On Generalizing Boson Field Theories

ALEX E. S. GREEN
Department of Physics, University of Cincinnati, Cincinnati, Ohio
(Received December 13, 1948)

If a boson field theory is generalized to admit in the field Lagrangian all the derivatives of the field coordinates up to the σth in a particularly symmetrical way, then we find that in the final expression for the interaction energy of fermions the usual interaction is replaced by a weighted sum of σ such interactions, where the relative weight factor and the boson mass associated with each interaction are given uniquely by algebraic relationships involving the constants appearing in the field Lagrangian. We thus are able to formulate a simple principle for obtaining the interaction energy according to a multiple-boson theory of the form suggested by our generalizations if the interaction energy is known for the one-boson case. The principle is then applied to the non-relativistic and relativistic interaction terms for electrons according to electrodynamic (photon) field theory, and for nucleons according to the meson field theories investigated by Kemmer. We find that in almost every case the weight factors are such that not only are the inadmissible R^{-3} and R^{-2} singularities removed, but also in three- and four-boson theory, the objectionable R^{-1} singularities. A closely related result is the fact that generalization justifies the neglect of short wave-lengths in the evaluation of the interaction and self-energy integrals.

1. THE BOSON FIELD

CONSIDER the quadratic Lagrangian density

$$L=(1/2a^2)\sum_\sigma c_\sigma a^{2\sigma}Q_{\lambda_\sigma}Q_{\lambda_\sigma}, \qquad (1.1)$$

where a is a length, the c's are dimensionless natural constants, $Q(x_\nu)$ the field coordinate is a real or imaginary potential, $x_\nu=(\mathbf{r}, ict)$, $\partial_{\lambda_1}=\partial/\partial x_{\lambda_1}$ and $Q_{\lambda_\sigma}=\partial_{\lambda_1}\partial_{\lambda_2}\cdots\partial_{\lambda_\sigma}Q$. Applying the variational equation

$$ic\delta W=ic\delta\int\int L\,dV\,dt=\delta\int L\,d\Omega=0, \qquad (1.2)$$

using

$$Q_{\lambda_\sigma}\delta Q_{\lambda_\sigma}=(-\Box)^\sigma Q\delta Q +\partial_\nu\left\{\sum_{\tau=0}^{\sigma-1}[(-\Box)^{\sigma-1-\tau}\partial_\nu Q_{\lambda_\tau}]\delta Q_{\lambda_\tau}\right\}, \qquad (1.3)$$

and assuming that the potential and all derivatives up to the order $\sigma-1$ are held constant on the boundaries of the four-dimensional space, we obtain the equation of motion

$$[\sum_\sigma c_\sigma a^{2\sigma}(-\Box)^\sigma]Q=0. \qquad (1.4)$$

Alternatively, in performing the variation we may obtain the same equation of motion by using in place of Eq. (1.3), the relation

$$Q_{\lambda_\sigma}\delta Q_{\lambda_\sigma}=(-\Box)^\sigma Q\delta Q +\sum_{\tau=1}^{\sigma}\left\{\binom{\sigma}{\tau}[(-\Box)^{\sigma-\tau}Q_{\lambda_\tau}]\delta Q\right\}_{\lambda_\tau}. \qquad (1.5)$$

Applying Gauss's theorem to convert successively the τ-dimensional integral of a divergence into $\tau-1$-dimensional integrals, we need only insist that the potential be held constant on the boundaries providing in four-space the Lagrangian contains only derivatives up to the fourth. This latter generalization is thus a less radical departure from the usual Hamilton's principle.

