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Radially Symmetric Distributions of Matter

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Several of the known solutions of the Einstein field equations are re-examined and it is shown that these solutions are special cases of more general solutions of the field equations. The explicit solution of the field equations in terms of known functions is also found for the case of Volkoff's massive spheres. In the last section of this paper a mathematical procedure for generating solutions of the field equations is outlined.

1. INTRODUCTION

IN a paper published some time ago R. C. Tolman[1] pointed out some of the difficulties in obtaining explicit solutions of Einstein's gravitational equations in terms of known analytic functions. It was pointed out how few such solutions were known, and in this paper methods were considered by means of which eight solutions were obtained. Some of these solutions had been discovered by other people using different methods. Tolman considered a sphere of fluid at rest in a coordinate system for which the line element has the form:

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad (1.1)$$

where λ, ν are unknown functions of r alone. For an isotropic sphere of fluid described by (1.1) the pressure p and density ρ must satisfy the relations:

$$8\pi p = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2, \quad (1.2)$$

$$8\pi p = e^{-\lambda}(\nu''/2 - \lambda'\nu'/4 + (\nu' - \lambda')/2r + \nu'^2/4), \quad (1.3)$$

$$8\pi\rho = e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2, \quad (1.4)$$

$$p' = -(p+\rho)\nu'/2, \quad (1.5)$$

where the prime denotes differentiation with respect to r.

It is well known that the equality of (1.2) and (1.3) automatically ensures that (1.5) is satisfied. For this reason the mathematical problem resolves itself into obtaining solutions of the equation resulting from the equality of (1.2) and (1.3). Since we have two unknowns λ and ν and only one equation the problem is indeterminate as it stands. In order to ensure a unique solution it is necessary to add a second relation involving λ and ν. The most satisfactory procedure would be to choose this relation by means of some physical condition, say an equation of state involving p and ρ which would then ensure that the resulting solution would be of physical interest. Tolman has pointed out however that such a procedure is almost impossible to carry through because of the complicated non-linear character of the expressions involved. For this reason, the usual procedure is to choose the auxiliary relation so that a mathematical simplification takes place in the equation to be solved. This procedure is not too satisfactory in that most of the solutions obtained in this way will not be of physical interest.

It is useful to point out one or two guiding principles in generating solutions of the field equations. If one assumes that the distribution of density $\rho = \rho(r)$ is known, then (1.4) can be integrated to determine $e^{-\lambda}$. For $e^{-\lambda}$ known, the equality of (1.2) and (1.3) results in a Riccati equation of the first order in ν'. It is of course well known that this equation is not in general solvable by quadratures but there exists an extensive literature on the equation which can be used to find solutions expressible in terms of known functions. If it is assumed that the gravitational potential e^ν is known, then the equality of (1.2) and (1.3) yields a linear equation of the first order in $e^{-\lambda}$. Such equations can always be solved by quadratures.

In our present paper we shall use the methods briefly outlined above to obtain several new solutions of the gravitational field equations. We shall also re-examine several of the known solutions and show how they can be generalized.

2. BOUNDARY CONDITIONS

For a line-element of the form (1.1), the Schwarzschild exterior solution is known to be:

$$e^\nu = e^{-\lambda} = 1 - 2m/r, \quad (2.1)$$

where m is the mass of the sphere as measured by its external gravitational field. For this reason any solution for e^ν and e^λ valid in the interior of this sphere must satisfy the relations:

$$e^\nu = e^{-\lambda} = 1 - 2m/r_b, \quad (2.2)$$

where r_b is the value of r at the boundary of the sphere. In addition we require the pressure p to vanish when $r = r_b$.

From (1.2) this implies:

$$e^{-\lambda}(r\nu' + 1) - 1 = 0, \quad \text{at} \quad r = r_b. \quad (2.3)$$

Since $e^{-\lambda}$ is continuous at the boundary, the three

[1] R. C. Tolman, Phys. Rev. **55**, 364 (1939).

boundary conditions are equivalent to the requirement that e^ν, $e^\nu\nu'$, and $e^{-\lambda}$ be continuous across the boundary of the sphere. From this it is easy to see that the solutions of the field equations which we seek will be such that e^ν contains two arbitrary constants, and $e^{-\lambda}$ contains one such constant.

