Division of the first of these equations by the second yields

$$\varphi_2=\varphi_1\cos k\beta+\frac{\beta k}{2}\varphi_1\sin ka$$

$$=\varphi_1\left(\cos k\beta+\frac{k\beta}{2}\sin k\beta\right)\cdot \quad (55)$$

Insertion into Eq. (52) yields

$$0=-k\varphi_1\sin k\beta$$
$$-4\pi\epsilon\beta\left(\frac{n_0\epsilon}{mV_W{}^2}\left(\cos k\beta+\frac{k\beta}{2}\sin k\beta\right)\right)\varphi_1$$
$$-4\pi\epsilon\beta n_1\bigg/1-\cos k\beta-\frac{k\beta}{2}\sin k\beta, \quad (56)$$

or

$$\varphi_1\left(k\sin k\beta+\frac{4\pi\epsilon^2}{m}\frac{\beta n_0}{V_W{}^2}\times\left(\cos k\beta+\frac{k\beta}{2}\sin k\beta\right)\right)$$
$$=\frac{-4\pi\epsilon\beta n_1}{1-\cos k\beta-\frac{k\beta}{2}\sin k\beta}.$$

We can solve for φ_1 from the above, then for φ_2 from (55), and finally for n_+-n_0 from (53). This provides a pulse solution, in which V_W, the wave velocity, and n_1, the number of trapped particles, may be specified arbitrarily.

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Theory of Plasma Oscillations. B. Excitation and Damping of Oscillations

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The theory of electron oscillations of an unbounded plasma is extended to take into account the effects of collisions and special groups of particles having well-defined ranges of velocities. It is found that as a result of collisions a wave tends to be damped in a time of the order of the mean time between collisions. If beams of sharply defined velocity or groups of particles far above mean thermal speeds are present, however, they introduce a tendency toward instability so that small oscillations grow until limited by effects not taken into account in the linear approximation. An estimate is made of the steady-state amplitude for plasma oscillations in which excitation occurs because of a peak at high velocities in the electron velocity distribution, and in which the main damping arises from collisions. It is also found that in variable density plasmas, waves moving in the direction of decreasing plasma density show even stronger instability.

In absence of plasma oscillations, any beam of well-defined velocity is scattered by the individual plasma electrons acting at random, but, when all particles act in unison in the form of a plasma oscillation, the scattering can become much greater. Because of the instability of the plasma when special beams are present, the beams are scattered by the oscillations which they produce. It is suggested that this type of instability can explain the results of Langmuir, which show that beams of electrons traversing a plasma are scattered much more rapidly than can be accounted for by random collisions alone. It is also suggested that this type of instability may be responsible for radio noises received from the sun's atmosphere and from interstellar space.

I. INTRODUCTION

In the preceding paper (referred to as A), we gave a theory of oscillations of an unbounded plasma, neglecting collisions, and treating in detail only ion gases with a continuous distribution of velocities, which decreases monotonically with increasing velocity. In this paper, we extend the theory to include effects of collisions and more general velocity distributions, showing how these can bring about excitation and damping of plasma oscillations.

II. EFFECTS OF COLLISIONS

A collision may be said to occur whenever two particles come so close together that a sudden transfer of momentum takes place, which is so rapid that for macroscopic phenomena, such as wave motion, it may be regarded as instantaneous. These momentum transfers occur at random relative to the phase of organized wave motion; hence, their general effect is to disrupt it and to cause damping. Because of persistence of velocity, not all of the organized motion will be lost, but in a close collision of an electron with a heavy object, such as a neutral atom or an ion, the persistence of velocity is not very important, and one can, in a rough quantitative treatment such as this, neglect it altogether. We therefore take a simplified model of these collision processes, and assume that particles emerge from a collision with no relation to their previous velocity, but with a velocity distri-

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bution, $f(\mathbf{V}_0)$, which is the same as that prevailing in the absence of plasma oscillations (usually very nearly Maxwellian). Between collisions, the particle moves in the average field of all the other particles, gaining an ordered component to its motion, which is lost in the next collision. As a result, there is a continual process of degradation of ordered energy by collisions. The description is completely analogous to the Lorentz theory of collision broadening of spectral lines.

Although this method provides a good representation of short range collisions, such as those between electrons and neutral molecules, it is not so good for Coulomb collisions, in which, because of the long range of the forces, there is a preponderance of distant impacts involving small momentum transfers and a great deal of persistence of velocities. As shown in A (Section II), the very distant collisions are best treated in terms of the smeared-out average field. A rough distinction between collision forces and the ordered average component of the force can be obtained by saying that momentum transfers occurring in less than a period of oscillation do not contribute to the ordered motion, but instead, tend to disrupt it. This provides an upper limit on the impact parameter, P, for collisions. To compute this limit, we note that the time of collision is of the order of P/V. Setting this equal to the period of a plasma oscillation, we get $P/V\sim\cdot 2\pi/(4\pi n_0\epsilon^2/m)^{\frac{1}{2}}$. With $V\sim(\kappa T/m)^{\frac{1}{2}}$, one obtains $P\sim 2\pi(\kappa T/4\pi n_0\epsilon^2)^{\frac{1}{2}}$. This is just 2π times the Debye length which is the characteristic shielding radius for the plasma (see A, Section VII).

The Debye length is usually much larger than an atomic radius; hence, there are a great many electron-ion collisions of range so long that the momentum transfer is very small. One can account for these collisions roughly by adding in a mean free path for electron-ion collisions. It is well known[1] that this mean free path depends on the logarithm of the maximum collision parameter, which should be taken equal to a Debye length. Because of the logarithmic dependence, the ambiguity introduced by the arbitrary separation of collisions from the ordered component of the force at this length will be unimportant.

In electron-electron collisions, one deduces from the conservation of momentum that the total current is not changed in a collision process. It can be shown, by an extension of our treatment, which is not, however, included in this paper, that for wavelengths appreciably longer than a Debye length, an electron-electron collision leaves the ordered component of the motion unaltered, to a very high degree of approximation. Hence, one can ignore the damping due to electron-electron collisions. At very high plasma densities, such collisions will, however, make possible the transmission of sound waves, by the same mechanism as occurs in a non-ionized gas.

It is instructive to estimate the various mean-free paths for typical cases. At a pressure of 5×10^{-3} mm, the free path for electron-gas collisions is about 10 cm, while that for electron-ion collisions (ion density $\sim 10^{11}/\text{cm}^3$) is about $4E^2$ cm, where E is the mean electronic energy in electron volts. At a typical electronic temperature of 3 ev, this is about 36 cm. Since the mean electronic speed is of the order of 10^8 cm/sec., the inter-collision time is about 10^{-7} sec., which is about 1000 times the period of a plasma oscillation, 10^{-10} sec. Hence, the effect of collisions will be small, and the approximation of paper A, neglecting them, is justified.

The lowest density plasmas that are known probably occur in the gas clouds of interstellar space, where the ion density is of the order of 1 per cm^3, the plasma frequency is about 10^4 c.p.s., and the mean free path about 10^{12} cm. The electronic temperature is of the order of a few ev, and the time between collisions is about 10^4 sec. Hence, the neglect of collisions is a good approximation here also.

In an actual plasma, the system is seldom in thermodynamic equilibrium, because ions and electrons are continually being generated by processes such as photo-ionization, ionization by impact, and direct injection from a cathode, while they leave by recombination, either in the body of the plasma or at the walls. The generation of electrons can alter the velocity distribution function, $f(\mathbf{V}_0)$ in such a way that peaks occur at high velocity. Injection from a cathode will introduce a beam of sharply defined velocity.

The electrons are usually liberated with more than the mean thermal energies, but they tend to lose their excess, either by collision with individual particles, or, as we shall see, by interaction with organized plasma oscillations which result from the instability of the plasma. Thus, a plasma is continually degrading a stream of energy into heat.