We define the energy momentum tensor $t_{\mu\nu}$ and the energy momentum four-vector P_μ by

$$\delta W=\int t_{\mu\nu}ds_\nu\delta x_\mu=P_\mu\delta x_\mu, \qquad (1.6)$$

where δx_μ represents a displacement of a space-like boundary, and the potential satisfies Eq. (1.4) throughout the four-dimensional domain. Using (1.3), Gauss theorem, and

$$\delta Q_{\lambda_\tau}=-(\partial_\mu Q_{\lambda_\tau})\delta x_\mu, \qquad (1.7)$$

we find that

$$ict_{\mu\nu}=L\delta_{\mu\nu}-(1/a^2)\sum_\sigma c_\sigma a^{2\sigma}\times\left\{\sum_{\tau=0}^{\sigma-1}[(-\Box)^{\sigma-1-\tau}\partial_\nu Q_{\lambda_\tau}]\partial_\mu Q_{\lambda_\tau}\right\}. \qquad (1.8)$$

These results may also be obtained by application of the formulas of Chang[1] to our particular Lagrangian. We may show without difficulty that $\partial_\nu t_{\mu\nu}=0$, thus insuring conservation of energy and momentum, and also that the antisymmetric part of $t_{\mu\nu}$ is a four-divergence.

The equation of motion of the field may be written as

$$[\Pi_\sigma(\Box-\xi_\sigma^2/a^2)]Q(x_\nu)=0, \qquad (1.9)$$

where ξ_σ^2 are the roots (dimensionless) of

$$F(\xi^2)=\sum_\tau c_\tau(-\xi^2)^\tau=0. \qquad (1.10)$$

In view of the basic position of wave motion in field phenomenon, it is satisfactory that our Lagrangian leads directly to a wave equation. A solution of Eq. (1.9) is the generalized Fourier

[1] T. S. Chang, Proc. Camb. Phil. Soc. **44**, 76 (1948).

integral

$$Q(x_\nu)=(1/2\pi)^{\frac{3}{2}}\sum_\sigma\int[Q_\sigma(\mathbf{k})\exp(ik_{\sigma\nu}x_\nu)+Q_\sigma^*(\mathbf{k})\exp(-ik_{\sigma\nu}x_\nu)]d\mathbf{k},\quad(1.11)$$

where

$$k_{\sigma\nu}=(\mathbf{k},\,i\boldsymbol{k}_\sigma)\ \text{and}\ \boldsymbol{k}_\sigma^2-\mathbf{k}\cdot\mathbf{k}=\xi_\sigma^2/a^2=\kappa_\sigma^2.\quad(1.12)$$

Comparing Eq. (1.12) with the energy momentum relationship of special relativity, we find that $\xi_\sigma=am_\sigma c/\hbar$ where m_σ are the masses of the bosons associated with the field. If the masses are to be real, then the c's in the Lagrangian must all have the same parity.

Using Eqs. (1.8) and (1.11) we may express the energy momentum four-vector in terms of Fourier amplitudes. As an aid to calculation we use a boundary corresponding to instantaneous space and discard all time-dependent terms. Both steps are permissible because $\partial_\nu t_{\mu\nu}=0$. We obtain

$$P_\mu=\sum_\sigma(\gamma_\sigma/c)\int k_{\sigma\mu}\boldsymbol{k}_\sigma[Q_\sigma(\mathbf{k})Q_\sigma^*(\mathbf{k})+Q_\sigma^*(\mathbf{k})Q_\sigma(\mathbf{k})]d\mathbf{k},\quad(1.13)$$

where

$$\gamma_\sigma=-\sum_\tau\tau c_\tau(-\xi_\sigma^2)^{\tau-1}=[dF/d\xi^2]_{\xi=\xi_\sigma}.\quad(1.14)$$

The γ's which arise naturally in our theory during this last calculation are factors which weight the fields associated with the different bosons. They alternate in parity, which means that alternate bosons make negative contributions to the total energy and momentum of the field. From (1.13) we see that the field Hamiltonian has the form

$$H=\sum_\sigma\gamma_\sigma\int\boldsymbol{k}_\sigma^2[Q_\sigma(\mathbf{k})Q_\sigma^*(\mathbf{k})+Q_\sigma^*(\mathbf{k})Q_\sigma(\mathbf{k})]d\mathbf{k}.\quad(1.15)$$

This same expression would be obtained from a field Hamiltonian of the form

$$H=\sum_\sigma(\gamma_\sigma/2)\int\{\kappa_\sigma^2[Q_\sigma(x_\nu)]^2+[\nabla Q_\sigma(x_\nu)]^2+\pi_\sigma^2\}dV,\quad(1.16)$$

where $\pi_\sigma=\partial Q_\sigma(x_\nu)/c\partial t$. Our Hamiltonian is thus a weighted sum of σ-terms, each of the form usually used as a starting point in the treatment of the boson field.

