beta v^2\right\}. \tag{20}$$

Computing $A$ and $B$ from their definitions, we find that

$$\beta A=B[1+(4/3)(\beta(\gamma\tau)^2)/\delta],$$
$$B=N_0\mathfrak{L}',$$

so that

$$f_0=N_0\exp(-mv^2/2k(T+T')), \tag{21}$$

where $\frac{3}{2}kT'=\frac{1}{2}M(\gamma\tau)^2$ is the excess of electron energy above the thermal energy $\frac{3}{2}kT$ of the gas molecule. The normalization constant $N_0$ is then

$$N_0=n[m/2\pi k(T+T')]^{\frac{3}{2}}.$$

The drift velocity $\bar{v}_z$ of the electrons is calculated from $f_1$ as given by (17), and is

$$\begin{aligned}\bar{v}_z&=\frac{1}{n}\int_{v\text{ space}} v_z^2 f_1 d\mathbf{v},\\ &=\frac{4\pi}{3n}\int f_1 v^4 dv,\\ &=\gamma\tau.\end{aligned} \tag{22}$$

This result shows that the energy by which the electron exceeds the molecule is just that which the molecule would have if streaming with the electron drift velocity.

It would appear that the electron temperature $T+T'$ were independent of the electron density. This is not the case, because in our formulation of the problem we have assumed equal ion and electron densities. Referring to (5), it is apparent that when $n>10^{-3}N$, the ions determine the magnitude of $\lambda$ in (3). Hence $\tau$ will decrease when $n$ increases above $10^{-3}N$, causing $T'$ to drop off with $n^2$. At sufficiently high electron densities, then, the electrons approach thermal equilibrium with the gas. It would not be proper to speak of the dependence of the drift velocity on $n$, since the dependence of $\tau$ on $n$ above is only qualitative. The actual electron-ion cross section goes not with $1/v$ but $1/v^4$, so that at very high densities the proper electron-ion cross section must be included in (19).

We next consider the case where $\lambda$ is constant. While it is possible to carry out the integration in (19), the resulting expression is too complicated for manipulation and will not be written down. It is preferable to investigate (19) numerically. In the variable $x=mv^2/2kT$,

$$f_0=N_0\exp\left\{-\int_0^x\frac{(10B\lambda/\delta)\beta^{\frac{1}{2}}x^{\frac{1}{2}}+x}{\alpha+(10(\beta A\lambda)/\delta)\beta^{\frac{1}{2}}x^{\frac{1}{2}}+x}dx\right\}. \tag{23}$$

The constant $\alpha$ replaces $4/3(\gamma\beta\lambda)^2/\delta$. Assuming a field strength of 1 volt/cm, a mean free path $\lambda$ of $10^{-2}$ cm, and a gas temperature of 300°K, $\alpha$ takes the value 100. In Fig. 1 are shown the normalized energy distribution curves for three illustrative electron densities. The function closely resembles the Davydov distribution for densities up to $n=10^9$/cc. Above $n=10^9$, the function makes a rapid change-over, as shown by the curve at $n=10^{10}$/cc. At $n=10^{11}$/cc the curve is very similar to $n=10^{12}$/cc, where the distribution has become Maxwellian at the temperature $T$.

The mean velocity $\bar{v}$, energy $\bar{\epsilon}$, and the drift velocity $\bar{v}_z$ have been computed and are shown in Table I. The mean energy and velocity are seen to decrease while the drift velocity increases, with increasing $n$. The act of increasing the density evidently lowers the entropy of the discharge in reducing the isotropic part of the distribution function while the drift velocity increases.

It is of interest to note that for electron densities of the order of $10^{12}$/cc the electrons in a gas having a $1/v$ cross section are $T'$ degrees hotter than the electrons in a gas having a constant cross section. This is because a $1/v$ cross section implies a reduced efficiency for momentum transfer at higher energies. Consequently, such a gas favors the population of higher electron energy

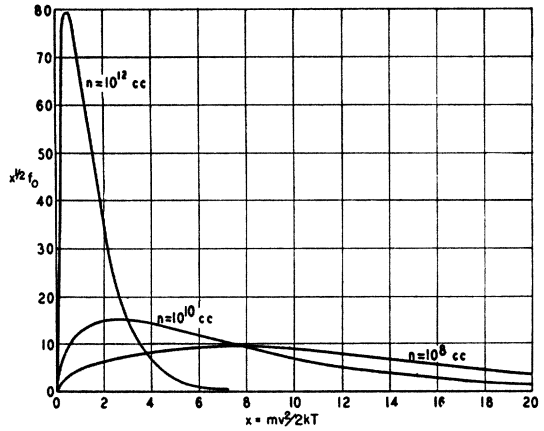


FIG. 1. Normalized energy distribution curves for three illustrative electron densities. $E=1$ volt/cm; $\lambda=10^{-2}$ cm; $\alpha=100$.

states over the gas in which the cross section is independent of energy.