III. MATHEMATICAL TREATMENT

We shall use a method here in which the motion of each particle is traced from the time at which it makes its last collision to the present. To do this, we label each particle according to the position and velocity $(\mathbf{x}_0, \mathbf{V}_0)$, which it had when it emerged from its last collision, at the time t_0. We assume that in the process of collision the particles lose all trace of any previous ordering in velocities. This means that one can write the distribution of particles emerging from collisions in unit time as a product of terms involving, respectively, the

[1] S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, 1942), p. 75.

spatial and velocity dependences, as follows:

$$dN=\frac{N(\mathbf{x}_0,t_0)}{\langle\tau\rangle_{\text{Av}}}f(\mathbf{V}_0)d\mathbf{V}_0d\mathbf{x}_0, \tag{1}$$

where $N(\mathbf{x}_0, t_0)$ is the total density of particles existing at the point $\mathbf{x}_0, t_0$, and $f(\mathbf{V}_0)$ is the velocity distribution with which the particles come off after collision. $f(\mathbf{V}_0)$ is normalized so that $\int f(\mathbf{V}_0)d\mathbf{V}_0=1$. If any particles are being liberated by ionization or injection, these should be included in $f(\mathbf{V}_0)$, and for these particles $\mathbf{x}_0$, $\mathbf{V}_0$ refer to the position and velocity with which the particles entered the plasma.[a] $\langle\tau\rangle_{\text{Av}}$ is the velocity average of the mean time between collisions.

We first seek wave-like solutions of the form

$$\varphi=\varphi_0\exp i(\mathbf{k}\cdot\mathbf{x}-\omega t),$$
$$N(\mathbf{x},t)=n_0+\delta N_0\exp i(\mathbf{k}\cdot\mathbf{x}-\omega t),$$

where n_0 is the equilibrium density of particles. As in A, we shall obtain the dispersion relation, defining ω as a function of $\mathbf{k}$.

In the presence of the average electric field, each electron changes its velocity according to the equation of motion

$$m(d\mathbf{V}/dt)=\epsilon\nabla\varphi=i\epsilon\mathbf{k}\varphi_0\exp i(\mathbf{k}\cdot\mathbf{x}-\omega t).$$

To solve for the velocity change, one must know $\mathbf{x}$ as a function of t. Because φ_0 is small, however, the velocity must remain close to its initial value, $\mathbf{V}_0$, so that to a first approximation, $\mathbf{x}=\mathbf{x}_0+\mathbf{V}_0(t-t_0)$. This approximation is adequate to insert in the right-hand side of the above equation, as the latter is already proportional to the first order term φ_0. The result is

$$m(d\mathbf{V}/dt)\cong i\epsilon\mathbf{k}\varphi_0\exp i(\mathbf{k}\cdot\mathbf{x}_0-\omega t_0)\times\exp i(\mathbf{k}\cdot\mathbf{V}_0-\omega)(t-t_0). \tag{2}$$

Integration, with the boundary condition that $\mathbf{V}=\mathbf{V}_0$ at $t=t_0$, yields

$$\mathbf{V}\cong\mathbf{V}_0-\frac{\epsilon\mathbf{k}\varphi_0}{m}\frac{\exp i(\mathbf{k}\cdot\mathbf{x}_0-\omega t_0)}{(\omega-\mathbf{k}\cdot\mathbf{V}_0)}\times(\exp i(\mathbf{k}\cdot\mathbf{V}_0-\omega)(t-t_0)-1).$$

In the subsequent work, we shall need to express $\mathbf{V}_0$ as a function of $\mathbf{V}$. To first order, one easily sees that this result is

$$\mathbf{V}_0=\mathbf{V}+\frac{\epsilon\mathbf{k}\varphi_0}{m}\frac{\exp i(\mathbf{k}\cdot\mathbf{x}_0-\omega t_0)}{(\omega-\mathbf{k}\cdot\mathbf{V})}\times(\exp i(\mathbf{k}\cdot\mathbf{V}-\omega)(t-t_0)-1)=\mathbf{V}-\delta\mathbf{V}, \tag{3}$$

[a] This schematic description is a way of treating the effects on plasma oscillations of collisions and high velocity particles without simultaneously entering into a detailed discussion of the mechanisms by which the steady-state distribution is maintained.

where we have replaced $\mathbf{V}_0$ by $\mathbf{V}$ in all first-order terms. One can find $\mathbf{x}$ in terms of $\mathbf{x}_0$ by integration of (3), but we shall not do so here, because the result is not needed in an explicit form.

Let us now consider all of the particles passing through the point $\mathbf{x}$, at the time, t. The probability that such particles suffered their last collisions between $\mathbf{x}_0$ and $\mathbf{x}_0+d\mathbf{x}_0$ will be denoted by $P(\mathbf{x},\mathbf{x}_0,\mathbf{V}_0,t)d\mathbf{x}_0$. To a first approximation, it is equal to $(1/l)\exp-|\mathbf{x}-\mathbf{x}_0|/l)$, where l is the mean free path. If the mean free path depends on the velocity, the expression is more complicated because the particle velocity changes as the particle moves under the action of the field. Since the motion is determined in terms of $\mathbf{x}$, $\mathbf{x}_0$, $\mathbf{V}_0$, t, the accurate expression for this probability is, at most a function of these. If there were no collisions, the number of particles starting between $\mathbf{x}_0$ and $\mathbf{x}_0+d\mathbf{x}_0$ with velocities between $\mathbf{V}_0$ and $\mathbf{V}_0+d\mathbf{V}_0$ would be given by Eq. (1) as

$$\delta N=N(\mathbf{x}_0,t_0)f(\mathbf{V}_0)d\mathbf{x}_0d\mathbf{V}_0. \tag{4}$$

One now applies Liouville's theorem, which states that the density of particles in phase space remains constant if one follows a moving particle for which the equations of motion can be derived from a Hamiltonian function. Hence, for a volume element which follows a particle, we can write $d\mathbf{x}d\mathbf{V}=d\mathbf{x}_0d\mathbf{V}_0$. Since the number of particles in this element does not change, we obtain from Eq. (4),

$$\delta N=N(\mathbf{x}_0,t_0)f(\mathbf{V}-\delta\mathbf{V})d\mathbf{x}d\mathbf{V}. \tag{5}$$

By dividing by $d\mathbf{x}$, one obtains the contribution to the density at the point $\mathbf{x}$, resulting from particles starting at the point $\mathbf{x}_0$, with the velocity $\mathbf{V}_0=\mathbf{V}-\delta\mathbf{V}$,

$$\rho(\mathbf{x},\mathbf{x}_0,\mathbf{V}-\delta\mathbf{V},t)d\mathbf{V}=N(\mathbf{x}_0,t_0)f(\mathbf{V}-\delta\mathbf{V})d\mathbf{V}. \tag{6}$$

To obtain the average density, one must multiply the above by the probability that the particle starts between $\mathbf{x}_0$ and $\mathbf{x}_0+d\mathbf{x}_0$, and integrate over $\mathbf{x}_0$ and $\mathbf{V}$. The result is

$$N(\mathbf{x},t)=\int\int N(\mathbf{x}_0,t_0)f(\mathbf{V}-\delta\mathbf{V})\times P(\mathbf{x},\mathbf{x}_0,\mathbf{V}-\delta\mathbf{V},t)d\mathbf{x}_0d\mathbf{V}. \tag{7}$$

Let us now expand $f(\mathbf{V}-\delta\mathbf{V})$ and $P(\mathbf{x},\mathbf{x}_0,\mathbf{V}-\delta\mathbf{V},t)$ as a series of powers of $\delta\mathbf{V}$. We also write

$$N(\mathbf{x}_0,t_0)=n_0+\delta N_0\exp i(\mathbf{k}\cdot\mathbf{x}_0-\omega t_0)=n_0+\delta N(\mathbf{x}_0,t_0).$$

With the retention of first-order terms, we get

$$N(\mathbf{x},t)=\int\int[P(\mathbf{x},\mathbf{x}_0,\mathbf{V},t)(n_0f(\mathbf{V})+\delta N(\mathbf{x}_0,t_0)f(\mathbf{V}))-n_0\delta\mathbf{V}\cdot\nabla_V(fP)]d\mathbf{x}_0d\mathbf{V}. \tag{8}$$