TABLE I. Parity of combinations for positive c's.

i	−2	−1	0	1	2	3	4	5	6
I	−	−	−	−	−	−	−	−	−
II	−	−	0	+	+	+	+	+	+
III	−	−	0	+	0	−	−	−	−
IV	−	−	0	+	0	−	0	+	+

2. THE INTERACTION OF FERMIONS

In the field theoretic treatment of the interaction of two fermions which are coupled by a boson field, one may assume that the interaction Hamiltonian involves only the potentials as (for example) in the charge-like interaction

$$H_i=gQ(x_j^a)+gQ(x_j^b),\quad(2.1)$$

or their first derivatives, as in the dipole-like interaction

$$H_i=-ga[\boldsymbol{\sigma}^a\cdot\nabla Q(x_j^a)+\boldsymbol{\sigma}^b\cdot\nabla Q(x_j^b)].\quad(2.2)$$

The g's are coupling constants having the dimension of charge, and a is a length which must now be identified with the Compton wave-length of the fermion. In the generalized cases we simply replace $Q(x_j)$ by $\sum_\sigma Q_\sigma(x_j)$. Taking either (2.1) or (2.2) in conjunction with (1.16), we obtain, after application of a classical[2] or quantum-mechanical formalism, the chief interactions

$$V_{ab}=-\sum_\sigma(g^2/4\pi\gamma_\sigma R)\exp(-\kappa_\sigma R),\quad(2.3)$$

$$V_{ab}=a^2(\boldsymbol{\sigma}^a\cdot\nabla)(\boldsymbol{\sigma}^b\cdot\nabla)\sum_\sigma(g^2/4\pi\gamma_\sigma R)\times\exp(-\kappa_\sigma R).\quad(2.4)$$

These two special cases demonstrate the general principle governing the change in the interaction energy between fermions which takes place in going from a one-boson theory to the corresponding generalized boson theory. The principle which also follows from general considerations may be stated in two ways.

1. If a one-boson theory gives an interaction energy V, then the corresponding result according to generalized theory is $\sum_\sigma V_\sigma/\gamma_\sigma$.

2. If a one-boson theory gives an interaction energy of the form oJ where o is an operator which does not involve the Compton wave-length of the boson, then the corresponding result according to generalized theory is $o\sum_\sigma J_\sigma/\gamma_\sigma$.

The principle is changed in no way when applied to a theory based upon a vector or tensor field having a plurality of components, or a field with two or three components in isotopic space such as the charged or symmetric theories, providing each component has been introduced and treated in a manner suitable for generalization. Unfortunately, it is sometimes difficult to determine whether these conditions are fulfilled because of the variety of complicated mathematical treatments given to theories which are actually identical in physical

[2] W. Pauli, *Meson Theory of Nuclear Forces* (Interscience Publishers, Inc., New York, 1946), pp. 4–7.

substance. Further, the Lagrangian used may appear quite different from a sum of our one-boson Lagrangians, yet only differ from such a sum by a four-divergence. It may also happen that actual differences in the field Lagrangian are compensated for by the auxiliary conditions or by the symmetry conditions imposed on the potentials. In cases which are doubtful, the principle may be applied on heuristic grounds.