3. VOLKOFF'S MASSIVE SPHERES

When the density ρ is taken to be constant Eq. (1.4) can be integrated to give

$$e^{-\lambda}=(x-x^3+K)/x, \tag{3.1}$$

where $x=r/R$ and $R^2=3/8\pi\rho$, and K is an arbitrary constant of integration. In order to avoid a singularity at the origin, Schwarzschild took $K=0$ and proceeded to find his well-known interior solution. By means of this solution it was possible to conclude that there must be an upper limit to the size of a sphere of given density. More recently G. M. Volkoff[2] has considered non-zero values of K and has shown that solutions exist for which the possible size of the sphere increases beyond that predicted by Schwarzschild. In fact, by increasing K indefinitely the possible size also increases indefinitely. Unfortunately several of the statements in Volkoff's paper are incorrect and he did not obtain his results in as complete a form as possible.

In the above mentioned paper Volkoff obtains the equation

$$d\mathrm{P}/dx=-[(\mathrm{P}+3)(\mathrm{P}x^3+v)/2x(x-v)], \tag{3.2}$$

where P is a quantity proportional to the pressure p and $v=x^3-K$. Volkoff states that Eq. (3.2) can not be explicitly integrated in terms of known functions. This statement is incorrect as (3.2) is an equation of Riccati type for which an obvious solution is $\mathrm{P}=-3$. It is well known that Riccati equations for which one solution is known can be integrated by quadrature. In fact we shall show that the integrals involved are elliptic integrals. It is also stated that solutions exist which behave near the origin like $\mathrm{P}\sim 7K/x^3$. Since all the solutions do not behave in this manner it is obvious that a behavior of this type at the origin implies some condition involving K. This condition is not given by Volkoff, but we shall be able to obtain it by carrying through the explicit integration of the field equations.

From the boundary condition to be satisfied by $e^{-\lambda}$, (3.1) implies

$$2m/R=x_b^3-K, \tag{3.3}$$

where $x_b=r_b/R$. Numerical integration of (3.2) for several different values of K showed that $x_b\sim x_1$, where x_1 is positive solution of the equation

$$x_1-x_1^3+K=0. \tag{3.4}$$

[2] G. M. Volkoff, Phys. Rev. **55**, 413 (1939).

From (3.3) it was argued that $x_b\sim x_1$ implied that

$$2m/R\sim x_1^3-K=x_1\sim x_b. \tag{3.5}$$

This argument is not correct. If we let $x=x_1-\delta$, where δ is a small correction term and drop all terms involving δ^2 we find

$$2m/R=x_1^3+3x_1^2\delta-K=x_1+3x_1^2\delta\sim x_b+3x_1^2\delta. \tag{3.6}$$

Thus for (3.5) to be correct it is necessary to show that for large value of x_1, $3x_1^2\delta$ is still small compared to x_b. As a numerical example we quote the value $K=80/7$ for which Volkoff obtains $x_b=2.39+$. Assuming that $2.39+$ means that x_b could probably be as large as 2.395 we find from (3.3) the largest possible value of $2m/R$ to be 2.31 and not $2.39+$ as quoted by Volkoff. The behavior of $2m/R$ for large values of K cannot be determined from (3.3) alone. It is not possible to discuss this behavior until the condition on K is determined which makes $\mathrm{P}\sim 7K/x^3$ for small values of x.

In order to find the general solution of the field equations we note that for constant ρ Eq. (1.5) can be integrated to give

$$8\pi(p+\rho)=ae^{-\nu/2}. \tag{3.7}$$

Adding (1.2) and (1.4) and using (3.7) we find

$$e^{-\lambda}(\nu'+\lambda')=ae^{-\nu/2}. \tag{3.8}$$

Hence

$$(d/dr)(e^{\nu/2})+(\lambda'/2)(e^{\nu/2})=(ar/2)e^{\lambda}. \tag{3.9}$$

This equation is a linear equation in $e^{\nu/2}$ whose solution is given by

$$e^\nu=e^{-\lambda}\left(b\int_{r_b}^{r} re^{3\lambda/2}dr+c\right)^2, \tag{3.10}$$

where $b=a/2$ and c is a second constant of integration. Thus the general solution of the field equations, for non-zero K, is:

$$e^{-\lambda}=1-(r/R)^2+KR/r,\quad e^\nu=e^{-\lambda}\left(b\int_{r_b}^{r} re^{3\lambda/2}dr+c\right)^2. \tag{3.11}$$

From the boundary condition $(e^{-\lambda})_{r=r_b}=1-2m/r_b$ we immediately find that

$$K=x_b^3-2m/R, \tag{3.12}$$

where $x_b=r_b/R$. Equation (3.12) gives an interesting physical interpretation of K. If we let $M=4\pi\rho r_b^3/3$, the Newtonian expression for the mass of the sphere, then

$$K=2(M-m)/R. \tag{3.13}$$

Thus K is a measure of the discrepancy between the Newtonian and relativistic values for the mass of the sphere. Since $e^{-\lambda}$ must be positive as $r\to 0$ we must have $K\geqslant 0$ and hence $m\leqslant M$. Thus the relativistic value of the mass is always less than or equal to the corresponding Newtonian value.