From Eqs. (8), and from the fact that $\int Pd\mathbf{x}_0=1$, we see that integration of the first term over $d\mathbf{x}_0 d\mathbf{V}$ in the integrand yields n_0. The third term may be simplified by noting from Eq. (3) that $\delta\mathbf{V}$ is in the $\mathbf{k}$ direction, so that $\delta\mathbf{V}\cdot\nabla_V$ can be written as $\delta V_k(\partial/\partial V_k)$ where V_k is the component of $\mathbf{V}$ in the $\mathbf{k}$ direction. One can now integrate this expression by parts over V_k, noting that the integrated parts vanish because $f(\mathbf{V})$ vanishes at infinity. We then obtain

$$N(\mathbf{x},t)=n_0+\int\int f(\mathbf{V})P\Bigg[\delta N(\mathbf{x}_0,t_0) -\frac{\epsilon\mathbf{k}\varphi_0 n_0}{m}\exp i(\mathbf{k}\cdot\mathbf{x}_0-\omega t_0)\frac{\partial}{\partial V_k} \times\left(\frac{\exp i(\mathbf{k}\cdot\mathbf{V}-\omega)(t-t_0)-1}{(\omega-\mathbf{k}\cdot\mathbf{V})}\right)\Bigg].$$

Since P now multiplies only first-order terms, we can evaluate it by its first approximation, $(1/l)$ $(\exp-|\mathbf{x}-\mathbf{x}_0|/l)$. To first order, one can also replace $\mathbf{x}-\mathbf{x}_0$ by $\mathbf{V}(t-t_0)$. We also set $l/V=\tau(V)$ $=$mean time between collisions. We get

$$N(\mathbf{x},t)=n_0+\exp i(\mathbf{k}\cdot\mathbf{x}-\omega t)\times\int\int_{t-t_0=-\infty}^{0} d\mathbf{V}\frac{dt_0}{\tau}\exp-(t-t_0)/\tau f(\mathbf{V}) \times\Bigg[\delta N_0\exp i(\omega-\mathbf{k}\cdot\mathbf{V})(t-t_0)-\frac{n_0\epsilon\varphi_0}{mV}k^2 \times\exp i(\omega-\mathbf{k}\cdot\mathbf{V})(t-t_0)\frac{\partial}{\partial k} \times\left(\frac{\exp-i(\omega-\mathbf{k}\cdot\mathbf{V})(t-t_0)-1}{\omega-\mathbf{k}\cdot\mathbf{V}}\right)\Bigg]. \quad (9)$$

The result of carrying out the integration over t_0 is

$$N(\mathbf{x},t)=n_0+\delta N_0\exp i(\mathbf{k}\cdot\mathbf{x}-\omega t)\int_{-\infty}^{+\infty}d\mathbf{V}_0 f(\mathbf{V}_0) \times\left\{\frac{n_0\epsilon k^2\varphi_0}{m(\omega-\mathbf{k}\cdot\mathbf{V}_0+i/\tau)^2}+\frac{i\delta N_0}{\tau(\omega-\mathbf{k}\cdot\mathbf{V}_0+i/\tau)}\right\}. \quad (10)$$

According to Poisson's equation, $-k^2\varphi_0=-4\pi\epsilon\delta N_0$. Eliminating δN_0 between these two equations gives the following dispersion relation

$$1=\int f(\mathbf{V}_0)d\mathbf{V}_0\left[\frac{4\pi n_0\epsilon^2}{m}\frac{1}{(\omega-\mathbf{k}\cdot\mathbf{V}_0+i/\tau)^2} +i/\tau\frac{1}{(\omega-\mathbf{k}\cdot\mathbf{V}_0+i/\tau)}\right]. \quad (11)$$

We note that as the collision time, τ, becomes infinite, this reduces to Eq. (9) of paper A obtained with the neglect of collisions.[b] We shall concern ourselves with situations in which the collision time is much larger than the period of the plasma oscillations. For collision times comparable with a period of oscillation, the use of an average field to describe the forces on plasma particles becomes a bad approximation and our treatment breaks down.[c] This means that we will neglect powers of the quantity $1/\omega_P\tau$ higher than the first. At long wave-lengths, $\mathbf{k}\cdot\mathbf{V}_0/\omega$ is small compared to unity for all $\mathbf{V}_0$ for which $f(\mathbf{V}_0)$ is appreciable. Expanding in powers of $\mathbf{k}\cdot\mathbf{V}_0/\omega$ as in paper A, Eq. (10), we obtain

$$\omega^2=\omega_P^2\int f(\mathbf{V}_0)d\mathbf{V}_0\left[1+\frac{2\mathbf{k}\cdot\mathbf{V}_0}{\omega} +3\left(\frac{\mathbf{k}\cdot\mathbf{V}_0}{\omega}\right)^2+\cdots\right]-\frac{i}{\omega\tau}[2\omega_P^2-\omega^2].$$

With the same assumptions for the distribution function as in A, we find as the dispersion relation for small k,

$$\omega^2=\omega_P^2+(3\kappa T/m)k^2-(i\omega_P/\tau)$$

or

$$\omega=(\omega_P^2+(3\kappa T/m)k^2)^{\frac{1}{2}}-(i/2\tau). \quad (12)$$

This shows that the effect of collisions is to damp the wave. The time necessary for the intensity to fall to $1/e$ of its initial intensity is just the collision time, τ.[d] This result is reasonable because each electron was assumed to lose all ordered motion on collision. As a result, the time needed to dissipate most of the ordered energy is just the mean time between collisions. It should be noted that unless there are specific excitation processes such as those described further ahead in this paper, plasma oscillations cannot persist.

IV. VALIDITY OF LINEAR APPROXIMATION

As shown in paper A, Section III, in the absence of collisions the linear approximation always breaks

[b] Note that Eq. (11) approaches the dispersion relation corresponding to organized motions. The non-medium-like oscillations considered in paper A, Section (VI), are produced mostly by a few particles near the wave velocity. In the collision treatment that we have just given, the density distribution is assumed at all times to be a continuous function of the velocity, so that there can be no waves in which most of the oscillation is carried by a few particles near a special velocity. In order to obtain waves of the latter type, one would have to start out with a density distribution in which a few particles of some definite velocity had a large trigonometric perturbation in density, much greater than that of particles of any other velocity. As shown in A (Section VI), such oscillations can have at most a very small amplitude.

[c] Our treatment is too rough to be able to lead to sound waves in the limit of short free path.

[d] If τ is a function of velocity, the above refers to a suitable average over velocities.

down for particles close to the wave velocity, because these are trapped in the trough of the potential, oscillating back and forth many times. It is easily seen that the effect of collisions is to improve the linear approximation for particles near the wave velocity. From Eq. (3), one sees that each particle suffers an oscillatory change of velocity, and that particles very close to the wave velocity (where $\omega=\mathbf{k}\cdot\mathbf{V}_0$) will in a long time suffer very large changes of velocity, so that the assumption that $\delta\mathbf{V}$ is small, tends to break down. If particles collide before they have time to do this, however, the linear approximation will be valid even for particles near the wave velocity. In the coordinate system in which the wave is at rest, the description of this effect is that the time between collisions is much less than the period of oscillation of a particle in the potential trough.

In order that the linear approximation be valid, it is necessary that for each group of particles of a given velocity, the charge density resulting from the response to the average potential be much less than the unperturbed initial density after collision, $n_0f(\mathbf{V}_0)$. According to Eq. (10), the density resulting from the response to the potential is

$$f(\mathbf{V}_0)\left\{\frac{\epsilon\varphi_0k^2n_0}{m(\omega-\mathbf{k}\cdot\mathbf{V}_0+i/\tau)^2}+i\frac{\delta N_0}{\tau(\omega-\mathbf{k}\cdot\mathbf{V}_0+i/\tau)}\right\}.$$

It is clear that the most unfavorable values for the validity of the linear approximation is $\omega-\mathbf{k}\cdot\mathbf{V}_0=0$. At this point, the first term in the above expression is usually larger than the second in the ratio $\tau^2\omega_P{}^2$, and $\omega_P\tau$ is usually much greater than unity. Hence, our criterion becomes[e]

$$\frac{\epsilon\varphi_0}{m}k^2\tau^2\ll1. \tag{13}$$

Thus, in accordance with our qualitative picture, a short collision time improves the linear approximation.