In ordinary meson theory, J is the Yukawa potential which may be written

$$J^{\mathrm{I}}=-(g^2/4\pi a c_1\rho)\exp(-\xi\rho)=-(g^2/4\pi a c_1)(1/\rho-\xi+\xi^2\rho/2-\xi^3\rho^2/6+\cdots), \quad (2.5)$$

where $\rho=R/a$ and $\gamma=-c_1$.

In ordinary electrodynamics, J is the Coulomb potential which is simply a special case of the above, corresponding to $\xi=0$.

For higher degrees of generalization we have, upon inserting the explicit expressions for the γ's,

$$J^{\mathrm{II}}=-(g^2/4\pi a c_2\nabla_{21}\rho)\times[\exp(-\xi_1\rho)-\exp(-\xi_2\rho)], \quad (2.6)$$

$$J^{\mathrm{III}}=-(g^2/4\pi a c_3\nabla_{21}\nabla_{31}\nabla_{32}\rho)[\nabla_{32}\exp(-\xi_1\rho)-\nabla_{31}\exp(-\xi_2\rho)+\nabla_{21}\exp(-\xi_3\rho)], \quad (2.7)$$

$$J^{\mathrm{IV}}=-(g^2/4\pi a c_4\nabla_{21}\nabla_{31}\nabla_{32}\nabla_{41}\nabla_{42}\nabla_{43}\rho)\times[\nabla_{32}\nabla_{42}\nabla_{43}\exp(-\xi_1\rho)-\nabla_{31}\nabla_{41}\nabla_{43}\exp(-\xi_2\rho)+\nabla_{21}\nabla_{41}\nabla_{42}\exp(-\xi_3\rho)-\nabla_{21}\nabla_{31}\nabla_{32}\exp(-\xi_4\rho)], \quad (2.8)$$

where $\nabla_{21}=\xi_2^2-\xi_1^2$, etc. It may be verified that these functions are static solutions of Eq. (1.4). The power series expansion of J^{I} suggests that in generalized theory we investigate the properties of $\sum_\sigma\xi_\sigma{}^i/\gamma_\sigma$ for various integral i's, a study which may be carried out by algebraic methods. It turns out that the parities of these combinations are independent of the magnitudes of the ξ's. They are given in Table I for positive c's (reverse each sign for negative c's).

Now in the various forms of field theory, J appears in interaction terms preceded by differential operators which result in combinations such as J'/R, $R(J'/R)'$, $R^{-2}(R^2J')'$, and $R^3[(J'/R)'/R]'$ (where prime denotes differentiation with respect to R). Upon carrying out these differentiations, we observe that the zeros of $\sum_\sigma \xi_\sigma{}^i/\gamma_\sigma$ occur at exactly the critical values of i, which cause the elimination of the R^{-3}, R^{-2}, and R^{-1} singularities which would ordinarily appear.[3] This remarkable result is closely related to a consequence of generalized theory which may be noted by examining the quantum field theory expression for our generalized potential, namely

$$J=[g^2/(2\pi)^3]\times\int\{\textstyle\sum_\sigma\exp[ik_{\sigma\nu}(x_\nu{}^b-x_\nu{}^a)]/2\mathbf{k}_\sigma{}^2\gamma_\sigma\}d\mathbf{k}. \quad (2.9)$$

Setting the times equal, and integrating over the angles in polar k space, we obtain

$$J=[g^2/(2\pi)^2]\times\int_0^\infty[\textstyle\sum_\sigma k\sin(kR)/\gamma_\sigma R(k^2+\kappa_\sigma{}^2)]dk, \quad (2.10)$$

for large values of k the integrand is

$$\textstyle\sum_\sigma[\sin(kR)/kR](1/\gamma_\sigma-\xi_\sigma{}^2/\gamma_\sigma k^2a^2+\xi_\sigma{}^4/\gamma_\sigma k^4a^4-\cdots). \quad (2.11)$$