From the boundary condition $(e^{\nu})_{r=r_b}=(e^{-\lambda})_{r=r_b}$ we immediately have $c=1$. Further the continuity of $e^{\nu}\nu'$ with the corresponding Schwarzschild value at $r=r_b$ gives us

$$b=3(1-2m/r_b)^{\frac{1}{2}}/2R^2. \tag{3.14}$$

At this stage we note that we have determined the constants K, b and c in terms of m, r_b, and R. We might thus expect that a knowledge of the boundary radius and the constant density would not determine the mass since m, r_b, and R are still independent variables. It is not until we examine the internal pressure that we find another relation connecting these three variables.

Using (1.2) and (3.11) we find

$$8\pi p=-3/R^2+\left[2be^{\lambda/2}\Big/\left(b\int_{r_b}^{r} re^{3\lambda/2}dr+1\right)\right]. \tag{3.15}$$

Since $e^{\lambda/2}\to 0$ as $r\to 0$ we see that the pressure will be negative near the center of the sphere unless

$$b\int_{r_b}^{0} re^{3\lambda/2}dr+1=0. \tag{3.16}$$

Since the boundary value r_b is the first place at which the pressure vanishes we see that the pressure would be negative throughout the sphere unless (3.16) holds. From the fact that we have assumed the sphere to be in equilibrium we see that the pressure could not be negative throughout and hence (3.16) must hold in order to give a solution of physical interest. From (3.14) and (3.16) we thus find that

$$\int_0^{r_b} re^{3\lambda/2}dr=2R^2(1-2m/r_b)^{-\frac{1}{2}}/3. \tag{3.17}$$

Using (3.16) we find

$$8\pi p=-3/R^2+\left[2e^{\lambda/2}\Big/\int_0^{r} re^{3\lambda/2}dr\right]. \tag{3.18}$$

For small values of r, (3.18) implies that the pressure behaves according to the law $8\pi p\sim 7KR/r^3$. Thus the only solutions with $K\neq 0$ which will be of physical interest are those in which the pressure behaves according to the law above. Moreover, in order to obtain this behavior, relation (3.17) connecting m, r_b, and R must be valid. This is the relation which was not obtained in the Volkoff paper, and it is the relation which enables us to discuss the behavior of $2m/R$ for large values of K.

Evaluation of $\int_0^r re^{3\lambda/2}dr$

If we let $\gamma(z)$ be the Weierstrass elliptic function with invariants 12, $4(27K^2-2)$, then it is well known that

$$(d\gamma/dz)^2=(\gamma'(z))^2=4\gamma^3-12\gamma-4(27K^2-2).$$

For this reason, the substitution

$$r=3KR/(\gamma(z)-1) \tag{3.19}$$

reduces the integral

$$I=\int_0^r re^{3\lambda/2}dr \tag{3.20}$$

to

$$I=216\sqrt{3}K^2R^2\int_0^z \frac{dz}{(\gamma')^2}. \tag{3.21}$$

From (3.19), $dr=-3KR\gamma'(z)dz/(\gamma(z)-1)^2$. Thus for positive dr and dz, $\gamma'(z)$ must be taken negative. For this reason we must take

$$(4\gamma^3-12\gamma-4(27K^2-2))^{\frac{1}{2}}=-\gamma'(z).$$

By using the properties of elliptic functions, (3.21) can be evaluated to give

$$I=[4\sqrt{3}R^2/3(27K^2-4)]\{\zeta(z)-\tfrac{1}{2}(27K^2-2)z +[(2\gamma^2-(27K^2-2)\gamma-4)/\gamma']\} \tag{3.22}$$

where $\zeta(z)$ is the Weierstrass zeta function defined by $\zeta(z)=-\int\gamma(z)dz$ and the condition $\zeta(z)-1/z\to 0$ as $z\to 0$. Although (3.15) is the expression of the integral in terms of known functions, it will not be much value for numerical computation. At the present time no tables of values of the Weierstrass elliptic functions seem to be in existence. For such work numerical integration of (3.20) is still necessary.