V. EXCITATION OF PLASMA OSCILLATIONS AS A RESULT OF INSTABILITY

We shall now study some of the processes which can lead to excitation of plasma oscillations, and which can overcome the collision damping. The first of these processes involves the effects of particles near the wave velocity $\mathbf{V}_W=(\omega/k^2)\mathbf{k}$. In Eq. (12) we expanded the dispersion relation, under the assumption that there were so few particles near the wave velocity that their effects could be neglected. If there are an appreciable number of such particles, this expansion is not permissible, and we shall see by a more accurate treatment that either excitation or further damping may be brought about.

If τ is large, the integrand in Eq. (11) has a very high and narrow peak centering at $\mathbf{V}_0=(\omega/k^2)\mathbf{k}$. For small k, this peak occurs at such a high value of $\mathbf{V}_0$ that $f(\mathbf{V}_0)$ is fairly small. It is convenient to divide the domain of integration over $\mathbf{V}_0$ into two regions, with the division line at a value of $\mathbf{V}_0$ large enough so that $f(\mathbf{V}_0)$ is already fairly small, but such that $|1/(\omega-\mathbf{k}\cdot\mathbf{V}_0+i/\tau)|^2$ is also fairly small. The contribution of the first region to the integral can then be obtained by expansion; in fact, the result is identical with that given Eq. (12). The contribution of the second region is now written down explicitly.[f]

$$S=\int_{V_1}^{\infty}\frac{f(V_0)\omega_P{}^2dV_0}{(\omega-kV_0+i/\tau)^2}+\frac{i}{\tau}\int_{V_1}^{\infty}\frac{f(V_0)dV_0}{(\omega-kV_0+i/\tau)}. \tag{14}$$

Since the frequency must, in general, be complex, we write $\omega=\omega_0+i\lambda$. We then get

$$S=\int_{V_1}^{\infty}f(V_0)dV_0\left[\omega_P{}^2\frac{\left((\omega_0-kV_0)^2-\left(\frac{1}{\tau}+\lambda\right)^2-2i(\omega_0-kV_0)\left(\frac{1}{\tau}+\lambda\right)\right)}{\left((\omega_0-kV_0)^2+\left(\frac{1}{\tau}+\lambda\right)^2\right)^2}+\frac{\frac{i}{\tau}(\omega_0-kV_0)+\frac{1}{\tau}\left(\frac{1}{\tau}+\lambda\right)}{(\omega_0-kV_0)^2+\left(\frac{1}{\tau}+\lambda\right)^2}\right]. \tag{15}$$

The dispersion relation is obtained by adding to S, the contribution of the low velocity region, obtained from Eq. (12). We get (for small k, large τ) where ω is close to ω_P,

$$1\cong\frac{\omega_P{}^2}{\omega^2}\left(1+\frac{3k^2\kappa T}{m\omega_P{}^2}\right)-\frac{i}{\omega_P\tau}+S. \tag{16}$$

One must satify both real and imaginary parts of the above equation. With $\omega=\omega_0+i\lambda$, the imaginary

[e] For a typical plasma in a discharge tube $\lambda\approx0.1$ cm, $m\omega_P{}^2/k^2\approx20$ ev, $l\sim10$ cm, so $\tau=l/V\sim10^{-7}$ sec and one obtains $\epsilon\varphi_{\max}\approx m(\omega_P{}^2/k^2)\cdot(1/\omega_P{}^2\tau^2)\sim2\cdot10^{-5}$ ev. In the interstellar plasma $l\approx10^{13}$ cm. With $\lambda=10^5$ cm we have $\epsilon\varphi_{\max}\sim10^{-17}$ ev. We see that near the wave velocity the linear approximation breaks down badly. The effects of particles near the wave velocity on observable plasma oscillations must be studied by the non-linear treatment given in A.

[f] In this work, we restrict ourselves to the one-dimensional case.

part yields

$$0=-\frac{2i\lambda\omega_0\omega_P^2}{(\omega_0^2+\lambda^2)^2}\left(1+\frac{3k^2\kappa T}{m\omega_P^2}\right)-\frac{i}{\omega_P\tau}-i\int_{V_1}^{\infty}f(V_0)dV_0\left[\frac{2(\omega_0-kV_0)\left(\frac{1}{\tau}+\lambda\right)\omega_P^2}{\left((\omega_0-kV_0)^2+\left(\frac{1}{\tau}+\lambda\right)^2\right)^2}-\frac{\frac{1}{\tau}(\omega_0-kV_0)}{(\omega_0-kV_0)^2+\left(\frac{1}{\tau}+\lambda\right)^2}\right]. \tag{17}$$

The real part yields

$$1=\frac{\omega_P^2(\omega_0^2-\lambda^2)}{(\omega_0^2+\lambda^2)^2}\left(1+\frac{3k^2\kappa T}{m\omega_P^2}\right)+\int_{V1}^{\infty}f(V_0)dV_0\left[\frac{\omega_P^2\left((\omega_0-kV_0)^2-\left(\frac{1}{\tau}+\lambda\right)^2\right)}{\left((\omega_0-kV_0)^2+\left(\frac{1}{\tau}+\lambda\right)^2\right)^2}+\frac{\frac{1}{\tau}\left(\frac{1}{\tau}+\lambda\right)}{(\omega_0-kV_0)^2+\left(\frac{1}{\tau}+\lambda\right)^2}\right]. \tag{18}$$

We observe that if $f(V_0)$ is small near the wave velocity, neither the real nor the imaginary part of the frequency will be greatly altered by the integral from V_1, to ∞. Equation (18) will describe mainly a small shift in the real part of the frequency, in which we are not interested, here.[g] Let us then study Eq. (17), from which one can estimate the imaginary part of the frequency. Since the main contribution to the integral comes from a small region near $V_0=\omega/k$, one can expand $f(V_0)=f(\omega_0/k)+(V_0-\omega_0/k)f'(\omega_0/k)+\cdots$. If $\omega_P\tau$ is large, the second term in the integral appearing in Eq. (17) can be shown to be very small compared with the first; hence, we neglect it. The integral then becomes

$$-2\left(\frac{1}{\tau}+\lambda\right)\omega_P^2k\int_{V_1}^{\infty}\frac{\left[f\left(\frac{\omega_0}{k}\right)\left(V_0-\frac{\omega_0}{k}\right)+f'\left(\frac{\omega_0}{k}\right)\left(V_0-\frac{\omega_0}{k}\right)^2+\dots\right]}{\left((\omega_0-kV_0)^2+\left(\frac{1}{\tau}+\lambda\right)^2\right)^2}dV_0.$$

If V_1 is far away from ω_0/k, the range of integration can, with small error, be extended to infinity. The first integral is then zero, because the integrand is an odd function of $V_0-(\omega_0/k)$. The second integral becomes $-2f'(\omega_0/k)(\omega_P^2/k^2)\cdot\pi/2$.