In generalized theories this integrand vanishes strongly even in the unfavorable case for $R=0$ which arises in the self-energy calculation. If the J in Eq. (2.10) is preceded by a differential operator, then after differentiation, terms with additional k's in the numerator of the integrand will appear. In these cases it will be the zeros of $\sum_\sigma \xi_\sigma{}^i/\gamma_\sigma$ for $i=2$, 4, etc. which will make the integrand vanish for short wave-lengths. Integrating (2.10) we get our generalized potential

$$J=\textstyle\sum_\sigma(g^2/4\pi a\gamma_\sigma\rho)\exp(-\xi_\sigma\rho).$$

3. THE INTERACTION OF ELECTRONS

The field Lagrangian often used as a starting point in ordinary electrodynamics is a sum of four terms of the form (1.1) corresponding to $c_1=-1$, and all the other c's equal to zero.[4,5] The interaction energy for one photon processes which follows from the usual quantum theory is Breit's interaction,

$$V=[1-\boldsymbol{\alpha}^a\cdot\boldsymbol{\alpha}^b/2+(1/2R)\times(\boldsymbol{\alpha}^a\cdot\mathbf{R})(\boldsymbol{\alpha}^b\cdot\mathbf{R})d/dR]J=\mathcal{B}J, \quad (3.1)$$

where J is the Coulomb potential and $\mathcal{B}$ is Breit's operator.

To obtain the interaction energy of electrons on a generalized theory which admits second derivatives of the field coordinates, as in the theories initiated by Bopp[6] and by Podolsky,[7] we replace the Coulomb potential by the non-singular potential given by (2.6) with $\xi_1=0$. The same result follows from a detailed investigation.[8] For further generalization of electrodynamics we use (2.7) or (2.8).

A reduction of the 16-component Dirac wave equation to a 4-component Pauli form[9] shows that

[3] The reader is invited to test Table I and these results with arbitrary numbers for the ξ's.

[4] L. Rosenfeld, Zeits. f. Physik **76**, 731 (1932), Eq. (8).
[5] J. Schwinger, Phys. Rev. **74**, 1442 (1948), Eq. (1.9).
[6] F. Bopp, Ann. d. Physik **38**, 345 (1940).
[7] B. Podolsky, Phys. Rev. **62**, 68 (1942).
[8] A. E. S. Green, Phys. Rev. **72**, 628 (1947).
[9] G. Breit, Phys. Rev. **51**, 258 (1937).

the relativistic interaction terms (retardation, spin orbit, spin-spin, tensor force, and Dirac character), which arise at this stage of approximation, are associated with the derivatives of J previously listed. Thus, we find that all the R^{-3} and R^{-2} singularities vanish upon generalization as well as the R^{-1} singularities (which are here not objectionable) in the three- and four-boson cases.

4. THE INTERACTION OF NUCLEONS

The most extensive investigation of the interaction of nucleons due to one-meson fields is that of Kemmer,[10] who derived eight distinct meson interactions. After making the necessary modifications in constants, his results for neutral mesons in the non-relativistic limit may be written (in obvious notation) as

$$V_{vg}=J=-V_{sg}, \quad (4.1) \qquad V_{sf}=V_{psf}=0, \quad (4.2)$$

$$V_{vf}=2\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b a^2R^{-2}(R^2J')'/3 - S_{ab}a^2R(J'/R)'/3=-V_{pvf}, \quad (4.3)$$

$$V_{psg}=\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b a^2R^{-2}(R^2J')'/3 + S_{ab}a^2R(J'/R)'/3=-V_{pvg}, \quad (4.4)$$

where $S_{ab}=3(\boldsymbol{\sigma}^a\cdot\mathbf{R}/R)(\boldsymbol{\sigma}^b\cdot\mathbf{R}/R)-\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b$.

We see that all of the usual Kemmer interactions become well behaved upon generalization because of the immunity of our generalized potential to singularities when subjected to the above operators. Here it is advantageous to go to the three- or four-meson case to eliminate the R^{-1} singularities which are now objectionable because of the infinite self-energies they imply.