We have previously stated that $K=0$ leads to the Schwarzschild interior solution which is expressible in terms of elementary functions. The only other value of K for which this is true is $K=2/3\sqrt{3}$. For this case

$$I=\frac{R^2}{27}\left[\frac{15z^4+25z^2+8}{z(1+z^2)^2}+15\tan^{-1}z-\frac{15\pi}{2}\right], \tag{3.23}$$

where $z=(2R-3^{\frac{1}{2}}r)^{\frac{1}{2}}/(3^{\frac{1}{2}}r)^{\frac{1}{2}}$. For this case the boundary values are $r_b/R=1.13$ and $2m/R=1.06$. For the Schwarzschild solution, $(K=0)$, the corresponding values are $r_b/R=0.944$ and $2m/R=0.838$. Thus we still find an upper limit to the size of a sphere of given density, but these upper limits are greater than those predicted by the Schwarzschild solution.

For general values of K we have from (3.1) that $x-x^3+K>0$ throughout the sphere. Thus if x_1 denotes the only positive root of $x-x^3+K=0$, we find for large values of K that $x_1<K^{\frac{1}{3}}+1/3K^{\frac{1}{3}}$.

For this reason $x_b < K^{\frac{1}{3}} + 1/3K^{\frac{1}{3}}$. Since $2m/R$ is positive, (3.12) implies $x_b > K^{\frac{1}{3}}$ and therefore $K^{\frac{1}{3}} < x_b < K^{\frac{1}{3}} + 1/3K^{\frac{1}{3}}$. With this restriction on x_b it is possible to show that

$$\lim_{K\to\infty} \int_0^{r_b} r e^{3\lambda/2} dr = \infty.$$

Hence from (3.17) $2m \to r_b$ and $2m/R \to x_b$ for large values of K.

4. DENSITY PROPORTIONAL TO A POWER OF r

In the previous section we assumed the density ρ to be constant. As a generalization we shall now assume that $\rho = ar^{N-2}$, where a and N are as yet unrestricted constants. From (1.4) we immediately find

$$e^{-\lambda} = 1 - (r/R)^N + K/r, \tag{4.1}$$

where $R^{-N} = 8\pi a/(N+1)$, and K is a constant of integration. In order to avoid a singularity at the origin we restrict $K = 0$ and $N > 0$. Thus

$$e^{-\lambda} = 1 - (r/R)^N. \tag{4.2}$$

Since $e^{-\lambda}$ must be positive we find that $r \leqslant R$. Equating (1.2) and (1.3) we find that the equation to determine ν is

$$(1-q)r^2(2\nu'' + (\nu')^2) - r\nu'(2 + (N-2)q) + 2q(2-N) = 0, \tag{4.3}$$

where $q = (r/R)^N$. The substitution

$$\nu = 2 \log y, \quad x = (1-q)^{\frac{1}{2}}$$

reduces this equation

$$(1-x^2)(d^2y/dx^2) - 2((N-2)/N)x(dy/dx) - 2((N-2)/N^2)y = 0. \tag{4.4}$$

For certain values of N, Eq. (4.4) does possess solutions which can be expressed in terms of elementary functions. However, for arbitrary values of N the general solution can only be expressed by means of the hypergeometric function $F(a, b; c; z)$ to be

$$y = AF(a, b; \tfrac{1}{2}; x^2) + BxF(a+\tfrac{1}{2}, b+\tfrac{1}{2}; \tfrac{3}{2}; x^2), \tag{4.5}$$

where

$$a = [(N-4) + (N^2 - 16N + 32)^{\frac{1}{2}}]/4N \tag{4.6}$$

$$b = [(N-4) - (N^2 - 16N + 32)^{\frac{1}{2}}]/4N \tag{4.7}$$

and A and B are arbitrary constants. For large values of N, the solution (4.4) approaches that given by

$$y = A + B \log\{(1-x)/(1+x)\}. \tag{4.8}$$

There are an infinite number of values of N for which Eq. (4.4) will have solutions which can be expressed in terms of elementary functions. For example, if one of a or b is a negative integer, at least one of the solutions can be expressed in terms of elementary functions. Similarly if one of $a + \frac{1}{2}$ or $b + \frac{1}{2}$ is a negative integer such a solution is possible. If $2(N-2)/N$ is an odd integer, it can be shown that the general solution can be expressed in terms of elementary functions. In addition to those mentioned there exist many other values of N for which a solution of this type exists. As an example we take the case $N = 4$. For this value of N the differential equation becomes