Noting that for small k and large τ, ω is close to ω_P, we obtain the following approximation to λ

$$\lambda\cong-\frac{1}{2\tau}+\frac{\omega_P^3}{k^2}\cdot\frac{\pi}{2}f'(\omega_0/k). \tag{19}$$

The term involving $1/2\tau$ leads to damping; it has already been obtained in Eq. (12). The second term, however, can lead either to further damping, if $f'(\omega_0/k)$ is negative, or to excitation, if it is positive. In a Maxwellian-like distribution, where f decreases monotonically with the velocity, $f'(\omega_0/k)$ is usually so small that the additional damping is negligible in comparison with that due to collisions. In a Maxwellian distribution, for example, one can write $f=(m/2\pi\kappa T)^{\frac{1}{2}}\exp-mV^2/2\kappa T$, obtaining

$$\lambda\cong-\frac{1}{2\tau}-\frac{\pi}{2}\frac{\omega_0}{(2\pi)^{\frac{1}{2}}}\frac{\omega_P^3}{k^3}\left(\frac{m}{\kappa T}\right)^{\frac{3}{2}}\exp-m\omega^2/2\kappa Tk^2.$$

For a typical value of $\omega_P=10^{10}$ c.p.s., $\kappa T\sim1$ ev, and a wave-length of 1 cm, the additional damping rate is $\cong10^{-26}$/sec., which is very small in comparison with 10^{-7} per second, resulting from collisions. For large k, however, the damping is very large; in fact, as one approaches the Debye wave-length, the damping time becomes less than the period of a plasma oscillation. This shows that when collisions are taken into account, one obtains an additional reason why organized oscillations shorter than a Debye length cannot occur; beyond those reasons given in A, Section (VII). The significance of this point will be discussed further ahead.

In order to obtain excitation, one must have a region above mean thermal speeds where $f'(\omega_0/k)$ is positive; this requires a peak in the distribution function at high velocities, or in other words, a special group of high energy particles. The reason

[g] If $f(V_0)$ is large near the wave velocity, then the shift may be large. We consider such cases later, in connection with teams of well-defined velocity.

why particles near the wave velocity can excite or damp plasma oscillations can be understood in terms of the transfer of energy from particles to waves. Let us recall that particles near the wave velocity can interact very strongly with the wave, because they stay in a force of the same phase for a very long time. It will be shown that particles which emerge from collision with a speed greater than that of the wave tend, on the average, to lose energy to the wave by the time they make their next collision, while particles starting out a little slower gain energy. If $(\partial f/\partial V)(\omega_0/k)$ is positive, this means that there are more faster than slower particles, so that the wave tends to be built up at the expense of the kinetic energy of the particles, while a negative $f'(\omega_0/k)$ produces the opposite tendency. In order to build up the wave in this way, it is necessary, then, that there be a group of particles of higher than average velocity, so that $f'(V_0)$ can be positive for a velocity, ω/k, at which a plasma oscillation can exist.[h] If there were no oscillation, this energy would be degraded by collision, but the plasma wave provides a means of intercepting this stream of energy before it is degraded, and transforming it into electrical energy.

Let us first calculate the energy transfer in the linear approximation. If V_0 is the velocity with which a particle emerges from the collision at the point x_0, and the time t_0, and δV is the change of velocity suffered by the particle before it undergoes another collision at the time, t, and the point, x, the kinetic energy lost by the particle to the wave is

$$\Delta E=\frac{m}{2}(V_0^2-(V_0+\delta V)^2)$$
$$=-mV_0\delta V-m\frac{(\delta V)^2}{2}. \quad (20)$$

According to Eq. (3), δV is to the first order, proportional to $\exp i(kx_0-\omega t_0)$; hence, its average will vanish when taken over all possible starting positions, x_0, leaving only second-order contributions. δV must therefore be obtained by solving the equations of motion to second order. Let us denote this solution as $\delta V=\delta V_1+\delta V_2$, where δV_1 is the first-order solution, given in Eq. (3), and δV_2 is the second-order correction. One must average ΔE over x_0 with the weighting factor

$$N(x_0, t_0)=n_0+Re\ \delta N_0 \exp i(kx_0-\omega t_0)=n_0+\delta N.$$

Since ΔE is already of first order, N need only be expressed to first order, in order to obtain a result correct to second order. The mean energy transfer per unit volume per second per unit velocity is then

$$\langle dW_1/dt\rangle_{\text{Av}}=-mf(V_0)/\tau\langle(n_0+\delta N)(V_0(\delta V_1+\delta V_2)+1/2(\delta V_1+\delta V_2)^2\rangle_{\text{Av}} \quad (21)$$

[h] In A, Section VII, it was shown that organized oscillations cannot exist when ω/k is below mean thermal velocities.

and to second order, noting that $\langle\delta V_1\rangle_{\text{Av}}$ vanishes, we get

$$\langle dW_1/dt\rangle_{\text{Av}}=-(mf/\tau)[n_0V_0\langle\delta V_2\rangle_{\text{Av}}+V_0\langle\delta N\delta V_1\rangle_{\text{Av}}+1/2n_0\langle(\delta V_1)^2\rangle_{\text{Av}}]. \quad (22)$$

After averaging over x_0, $\langle dW_1/dt\rangle_{\text{Av}}$ must be multiplied by the probability that the particles go for a time, without collision, which is

$$(t-t_0)\exp-(t-t_0)/\tau,$$

and integrating over t_0. These calculations are carried out in the appendix. The result is, for sufficiently long collision times,

$$\langle dW_1/dt\rangle_{\text{Av}}=\frac{f(V_0)}{\tau}\frac{\epsilon^2\varphi_0^2k^4}{m}\frac{n_0V_0(V_0-\omega/k)}{[(\omega-kV_0)^2+1/\tau^2]^2}. \quad (23)$$

For large τ and V_0 close to ω/k, the energy transfer becomes large, and it is clear that for $V_0>\omega/k$, it is positive, while for $V_0<\omega/k$ it is negative.[i]

For large φ_0 or large τ, the energy transfer implied by the linear approximation may be much larger than the energy available. In this case, one must go to a non-linear treatment. In A (Section III), we already saw that particles near the wave velocity are trapped and oscillate in the trough of the potential, so that after a long time, their gains and losses of energy tend to balance, thus preventing the indefinitely large transfer predicted for large τ, by the linear theory. Yet, in the non-linear limit, the same qualitative result holds. To see this, we note that particles, emerging from their last collision faster than the wave, will, when the potential is large or the mean free path long, end up by oscillating about a mean velocity equal to that of the wave, and hence, lose energy on the average, while particles starting slower than the wave will do the opposite. The non-linear effects merely set a limit to the amount of energy that a given particle can deliver.

In paper A, where it was assumed that no collisions took place, no excitation or damping by particles near the wave velocity was obtained in the exact non-linear treatment. The reason is that in A, the assumption was made that each particle had come to equilibrium with the wave, so that over a long period of time, transfer of energy between

[i] In studying the excitation of plasma waves, we have adopted the procedure of regarding the particles near the wave velocity, as a separate system, interacting with the rest of the wave. To justify this, we use the results of A, (Section VI), where it was shown that for organized plasma oscillations, the contribution to the charge density of particles near the wave velocity was negligible. Hence, it is permissible as a first approximation to discuss the wave motion, independently of these special particles, and then to bring in the interaction between these special particles and the rest of the wave. For the non-plasmatic oscillations, also treated in A, (Section VI), this procedure would be impossible, because the oscillation is supported mainly by particles near the wave velocity.

particle and wave averaged out to zero. But in the collision treatment, particles are assumed to emerge with no particular phase relation to the wave; in obtaining such a phase relation as a result of the forces, the particles either lose or gain energy, depending on whether they are faster than the wave or slower, and thus can excite or damp the wave. Landau[2] has given a treatment neglecting collisions in which the particles are assumed to start out at $t=0$ with no particular phase relation to the wave; he obtains the same type of excitation and damping due to particles close to the wave velocity, that we have obtained here. As one approaches the Debye length, where there are many particles near the wave velocity, it becomes necessary to bring so many particles into phase with the wave that heavy damping results.