## CONCLUSION

The functional behavior of (19) amply justifies the interaction terms (12). We are now in a position to use (12) in the further investigation of inelastic processes in d.c. discharges and in extending work of Margenau and Hartman[12] on high frequency discharges to high current densities.

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## APPENDIX I

### Angular Integration of J

It was found necessary to employ the coordinate system devised by Hylleraas,[10] as shown in Fig. 2. The velocity $\mathbf{v}$ of the incident electron is referred to $xyz$ coordinates, while the velocity $\mathbf{v}'$ of the scattering electron is referred to a coordinate system having $\mathbf{v}$ as polar axis. The azimuth of $\mathbf{v}'$ is measured by the angle $\chi$. The purpose of this choice of coordinate systems is to permit integration over all $\mathbf{v}'$ for arbitrary $\mathbf{v}$, thereby preserving the functional dependence of $\mathbf{J}$ on $\mathbf{v}$. The volume element for the $\mathbf{v}'$ integration is $d\mathbf{v}'=v'dv'ududx/v$. Because of the appearance of the rectangular components $v_x'$, $v_y'$, and $v_z'$ of $\mathbf{v}'$, Table II is used to transform to the system having $\mathbf{v}$ as polar axis, in which the rectangular components of $\mathbf{v}'$ are

$$v_1'=v'\sin\psi\cos\chi,$$
$$v_2'=v'\sin\psi\sin\chi, \text{ and}$$
$$v_3'=v'\cos\psi.$$

We note that

$$\nu_1=v_x/v, \quad \nu_2=v_y/v, \quad \nu_3=v_z/v,$$

and

$$\cos\psi=v^2+v'^2-u^2/2vv'.$$

The limits of integration for the respective variables are:

| Variable | Lower limit | Upper limit |
|---|---|---|
| $v'$ | 0 | $\infty$ |
| $u$ | $v'-v$ | $v'+v$ |
| $\chi$ | 0 | $2\pi$. |

It was found convenient to define the following quantities:

$$U^{kl}\equiv f_k'/v(\partial f_l)/(\partial v)-f_l/v'(\partial f_k')/(\partial v'),$$
$$V_i^{kl}\equiv f_k'(\partial f_l)/(\partial v_i)-f_l(\partial f_k')/(\partial v_i'),$$

and

$$V_i\equiv f'(\partial f)/(\partial v_i)-f(\partial f')/(\partial v_i'),$$

where $i$, $j$, $=x$, $y$, and $z$, and $k$, $l$, $=0$ and 1. With these abbreviations, we may write (9) as

$$\mathbf{J}=\mathfrak{L}\int V_j(u^2\delta_{ij}-u_iu_j)/(u^3)dv'. \qquad (I.1)$$

Assuming the explicit form of $f(\mathbf{v})$ as given by (6),

$$\mathbf{V}=\mathbf{V}^{00}+v_z'\mathbf{V}^{10}+v_z\mathbf{V}^{01}+v_zv_z'\mathbf{V}^{11}+\mathbf{k}(f'f_1-ff_1'), \qquad (I.2)$$

where $\mathbf{k}$ is the unit vector in the $z$ direction. When (I.2) is substituted in (I.1), the resulting

TABLE II. Transformation scheme to relative coordinate system in which $\mathbf{v}$ serves as polar axis.

| | $v_x'$ | $v_y'$ | $v_z'$ |
|---|---|---|---|
| $v_1'$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
| $v_2'$ | $\mu_1$ | $\mu_2$ | $\mu_3$ |
| $v_3'$ | $\nu_1$ | $\nu_2$ | $\nu_3$ |

[12] H. Margenau and L. M. Hartman, Phys. Rev. **73**, 309 (1948).