$$(1-x^2)(d^2y/dx^2) - x(dy/dx) - (1/4)y = 0. \tag{4.9}$$

The general solution of this equation is

$$y = A \sinh(\tfrac{1}{2} \arcsin x + B), \tag{4.10}$$

where A and B are arbitrary constants. This leads to

$$e^{-\lambda} = 1 - (r/R)^4,$$
$$e^{\nu} = A^2 \sinh^2[\tfrac{1}{2} \arcsin(1-(r/R)^4)^{\frac{1}{2}} + B] \tag{4.11}$$

as a rigorous solution of the field equations. For this solution the distribution of density ρ is given by

$$8\pi\rho = 5r^2/R^4. \tag{4.12}$$

The distribution of pressure can be exactly obtained from (1.2) and (4.11) but the expression is complicated. From the boundary condition $(e^{-\lambda})_{r=r_b} = 1 - 2m/r_b$ we find that $m = r_b^5/2R^4$. Thus we see that the relativistic expression for the mass is the same as the Newtonian expression for the mass of a sphere of radius r_b corresponding to a distribution of density given by (4.12).

For most bodies the quantity r/R is extremely small. Thus one can expand the expression for e^{ν}, given in (4.11), in powers of r/R and obtain a simplified approximate solution for e^{ν}.

5. TOLMAN'S SOLUTION VI

In the paper previously mentioned, Professor Tolman determines eight solutions of the field equations, five of which were new at that time. We propose to show in the next sections of this paper, that three of these solutions can be generalized.

The sixth solution given by Professor Tolman is

$$e^{\lambda} = 2 - n^2, \quad e^{\nu} = (Ar^{1-n} - Br^{1+n})^2, \tag{5.1}$$

where A, B, and n are arbitrary constants. Corresponding to this solution the pressure p and density ρ are given by

$$8\pi p = \frac{(1-n)^2 A - (1+n)^2 B r^{2n}}{(2-n^2)r^2(A - Br^{2n})} \tag{5.2}$$

$$8\pi\rho = \frac{(1-n^2)}{(2-n^2)r^2}. \tag{5.3}$$

Before proceeding to the generalization of this solution we should like to note some restrictions on the parameters occurring in the solution which have not been pointed out before. Since the solution is symmetric with respect to n and $-n$, there is no loss in restricting $n \geqslant 0$. Further, to ensure that e^{λ} is positive we must have $0 \leqslant n < \sqrt{2}$. For this range of values the density given by (5.3), will be positive only if $0 \leqslant n \leqslant 1$. In the particular case $n=1$ this solution reduces to the solution corresponding to special relativity. If one leaves n, A and B as constants to be determined by the boundary conditions, then the solution does contain the proper number of arbitrary constants. However, if we desire to fix n arbitrarily independently of the boundary conditions then, of course, only two constants remain with which to satisfy the three boundary conditions. This procedure would of course immediately place a condition on the boundary conditions. In the detailed discussion of this solution Professor Tolman arbitrarily assigns the value $\frac{1}{2}$ to n. On this basis the mass m must satisfy $m=3r_b/14$. Thus the mass m becomes completely determined by the boundary radius of the sphere and no arbitrary constant remains that could be interpreted in terms of the average density of the sphere. If we choose $n=\frac{1}{2}$ we have from (5.3), $8\pi\rho=3/7r^2$, and hence the distribution of density becomes completely determined, and therefore the corresponding solution can apply at most to one sphere. We shall show that there exists a solution of the field equations which includes the present solution as a special case and in which the constant n can be arbitrarily assigned without imposing a condition on the boundary conditions. Moreover we shall show that in our new solution the range of n is not restricted to be $0 \leqslant n \leqslant 1$.

We shall retain the expression

$$e^{\nu}=(Ar^{1-n}-Br^{1+n})^2 \tag{5.4}$$

since, for fixed n, the expression still retains the proper number of arbitrary constants. Equating (1.2) and (1.3) we find that the equation to determine $e^{-\lambda}$ is

$$(r^2\nu'+2r)(d/dr)e^{-\lambda}+(2r^2\nu''+r^2\nu'^2-2r\nu'-4)e^{-\lambda}=-4. \tag{5.5}$$