To estimate the amplitude to expect when the linear approximation breaks down, we see that if $f'(\omega_0/k)$ is positive, and large enough to overcome collision damping, the wave amplitude will grow until the energy dissipated by collisions is equal to that made available by the fast particles as they are slowed down to the wave velocity. Only particles which can be trapped will exchange appreciable amounts of energy with the wave over a long period of time. According to A (Section III), the range of trapped velocities is $(V_0-\omega/k)^2 \leqq 2\epsilon\varphi_0/m$. Since the particle ends up with an average velocity equal to that of the wave, $V_W=\omega/k$, the average energy gained by the particle over a long time will be

$$\Delta E=\frac{m}{2}\left(V_0^2-\frac{\omega^2}{k^2}\right)=\frac{m}{2}\left(V_0-\frac{\omega}{k}\right)\left(V_0+\frac{\omega}{k}\right).$$

If φ is not too large, V_0 will be close to ω/k, so that to a first approximation, the above becomes

$$\Delta E=m(\omega/k)(V_0-\omega/k). \quad (24)$$

If $R(V_0)dV_0$ is the mean number of particles entering the range dV_0 per cm^3 per second, then the mean rate at which the particles gain energy is

$$\frac{dW_1}{dt}=\int_{-(2\epsilon\varphi_0/m)^{\frac{1}{2}}}^{+(2\epsilon\varphi_0/m)^{\frac{1}{2}}} \times m\frac{\omega}{k}\left(V_0-\frac{\omega}{k}\right)R(V_0)d\left(V_0-\frac{\omega}{k}\right), \quad (25)$$

where the integration is carried out over the trapped range of velocities. If the gas is in a steady state, the number of particles entering a given velocity range is also equal to the number leaving as a result of collision. Thus, we obtain

$$R(V_0)dV_0=\frac{n_0 f(V_0)}{\tau_f}dV_0, \quad (26)$$

[2] L. Landau, J. Phys., U.S.S.R., 10, 25 (1946).

where τ_f is the mean time between collisions for fast particles. We then get

$$\frac{dW_1}{dt}\cong\int_{-(2\epsilon\varphi_0/m)^{\frac{1}{2}}}^{+(2\epsilon\varphi_0/m)^{\frac{1}{2}}} \times m\frac{\omega}{k}\left(V_0-\frac{\omega}{k}\right)\frac{n_0 f(V_0)}{\tau_f}d(V_0-\omega/k).$$

If the range of velocities is not too broad, $f(V_0)$ can be approximated by expansion as a series of $V_0-\omega/k$. The result is

$$\frac{dW_1}{dt}\cong m\frac{\omega}{k}\frac{n_0}{\tau_f}\cdot\frac{2}{3}\left(\frac{2\epsilon\varphi_0}{m}\right)^{\frac{3}{2}} f'\left(\frac{\omega}{k}\right). \quad (27)$$

To obtain the rate at which the wave dissipates energy by collision, we use the result (Eq. (12)) that the wave dies out to $1/e$ of its value in a time $1/\tau_s$. From this, one concludes that $dW_2/dt=-W/\tau_s$ is the rate of loss of energy, where W is the energy density in the wave, and τ_s is the mean time between collisions for slow particles. Since the total energy of any harmonic oscillation is always twice the mean potential energy, we get $W=\langle E^2\rangle_{\text{Av}}/4\pi=k^2\varphi_0^2/8\pi$ where E is the electric field.

Setting $(dW/dt)+(dW_2/dt)=0$ for a steady state, we obtain

$$\epsilon\varphi_0=\frac{m}{2}\left(\frac{\omega}{k}\right)^2\left(\frac{16}{3}\frac{\omega_P^2}{k^2}\frac{\tau_s}{\tau_f}f'\left(\frac{\omega}{k}\right)\right)^2. \quad (28)$$

In order to get appreciable amplitudes, one needs a region in which f is increasing rapidly with increasing velocity; this is possible only if there is a group of high energy particles. In a typical case, one may take $\omega_P/k\approx 10V_{th}$, and consider a beam of high energy particles containing 0.01 of the total number, and with a velocity spread of the order of mean thermal velocities. Take $\tau_f/\tau_s\approx 100$. Then $(\omega_P^2/k^2)f'(\omega/k)\sim 1$. With thermal energies of the order of 1 ev, one obtains $\epsilon\varphi_0\sim 0.3$ ev.

It is of interest to estimate how much energy a wave of a given potential can exchange with a particle. The maximum value of this quantity can be obtained from Eq. (24) by setting $|V_0-(\omega/k)|=(2\epsilon\varphi_0/m)^{\frac{1}{2}}$. One gets

$$\Delta W_{\text{max}}\approx m\frac{\omega}{k}\left(\frac{2\epsilon\varphi_0}{m}\right)^{\frac{1}{2}}=2\epsilon\varphi_0\left(\frac{m}{2}\frac{(\omega/k)^2}{\epsilon\varphi_0}\right)^{\frac{1}{2}}. \quad (29)$$

One sees that for small $\epsilon\varphi$, the possible energy transfer is much larger than $\epsilon\varphi$ itself. This is because a particle moving at the speed of the wave feels a force of the same sign for a very long time. For example, consider a wave with $\epsilon\varphi_0=0.1$ ev, and wave-length 0.1 cm, so that $k=2\pi/\lambda=20\pi\ \text{cm}^{-1}$. The maximum electric field is $E=|k\varphi_0|=2\pi$ v per cm, so that if the particle moves with the wave for 1 cm, it can deliver 2π v, although no potentials larger than 0.1 v exist; hence, relatively small wave potentials may provide a very effective means of exchanging energy with particles.

VI. PLASMAS OF VARYING DENSITY

If the plasma density varies slowly with position, as, for example, it might do in the neighborhood of a wall, where ions and electrons are being lost by recombination, then there are much more powerful mechanisms for excitation of plasma oscillations. Let us refer to Eq. (12) for the plasma frequency, but express k as a function of ω. We get

$$(3\kappa T/m)k^2=\omega^2-\omega_P{}^2=\omega^2-(4\pi\epsilon^2/m)n_0(x). \quad (30)$$

Suppose that we have a plasma oscillation of a definite frequency, which is moving in a direction in which $n_0(x)$ is decreasing. The wave-length, $\lambda=2\pi/k$, then decreases in the direction of motion of the wave; and the phase velocity, $V_{Ph}=\omega/k$, also decreases.

Consider now a particle which enters the velocity range in which it can be trapped so that it oscillates back and forth about the potential trough. If the wave velocity decreases slowly enough, the particle will respond adiabatically as it moves with the wave, and it will continue to be trapped, oscillating about slower and slower speeds, and giving up more and more energy. When the wave velocity decreases to mean thermal speeds, however, wave motion becomes impossible (see A (Section VII)) so that this mechanism can operate only on particles considerably faster than the mean thermal speed, but where it does operate, practically the entire energy of the particle can be given up.

If the wave moves in the direction of increasing density, its phase velocity increases, and trapped particles can be accelerated. We have suggested this as a possible cause of acceleration of cosmic-ray particles.[3]

To estimate the wave amplitudes that might be produced in this way, we set the rate of loss of energy by particles to the wave equal to

$$\frac{dW_1}{dt}\cong\int_{-(2\epsilon\varphi_0/m)^{\frac{1}{2}}}^{+(2\epsilon\varphi_0/m)^{\frac{1}{2}}}\frac{m}{2}\left(\frac{\omega}{k}\right)^2R(V_0)d\left(V_0-\frac{\omega}{k}\right). \quad (31)$$

This expression implies that every trapped particle liberates all of its kinetic energy. $R(V_0)$ is evaluated in Eq. (26). Here, for small φ_0 we need retain only the first term in the power series. The result is

$$\frac{dW_1}{dt}=\frac{n_0}{\tau_f}m\left(\frac{\omega}{k}\right)^2f\left(\frac{\omega}{k}\right)\left(\frac{2\epsilon\varphi_0}{m}\right)^{\frac{1}{2}}. \quad (32)$$

[3] D. Bohm and E. P. Gross, Phys. Rev. **74**, 624 (1948).