expression may be represented by

$$\mathbf{J}=\mathbf{J}^{00}+\mathbf{J}^{01}+\mathbf{J}^{10}+\mathbf{J}^{11}$$
$$+\mathfrak{L}\int (f'f_1-ff_1')\frac{\mathbf{k}u^2-u_z\mathbf{u}}{u^3}d\mathbf{v}'. \quad (I.3)$$

The computation of individual terms $J_i^{kl}$ is facilitated by the identity

$$V_j^{kl}(u^2\delta_{ij}-u_iu_j)/(u^3)$$
$$\equiv f_k'/v(\partial f_l)/(\partial v)(v_i/u-(\mathbf{v}\cdot\mathbf{v}'-v^2)/u^3)$$
$$-f_l/v'(\partial f_k')/(\partial v')(v_i'/u-(\mathbf{v}\cdot\mathbf{v}'-v'^2)/u^3).$$

After substituting the angular expressions for $v_x'$, $v_y'$, and $v_z'$ and integrating over the angles $\chi$ and $\psi$, we obtain for $J_i^{00}$ the result

$$J_i^{00}=\mathfrak{L}\int V_j^{00}(u^2\delta_{ij}-u_iu_j)/(u^3)d\mathbf{v}'$$
$$=8\pi v_i\mathfrak{L}/3\int_0^\infty U^{00}v'dv'.$$

In computing $J_i^{10}$ we discover that cylindrical symmetry causes $J_x^{10}$ and $J_y^{10}$ to be alike except for subscript, and different from $J_z^{10}$. We obtain

$$J_x^{10}=\mathfrak{L}\int v_z'V_j^{10}(u^2\delta_{xj}-u_xu_j)/(u^3)d\mathbf{v}'$$
$$=16\pi v_xv_z\mathfrak{L}/15\int_0^\infty U^{10}v'dv',$$
$$J_y^{10}=16\pi v_yv_z\mathfrak{L}/15\int_0^\infty U^{10}v'dv',$$

and

$$J_z^{10}=16\pi/15(v_z^2+\tfrac{1}{2}v^2)\mathfrak{L}\int_0^\infty U^{10}v'dv'.$$

In addition to these integrations, the following must be computed:

$$\mathfrak{L}\int (f'f_1-ff_1')(\mathbf{k}u^2-u_z\mathbf{u})/(u^3)$$
$$=8\pi\mathfrak{L}/15\mathbf{v}\int_0^\infty f_1'f_1v'dv'$$
$$+8\pi\mathfrak{L}/15\mathbf{k}\int_0^\infty [5(f_0'f_1-f_0f_1')-3v_zf_1'f_1]v'dv'.$$

It is not necessary to compute more of the $J_i^{kl}$,

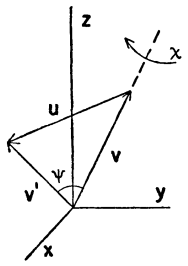


FIG. 2. Coordinates for J integrations.

as they may be obtained by changes of indices. Combining the results of the computation into the single vector $\mathbf{J}$, we have

$$\mathbf{J}(\mathbf{v})=\mathfrak{L}'\mathbf{v}\int_0^\infty (5U^{00}+5v_zU^{01}$$
$$+2v_zU^{10}+2v_z^2U^{11}+f_1f_1')v'dv'$$
$$+\mathbf{k}\mathfrak{L}'\int_0^\infty [v^2U^{10}+v^2v_zU^{11}$$
$$+5(f_0'f_1-f_0f_1')-3v_zf_1'f_1]v'dv',$$

which is Eq. (10). The constant $\mathfrak{L}'$ replaces $8\pi/15\mathfrak{L}$.

## APPENDIX II

### Calculation of $\nabla\cdot\mathbf{J}$

The expression of $\mathbf{J}$ as a vector having components in spherical coordinates was found useful in carrying out the divergence operation. In terms of its rectangular components, the spherical components of $\mathbf{J}$ are

$$J_v=J_x\sin\theta\cos\varphi+J_y\sin\theta\sin\varphi+J_z\cos\theta,$$
$$J_\theta=J_x\cos\theta\cos\varphi+J_y\cos\theta\sin\varphi-J_z\sin\theta, \quad (II.1)$$
$$J_\varphi=-J_x\sin\varphi+J_y\cos\varphi.$$