When ν is given by (5.4), Eq. (5.5) becomes a linear equation of the first order in $e^{-\lambda}$. Its solution can easily be determined to be

$$e^{-\lambda}=(2-n^2)^{-1}+ar^b[A(2-n)-B(2+n)r^{2n}]^c, \tag{5.6}$$

where a is an arbitrary constant of integration, $b=2(n^2-2)/(n-2)$ and $c=2(2-n^2)/(n^2-4)$. This solution is valid providing n does not have the value $2^{\frac{1}{2}}$ or 2. For the case $n=2$ we find

$$e^{-\lambda}=-\tfrac{1}{2}+ar\,e^{-(A/4Br^4)}, \tag{5.7}$$

$$e^{\nu}=(Ar^{-1}-Br^3)^2. \tag{5.8}$$

Similarly for $n=2^{\frac{1}{2}}$ we find

$$e^{-\lambda}=a+\log[A(2-2^{\frac{1}{2}})-B(2+2^{\frac{1}{2}})r^{2^{\frac{3}{2}}}]/r(2+2^{\frac{1}{2}}), \tag{5.9}$$

$$e^{\nu}=(Ar^{1-2^{\frac{1}{2}}}-Br^{1+2^{\frac{1}{2}}})^2, \tag{5.10}$$

where a is again an arbitrary constant of integration.

It will be noticed of course that $a=0$ in (5.6) will give the Tolman solution, and in this sense our solution will include the Tolman solution as a special case.

We have pointed out that the Tolman solution can apply only in the range $0 \leqslant n \leqslant 1$. Moreover for $n=1$ this solution reduces to that of spacial relativity. When $n=1$ in our generalized solution we find

$$e^{\nu}=(A-Br^2)^2, \tag{5.11}$$

$$e^{-\lambda}=1+ar^2(A-3Br^2)^{-\frac{2}{3}}. \tag{5.12}$$

This gives a new solution which does not reduce to the solution corresponding to special relativity unless $a=0$. Satisfying the boundary conditions we find

$$A=(1-5m/2r_b)/(1-2m/r_b)^{\frac{1}{2}}, \tag{5.13}$$

$$B=-m/2r_b^3(1-2m/r_b)^{\frac{1}{2}}, \tag{5.14}$$

$$a=-2m(1-m/r_b)^{\frac{2}{3}}/r_b^3(1-2m/r_b)^{\frac{1}{3}}, \tag{5.15}$$

where m is the mass and r_b is the boundary of the sphere. From (1.4) we find that the density ρ is given by

$$8\pi\rho=-a(A-3Br^2)^{-5/3}(3A-5Br^2). \tag{5.16}$$

For small values of m/r_b we notice that $A\sim 1$, $B\sim -m/2r_b^3$, and $a\sim -2m/r_b^3$. Thus the boundary density ρ_b will be given, approximately, by (5.16), to be $8\pi\rho_b\sim 6m/r_b^3$. Hence $m\sim 4\pi\rho_b r_b^3/3$. Thus in our solution the mass is determined by the boundary radius r_b and a physical constant ρ_b which will be determined by the physical constitution of the sphere. In the similar example of Tolman's solution it was found that the mass was determined only by the radius of the sphere, and it was shown that this solution would apply at most to one sphere. Since the parameter ρ_b still remains undetermined in our solution it will be possible to apply this solution to spheres of different physical composition.

In addition to the above we have, in our solution, been able to remove the restriction that $0 \leqslant n \leqslant 1$ which was found necessary in order that the Tolman solution should have physical significance.

6. TOLMAN'S SOLUTION V

In the same paper as mentioned before Professor Tolman gives

$$e^{-\lambda}=a-(r/R)^N, \quad e^{\nu}=b^2r^{2n}, \tag{6.1}$$

where $a=(1+2n-n^2)^{-1}$ and $N=2(1+2n-n^2)/(n+1)$ as a new solution of the field equations. In the above n, R, and b are arbitrary constants. If we take the point to view, as was taken in the previous section, that n is an arbitrary constant to be assigned independently of the boundary conditions, then the solution (6.1) does not contain the proper number of arbitrary constants to fulfill the boundary conditions. We propose to show that a solution of the field equations does exist which contains (6.1) as a special case and which contains three arbitrary constants in addition to n.