The rate of loss of energy by collisions is still given by $dW_2/dt=-W/\tau_s$. Setting $dW_1/dt+dW_2/dt=0$, we obtain

$$\frac{\epsilon\varphi_0}{m}=2\left[\frac{\tau_s}{\tau_f}\frac{\omega_P{}^2}{k^2}\left(\frac{\omega}{k}\right)^2f\left(\frac{\omega}{k}\right)\right]^{\frac{2}{3}}. \quad (33)$$

The ratio, σ, of this potential to that obtained in Eq. (28), is

$$\sigma=\frac{9}{2}\left[\frac{\tau_s}{\tau_f}\frac{\omega_P{}^2}{k^2}\frac{\omega}{k}\right]^{-4/3}\frac{\left[\frac{\omega}{k}f\left(\frac{\omega}{k}\right)\right]^{\frac{2}{3}}}{[f'(\omega/k)]^2}. \quad (34)$$

One readily sees that for small f and f', σ can become very large. In this mechanism, the energy for excitation of the wave comes from a non-equilibrium property of a plasma of non-uniform density. There is a continual diffusion of particles from regions of high to low density, and this dissipates free energy. The wave can intercept this stream of energy, and transform part of it into energy of ordered motion.

VII. INSTABILITY OF BEAMS OF WELL-DEFINED VELOCITY

The treatment of instability given thus far applies only to a continuous and differentiable velocity distribution. Let us now consider a plasma in which there may be groups of particles within a narrow range of velocities, so that the expansion of Eq. (18) is no longer permissible.

The first problem that we shall consider here is that of a one-dimensional plasma consisting of two beams of the same speed, but moving in opposite directions through a plasma positive ion background with charge density compensating that of the beams in the absence of oscillations. Although this problem is somewhat abstract, it demonstrates in a simple way the most important effects of a sharply defined distribution function. Let $f(V_0)$ have the following values: $f(V_0)=1/2\delta$ when $|V_0|$ lies between $|V_0|=a$, and $|V_0|=a-\delta$; $f(V_0)=0$ for all other values of V_0. This means that each beam has a velocity spread of width, δ. This system has a zero mean velocity.

We assume a very long free path, so that collisions can be neglected, and the treatment given in paper A can be used. According to A (Eq. (9)), the dispersion relation becomes

$$1=\frac{\omega_P{}^2}{2\delta}\left[\left(\int_{-a}^{-a+\delta}+\int_{a-\delta}^{a}\right)\frac{dV_0}{(\omega-kV_0)^2}\right]$$

$$=\frac{\omega_P{}^2}{2k\delta}\left[\frac{1}{\omega-ka}-\frac{1}{\omega-k(a-\delta)}-\frac{1}{\omega+ka}+\frac{1}{\omega+k(a-\delta)}\right] \quad (35)$$

or

$$\frac{\omega_P^2(\omega^2+k^2a(a-\delta))}{(\omega^2-k^2a^2)(\omega^2-k^2(a-\delta)^2)}=1.$$

This is a second-order equation for ω^2, the roots are

$$\omega^2=\frac{\omega_P^2+k^2(a^2+(a-\delta)^2)}{2}\pm\tfrac{1}{2}[(\omega_P^2+k^2(a^2+(a-\delta)^2))^2-4k^2a(a-\delta)(k^2a(a-\delta)-\omega_P^2)]^{\frac{1}{2}}. \quad (36)$$

We shall restrict ourselves here to the case of small k. One root is (for small k):

$$\omega^2\cong\omega_P^2+k^2[a^2+(a-\delta)^2+a(a-\delta)]. \quad (37)$$

This root corresponds to the usual plasma frequency, obtained in A (Eq. (11)). The other root is

$$\omega^2\cong-k^2a(a-\delta). \quad (38)$$

Unless $\delta=a$, this root corresponds to an imaginary frequency; hence, it represents an instability of the system. (When $\delta=a$, the two beams fuse, and this root approaches $\omega=0$. It is readily verified that the root corresponding to the negative sign of the square root is always imaginary, as long as $k^2<\omega_P^2/a(a-\delta)$.

The physical reason for the instability of this system is as follows: A very small perturbation away from zero field at a given point causes a velocity modulation of each beam. In time, this produces a bunching of space charge, in the direction of motion of each beam, which creates a much larger potential than that due to the original perturbation. The fields due to any one beam modulate the other beam, which then feeds the disturbance back to the source in a highly amplified form. Thus, the perturbation builds up cumulatively, and instability results. If the beams have a spread of velocity, however, particles of different velocity tend to bunch in different places, so that the amplification of the original perturbation is less, and the instability is reduced. When the two beams fuse, the instability is removed altogether. We shall see that, in general, beams of well-defined velocity lead to instability which can be reduced or eliminated by a velocity spread.

Another illustrative problem arises if one sends a beam of fairly well-defined velocity into a main plasma having a continuous and more or less Maxwellian velocity distribution. In paper A, (Section VI), it was shown that a beam of sharply defined velocity adds another degree of freedom to the organized motion of a plasma; in fact, such a system is best described as being made up of two interpenetrating and interacting plasmas. One may therefore expect a corresponding increase in the number of possible frequencies of oscillation, in a manner which is analogous to the behavior of two coupled harmonic oscillators.

Let n_1 denote the density of the main plasma, n_2 that of the special beam. We take $\mathbf{V}_2$ as the mean velocity of the beam, and assume a uniform symmetric spread in velocity of $\boldsymbol{\delta}$ about $\mathbf{V}_2$. We seek solutions where all quantities vary as $\exp(\mathbf{k}\cdot\mathbf{x}-\omega t)$, but restrict outselves to waves long enough so that expansions in kV_0/ω are possible for the main plasma. We also assume infinite free path. We take the x axis of coordinates to be in the $\mathbf{V}_2$ direction. $\boldsymbol{\delta}$ is taken parallel to $\mathbf{V}_2$.

From Eq. (10) the change in density for the main plasma, resulting from the potential $\varphi=\varphi_0\exp(\mathbf{k}\cdot\mathbf{x}-\omega t)$ is

$$\delta N_1=-\frac{\epsilon n_1k^2}{m\omega^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{V}_0)^2}{\omega^2}\right)\varphi. \quad (39)$$

For the beam particles

$$\delta N_2=-\frac{\epsilon n_2}{m}\frac{k^2}{2\delta}\varphi\int_{V_2-\delta}^{V_2+\delta}\frac{dV_x}{(\omega-k_xV_2)^2}. \quad (40)$$

From Poisson's equation we obtain the dispersion relation

$$1=\frac{\omega_1^2}{\omega^2}\left(1+\left\langle\left(\frac{\mathbf{k}\cdot\mathbf{V}_0}{\omega}\right)^2\right\rangle_{\text{Av}}\right)+\frac{\omega_2^2}{(\omega-k_xV_2)^2-\delta^2k_x^2}, \quad (41)$$

where $\omega_1^2=4\pi n_1\epsilon^2/m$; $\omega_2^2=4\pi n_2\epsilon^2/m$. We shall consider here only the case of a weak beam where $n_2\ll n_1$ so that $\omega_2\ll\omega_1$.

The above equation has six roots, when $\langle(\mathbf{k}\cdot\mathbf{V}_0/\omega)^2\rangle_{\text{Av}}$ is not neglected. Even in the absence of a special beam, this term would introduce additional roots, which are brought in by the application of the expansion in powers of k^2 beyond the region of its validity. The only roots to be retained are, therefore, those which go continuously into the four obtained by neglecting this term.

Let us divide the discussion of the roots into three cases.

Case (a) $(k_xV_2)^2\gg\omega_1^2$.

One can obtain two of the roots by first noting that when $\omega_2=0$, the roots are

$$\omega\cong\pm\omega_1\left(1+\left\langle\left(\frac{\mathbf{k}\cdot\mathbf{V}_0}{\omega}\right)^2\right\rangle_{\text{Av}}\right)^{\frac{1}{2}}=\pm\omega_a.$$

For small ω_2 the solutions will be changed only slightly. The result is

$$\omega \cong \pm \left[\omega_a^2 + \frac{\omega_2^2}{\left(1 - \frac{k_x V_2}{\pm \omega_1}\right)^2 - \frac{\delta^2 k_x^2}{\omega_1^2}} \right]^{\frac{1}{2}}. \tag{42}$$

We have assumed that both $\omega_1/k_x V_2$ and $\langle(\mathbf{k}\cdot\mathbf{V}_0/\omega_1)^2\rangle_{\text{Av}}$ are small. These roots correspond to the normal modes in which the main plasma carries most of the energy, and the special group undergoes a small forced oscillation of velocity and density in the field of the main plasma.