The polar axis is taken in the $z$ direction so that we can take advantage of the symmetry of the problem. This causes $J_\varphi$ to vanish, as can be readily verified. Substitution of the rectangular components of $\mathbf{J}$ in (II.1) immediately gives

$$J_v=\mathfrak{L}'\int_0^\infty [5vU^{00}+5v^2U^{01}\cos\theta+3v^2U^{10}\cos\theta$$
$$+3v^3U^{11}\cos^2\theta+5(f_0'f_1-f_0f_1')\cos\theta$$
$$+vf_1'f_1(1-3\cos^2\theta)]v'dv',$$

and

$$J_\theta = \mathfrak{L}\int_0^\infty \{[v^2U^{10}+5(f_0'f_1-f_0f_1')]\sin\theta + \tfrac{1}{2}[v^3U^{11}-3vf_1'f_1]\sin 2\theta\}v'dv'.$$

The divergence can then be calculated to be

$$\begin{aligned}\nabla\cdot\mathbf{J} &= 1/v\sin^2\theta\left(\frac{\partial}{\partial v}v^2\sin\theta J_v+\frac{\partial}{\partial\theta}v\sin\theta J_\theta\right)\\ &= \mathfrak{L}'\int_0^\infty \{15U^{00}+5v(\partial U^{00})/(\partial v)+5v^2U^{11}\\ &\quad +v^3(\partial U^{11})/(\partial v)+[20vU^{01}+5v^2(\partial U^{01})/(\partial v)\\ &\quad +12vU^{10}+3v^2(\partial U^{10})/(\partial v)\\ &\quad +5(f_0'(\partial f_1)/(\partial v)-f_1'(\partial f_0)/(\partial v))\\ &\quad -2vU^{10}]\cos\theta\}v'dv'. \quad \text{(II.2)}\end{aligned}$$

In the process of calculation, wherever a non-linear trigonometric expression appeared, it was expanded in Legendre Polynomials, the constant and linear terms alone being retained. It was found convenient to work with the variable $s=v^2$, in terms of which (II.2) becomes

$$\begin{aligned}\nabla\cdot\mathbf{J} &= \mathfrak{L}'\int_0^\infty \{15U^{00}+10s(\lambda U^{00})/(\partial s)+5sU^{11}\\ &\quad +2s^2(\partial U^{11})/(\partial s)+[20U^{01}+10s(\partial U^{01})/(\partial s)\\ &\quad +10U^{10}+6s(\partial U^{10})/(\partial s)\\ &\quad +5(f_0'(\partial f_1)/(\partial s)-f_1'(\partial f_0)/(\partial s))]v_z\}ds', \quad \text{(II.3)}\end{aligned}$$

where now

$$U^{kl}=f_k'(\partial f_l)/(\partial s)-f_l'(\partial f_k')/(\partial s').$$

We are forced to define the following functional constants:

$$A=\mathfrak{L}'\int_0^\infty f_0ds, \quad C=\mathfrak{L}'\int_0^\infty f_1ds,$$

$$B=\mathfrak{L}'f_0(0), \qquad D=\mathfrak{L}'f_1(0).$$

These appear in the final solution, so that in solving the differential aspect of the Boltzmann equation we still are faced with the solution of a non-linear integral equation.

If we separate the isotropic and anisotropic terms of $\nabla\cdot\mathbf{J}$ in the form

$$\nabla\cdot\mathbf{J}=(\partial_e f_0/\partial t)_1+(\partial_e v_z f_1/\partial t)_1,$$

we obtain

$$\begin{aligned}\left(\frac{\partial_e f_0}{\partial t}\right)_1 &= (10)/(s^{\frac{1}{2}})\frac{d}{ds}s^{\frac{1}{2}}(Af_0'+Bf_0)\\ &\quad +(2)/(s^{\frac{1}{2}})\frac{d}{ds}s^{5/2}(Cf_1'+Df_1)\end{aligned}$$

and (II.4)

$$\begin{aligned}\left(\frac{\partial_e v_z f_1}{\partial t}\right)_1 &= v_z\{10/s^{\frac{1}{2}}\frac{d}{ds}s^{5/2}(Af_1'+Bf_1)\\ &\quad +6/s^{\frac{1}{2}}\frac{d}{ds}s^{5/2}(Cf_0'+Df_0)\\ &\quad -5(2Cf_0'+Df_0+Bf_1)\},\end{aligned}$$

the primes in (II.4) indicating differentiation with respect to $s$.