If we consider the value of $e^{-\lambda}$ to be given by the expression in (2.1) then from the equality of (1.2) and (1.3) the equation to determine ν is

$$\nu''/2-(\lambda'/4+1/2r)\nu'+\tfrac{1}{4}\nu'^2 + (e^\lambda/r^2-\lambda'/2r-1/r^2)=0. \quad (6.2)$$

The substitution $\nu=2\log y$ reduces this equation to

$$y''-(\lambda'/2+1/r)y'+(e^\lambda/r^2-\lambda'/2r-1/r^2)y=0. \quad (6.3)$$

When λ is known, this equation is a homogeneous linear equation of the second order in y. For the particular λ which we have chosen we know, from Tolman's work, that $y=r^n$ is a particular solution of (6.3). From the theory of linear equations of the second order we know that the general solution of (6.3) can therefore be obtained by a quadrature. It is not difficult to show that this general solution is

$$y=cr^n\int x^p dx/(1-x)^{\frac{1}{2}}+Br^n, \quad (6.4)$$

where $x=(1+2n-n^2)(r/R)^N$ and $p=2n/(n^2-2n-1)$. The integral occurring in (6.4) can be evaluated in terms of elementary functions only if p or $p-\frac{1}{2}$ is an integer, either positive, negative, or zero. For all values of p, (6.4) can be expressed by the hypergeometric function to be

$$y=cr^{2-n}F(\tfrac{1}{2},p+1;p+2;(1+2n-n^2)(r/R)^N)+Br^n. \quad (6.5)$$

For this case the resulting solution of the field equations is

$$e^{-\lambda}=(1+2n-n^2)^{-1}-(r/R)^N, \quad (6.6)$$

$$e^\nu=[cr^{2-n}F(\tfrac{1}{2},p+1;p+2;(1+2n-n^2)(r/R)^N)+br^n]^2. \quad (6.7)$$

Since the above solution degenerates to the Tolman solution of $c=0$ we thus have obtained a generalization of the Tolman solution. We note that the constants n, R, c, and b are all arbitrary. Again we can assign n in arbitrarily, and determine R, c, and b from the boundary conditions.

As examples of elementary solutions we might point out that $n=0$ results in the Schwarzschild interior solution. To illustrate a new solution we take $n=1$. Thus $N=2$, $p=-1$. For this case the hypergeometric function of (6.7) degenerates into a constant and the two particular solutions as given in (6.7) are not linearly independent. Returning to (6.4) we easily find that

$$e^{-\lambda}=\tfrac{1}{2}-(r/R)^2,$$
$$e^\nu=(cr\log\{(1-(1-2(r/R)^2)^{\frac{1}{2}})/(1+(1-2(r/R)^2)^{\frac{1}{2}})\}+br)^2. \quad (6.8)$$

Similarly one can obtain from (6.4) many other examples of solutions of the field equations which can be expressed in terms of the known elementary functions.

7. TOLMAN'S SOLUTION VIII

By making the mathematical assumption that $e^{-\lambda}=\text{const.}r^{-2b}e^\nu$, Professor Tolman was able to obtain a new solution of the field equations. The results as given by Professor Tolman are not quite complete as we shall show when we generalize the procedure introduced by him. If one replaces the above mathematical assumption by the requirement

$$e^{-\lambda}=f(r)e^\nu, \quad (7.1)$$

where $f(r)$ is to considered an arbitrary but known function, then it is possible to show that the final problem of solving the field equation reduces to the solution of a second-order non-homogeneous linear differential equation. Since the theory of such equations is well known this provides another general procedure worth investigating in our attempt to find solutions which are of physical interest.

Adding (1.2) and (1.4) we find

$$8\pi(p+\rho)=e^{-\lambda}(\lambda'+\nu')/r. \quad (7.2)$$

From (7.1)

$$\lambda'+\nu'=-f'(r)/f(r). \quad (7.3)$$

Hence

$$8\pi(p+\rho)=-f'(r)e^\nu/r. \quad (7.4)$$

From (1.5) and (7.4)

$$8\pi p'=f'(r)e^\nu\nu'/2r. \quad (7.5)$$

Using (1.2) and (7.1) our expression for the pressure p is given by

$$8\pi p=[f(r)e^\nu\nu']/r+[e^\nu f(r)]/r^2-1/r^2. \quad (7.6)$$

Differentiating (7.6) with respect to "r" and equating the result to the right-hand side of (7.5) we find

$$r^2f(r)(d^2e^\nu/dr^2)+\tfrac{1}{2}r^2f'(r)(de^\nu/dr) + (rf'(r)-2f(r))e^\nu=-2. \quad (7.7)$$

Thus with $f(r)$ considered as a known function, Eq. (7.7) becomes a linear differential equation of the second-order to determine e^ν.