Consider for further illustration several of Kemmer's relativistic forms.

$$V_{sg}'=-\beta^a\beta^bJ, \quad (4.5)$$

$$V_{sf}'=a^2(\boldsymbol{\alpha}^a\cdot\nabla)(\boldsymbol{\alpha}^b\cdot\nabla)J, \quad (4.6)$$

$$V_{vg}'=[1-\boldsymbol{\alpha}^a\cdot\boldsymbol{\alpha}^b+(1/\kappa^2)(\boldsymbol{\alpha}^a\cdot\nabla)(\boldsymbol{\alpha}^b\cdot\nabla)]J. \quad (4.7)$$

The relativistic terms due to V_{sg}', V_{sf}', and the first two parts of V_{vg}' are well behaved in the generalized cases, however, a retardation term and spin orbit term arising from $(1/\kappa^2)(\boldsymbol{\alpha}^a\cdot\nabla)(\boldsymbol{\alpha}^b\cdot\nabla)J$ have inadmissible singularities which do not vanish upon generalization. By an alternate formalism involving, among other things, a different treatment of the supplementary conditions, the writer has obtained in this case[11] the interaction $V_{vg}'=\mathcal{B}J$, whose relativistic terms are well behaved in the generalized cases. The complications which arise in the treatment of the supplementary conditions, however, require that the efficacy of generalization for the relativistic terms in the vector and also in the pseudo-vector case be regarded as provisional.

The fact that the generalized meson potential and its important derivatives have simple, well-behaved shapes enables us to compare qualitatively the results of the numerous possible generalized meson theories with the more successful phenomenological theories. We can thus readily find one-meson interactions which, upon generalization, will account roughly for the position of the ^3S and ^1S levels of the deuteron and the sign of the quadrupole moment. However, to arrive at quantitative conclusions concerning these and other equations would require a specific assignment of the meson masses and the g or f (or g's and f's if more than one force is used) followed by detailed numerical computations. Such a task involving a many-parameter set of numerical computations might appropriately be undertaken by a computation laboratory.

Examples of generalized potentials which correspond to meson masses which have been reported are[12,13]

$$J=\mp(A/\rho)[\exp(-0.10\rho)-1.07\exp(-0.16\rho)+0.07\exp(-0.5\rho)], \quad (4.8)$$

$$J=\mp(B/\rho)[\exp(-0.05\rho)-1.527\exp(-0.10\rho)+0.530\exp(-0.16\rho)-0.003\exp(-0.5\rho)]. \quad (4.9)$$

There is an intimate connection between generalized meson theories and mixture theories. Thus, a two-meson generalization of V_{sg} will give *exactly* the interaction obtained by mixing V_{sg} with V_{vg} where the masses are different but the couplings are equal. A three-meson generalization of V_{sg} will give *exactly* the interaction obtained by mixing two V_{sg}'s with one V_{vg} where the masses are different and the relative couplings are determined by (2.7). A similar conclusion follows for the four-meson generalization and for the other sets.

It must be noted that, although the mixture theories which have been used are the least objectionable of the relativistic theories of nuclear forces, they may be criticized (1) for their *ad hoc* nature, (2) for the inadmissible singularities which persist amongst the relativistic terms,[14] and (3) for their failure in quantitative details.[15] Generalized theories are certainly free from the first objection and quite possibly from the second. It remains to be seen whether other theoretical objections will be raised to generalization, and whether a generalized meson theory will succeed in quantitatively accounting for nuclear phenomenon.

---

[10] N. Kemmer, Proc. Roy. Soc. **A166**, 127 (1938).
[11] A. E. S. Green, Phys. Rev. **73**, 26 (1948).
[12] Lattes, Occhialini, and Powell, Nature **160**, 453 (1947).
[13] L. Leprince-Ringuet and M'Lheritier, J. de Phys. et rad. **7**, 66 (1946).
[14] Ning Hu, Phys. Rev. **67**, 339 (1945).
[15] W. H. Ramsey, Proc. Phys. Soc. **61**, 297 (1948).