Since the procedure outlined is purely mathematical, many of the solutions obtained in this way will be of no physical interest. Whether or not a solution will be of physical interest will depend on a suitable choice of $f(r)$. Although we have very little to guide us in making a suitable choice of $f(r)$, there are a few clues worth pointing out. Since $e^{-\lambda}$ and e^{ν} are positive we must have that $f(r)$ is a positive function. Moreover, the boundary condition $(e^{-\lambda})_{r=r_b}=(e^{\nu})_{r=r_b}$ immediately implies $f(r_b)=1$. From Eq. (7.4) we can also see that $f'(r)$ must be negative since p, ρ, e^{ν}, and r are all positive quantities. These requirements of course provide very little restriction on our choice of $f(r)$ but at least they do provide a guide in making such a choice.

In the solution obtained by Professor Tolman, $f(r)$ was given by $f(r)=c^2r^{-2b}$. With this choice (7.7) becomes

$$r^2(d^2e^{\nu}/dr^2)-br(de^{\nu}/dr)-2(b+1)e^{\nu}=-2r^{2b}/c^2. \quad (7.8)$$

Tolman's results are complete except for the one case in which r^{2b} happens to be a solution of the homogeneous equation. The only positive value of b for which this is true is $b=1+2^{\frac{1}{2}}$. For this case the general solution of (7.8) is

$$e^{\nu}=-\{2r^{2b}\log r/c^2(3b-1)\}+\alpha r^{2b}+\beta r^{1-b}, \quad (7.9)$$

where α and β are constants of integration and $b=1+2^{\frac{1}{2}}$. From this

$$e^{-\lambda}=\{-2\log r/(3b-1)\}+\alpha c^2+\beta c^2 r^{1-3b}. \quad (7.10)$$

With the solution given by (7.9) and (7.10) Tolman's results become complete for all positive values of b. Negative values of b are of course excluded because of the fact that $f'(r)$ must be negative.

As an example of a new solution found by this method we shall take $f(r)=a-br^2$ where a and b are both positive constants. With this choice of $f(r)$ Eq. (7.7) becomes

$$r^2(a-br^2)(d^2e^{\nu}/dr^2)-br^3(de^{\nu}/dr)-2ae^{\nu}=-2. \quad (7.11)$$

An obvious solution of the non-homogeneous equation is $e^{\nu}=1/a$. In order to complete the problem we must therefore find the general solution of the homogeneous equation. For the homogeneous equation the substitution

$$e^{\nu}=r^2y, \quad x=br/a$$

reduces this equation to

$$x(1-x)(d^2y/dx^2) +((5/2)-3x)(dy/dx)-y=0. \quad (7.12)$$

This is a hypergeometric equation whose general solution can be put into the form

$$y=AF(1,1;5/2;x) +Bx^{-\frac{3}{2}}F(-\tfrac{1}{2},-\tfrac{1}{2};-\tfrac{1}{2};x), \quad (7.13)$$

where A and B are again arbitrary constants of integration. However, both of the hypergeometric functions of (7.13) can be expressed in terms of elementary functions and we find

$$y=(A/x)[(1-(1-x)/x)^{\frac{1}{2}}\text{ arc sin}x^{\frac{1}{2}}] +Bx^{-\frac{3}{2}}(1-x)^{\frac{1}{2}}. \quad (7.14)$$

This leads to the following solution of the non-homogeneous equation

$$e^{\nu}=A[1-(R^2/r^2-1)^{\frac{1}{2}}\text{ arc sin}(r/R)] +(B/r)(1-(r/R)^2)^{\frac{1}{2}}+1/a, \quad (7.15)$$

$$e^{-\lambda}=a(1-(r/R)^2)e^{\nu}. \quad (7.16)$$

In this section we have given a brief outline of a procedure that can be used to generate new solutions of the field equations. We have pointed out that the procedure is unsatisfactory in that a judicious choice of $f(r)$ must be made. It has however been possible to point out a few requirements that $f(r)$ must satisfy which help in making this choice of $f(r)$. Functions of the form $(a-br^n)^m$ and ae^{-br^n} would satisfy these restrictions, and the resulting solutions of the field equations would be worth investigating. However, in any case in which the procedure of this section is followed, a thorough investigation of the physical consequences of the solution should be made in order to determine whether or not the solution is of physical interest.

8. CONCLUSION

For the main part our paper has been a critical examination of some of the known solutions of the gravitational field equations. In many cases we were able to show that the known solutions were particular cases of a more general class of solutions of the field equations. A new method of generating solutions was outlined in Section 7 but it was pointed out that this method is still not too satisfactory from the physical point of view.