To find the second type of root, we notice that if $\omega_1 = 0$, the solution would be

$$\omega = k_x V_2 \pm (\omega_2^2 + \delta^2 k_x^2)^{\frac{1}{2}} = \omega_{b\pm}. \tag{43}$$

This represents an oscillation involving the special group of particles. The frequency is just the plasma frequency for the special group. For large $k_x V_2$ it is readily verified that the main plasma is only slightly perturbed by the fields due to the special group. We obtain

$$(\omega - k_x V_2)^2 = \delta^2 k^2 + \frac{\omega_2^2}{1 - \frac{\omega_1^2}{\omega_{b\pm}^2}\left(1 + \left\langle\left(\frac{\mathbf{k}\cdot\mathbf{V}_0}{\omega_{b\pm}}\right)^2\right\rangle_{\text{Av}}\right)}. \tag{44}$$

Since $\omega_{b\pm}^2 \gg \omega_1^2$, the above implies only a small frequency shift away from $\omega_{b\pm}$. This is, therefore, the normal mode in which the beam oscillates almost independently of the rest of the plasma. The frequency of oscillation may be, for this case, far above the natural frequency, ω_1, of the main plasma. Physically, this happens because the special group of particles has an oscillation with a very short wave-length, which together with the Doppler shift results in a frequency so high that the main plasma cannot respond appreciably.

The behavior of this system is very analogous to that of two weakly coupled harmonic oscillators. The case considered above corresponds to a situation in which the natural frequencies of each oscillator are very different. The normal modes then involve the excitation mainly of one oscillator to a time, while the other oscillator responds weakly.

Case (b) $k_x V_2 \ll \omega_1$.

We can obtain the approximate solutions in a way similar to Case (a). One finds for the solution oscillating near the main plasma frequency

$$\omega^2 \cong \omega_1^2 + \langle(\mathbf{k}\cdot\mathbf{V}_0)^2\rangle_{\text{Av}} + \frac{\omega_2^2\omega_1^2}{(\omega_1 - k_x V_2)^2 - \delta^2 k_x^2}. \tag{45}$$

To obtain the oscillation near the beam frequency, we write $\alpha = \omega - k_x V_2$. From Eq. (41) we then obtain

$$\alpha^2 = k_x^2\delta^2 + \frac{\omega_2^2}{1 - \frac{\omega_1^2}{\omega^2}\left(1 + \left\langle\left(\frac{\mathbf{k}\cdot\mathbf{V}_0}{\omega}\right)^2\right\rangle_{\text{Av}}\right)}. \tag{46}$$

We now assume $\alpha/k_x V_2 \ll 1$. The above equation yields, with the neglect of α on the right-hand side,

$$\alpha^2 \cong k_x^2\delta^2 - \left(\frac{k_x V_2}{\omega_1}\right)^2 \frac{\omega_2^2}{\left(1 + \left\langle\left(\frac{\mathbf{k}\cdot\mathbf{V}_\theta}{k_x V_2}\right)^2\right\rangle_{\text{Av}}\right)}. \tag{47}$$

Since $\omega_2/\omega_1 \gg 1$, and $k_x\delta \ll k_x V_2$, we see that α is indeed much less than $k_x V_2$.

The oscillation frequency[j] has not been changed much from $k_x V_2$, but there is a qualitatively new effect; namely, α can become imaginary, so that the system becomes unstable. We note that the spread of velocities, δ, opposes this instability; in fact, if $\delta^2 > (\omega_2^2/\omega_1^2) V_2^2 \cos^2\theta$, where θ is the angle between $\mathbf{k}$ and $\mathbf{V}_2$, the distribution is stable. This result is very analogous to that obtained with the plasma consisting of interpenetrating beams moving in opposite directions. In this case, the instability is due to amplification of a perturbation by bunching of the beam, with subsequent feedback through the main plasma.

Case (c) $k_x V_2 \cong \omega_1$.

This corresponds to the resonant case for harmonic oscillators, because ω is close both to $k_x V_2$ and to ω_1. The interactions of the two systems will be greatest in this region. It is also in this region that the change from real to complex roots occurs, since, when $k_x V_2 \gg \omega_1$, all the roots are real, while two of them are complex when $k_x V_2 \ll \omega_1$. Furthermore $|Im\,\alpha|$ must have a maximum for some value of $k_x V_2$, because it rises steadily as $k_x V_2$ is increased, but must fall to zero again at the point of transition between real and complex roots. The maximum can be reached only in the region where $k_x V_2/\omega_1$ is neither large nor small, hence, it must occur where $k_x V_2 \cong \omega_1$. An order of magnitude estimate of this maximum can be obtained by setting $k_x V_2 = \omega_1/2$ in Eq. (47). The result is

$$\alpha^2_{\max} \cong k_x^2\delta^2 - (\omega_2^2/4).$$

To prevent instability, the spread of velocities must be at least $\delta = \omega_2/2k_x$. This means that sharply defined beams cannot exist in equilibrium for any great length of time; they create the oscillations which scatter them.

[j] Note $k V_2 \ll \omega_1$, so that we have oscillations far below the main plasma frequency. These can exist only by virtue of the beam charges.

VIII. CONCLUSION

Whenever a plasma contains beams of particles of well-defined velocity, or groups of particles for above mean thermal speeds, the system can become unstable, and small oscillations grow until they are limited by the appearance of non-linear effects. Collisions tend to damp the wave, and thus provide a stabilizing effect. In a sense, special groups of particles may be said to create the oscillations which scatter them since plasma electrons can scatter much more effectively when they move coherently than when they set at random. This effect must be taken into account in estimating the rate at which a Maxwellian distribution among the electrons tends to be established. With these results, one can explain the experiments of Langmuir[4] which showed that groups of fast electrons are scattered much more rapidly than can be accounted for by random collisions alone.

Thus far, we have studied only a few of the possible causes of instability. In Paper C, we shall treat the effects of boundary conditions, which are responsible for many important instabilities. A complete investigation, extending the treatment to include effects of variations of density and temperature, non-linear phenomena, positive ions, and magnetic fields remains to be carried out.

Plasmas occur not only in discharge tubes, but also in the ionosphere of the earth, the atmosphere of the sun, and interstellar gas and dust clouds. Since streams of fast particles are continually being emitted from below into the atmosphere of the sun, one may expect a great deal of instability, particularly because the plasma density decreases as one leaves the surface of the sun. Vlasov,[5] Martyn,[6] and Haeff[7] have suggested also that the motion of plasma particles in the magnetic fields of sunspots can cause oscillations, which may be able to account for the high intensity of radio waves that the sun is known to emit. When there is a magnetic field, transverse and longitudinal waves are coupled, so that all plasma oscillations can produce radio waves, which are certain to escape, because the plasma density decreases as one leaves the surface, so that the waves cannot be reflected back in.

It is not unlikely that the plasmas of interstellar space, are also highly unstable. For example, photo-ionization should be able to supply high energy electrons which can excite plasma waves by the mechanism described in this paper. There may also be a galactic magnetic field[8] which produces further instabilities. Any two ion gases with different average velocities will also be unstable if they collide and interpenetrate. Some of these mechanisms may perhaps explain the observed radio waves coming from interstellar space.

IX. APPENDIX

The expression for the average transfer of kinetic energy from particle to wave is given in Eq. (23). The first step is to evaluate $\langle\delta V_2\rangle_{\text{Av}}$, the average value of the second order correction to the velocity. To do this, we begin with the equation of motion

$$m\frac{dV}{dt}=Re\ i\epsilon k\varphi_0 e^{i(kx-\omega t)}. \tag{48}$$

Writing $x\cong x_0+V_0(t-t_0)+\delta x_1$, where δx_1 is the first-order correction to x, we get the following expression for dV/dt, accurate to the second order

$$\frac{dV}{dt}\cong Re\frac{i\epsilon k\varphi_0}{m}e^{i(kx_0-\omega t_0)}e^{i(kV_0-\omega)(t-t_0)}e^{ik\ Re\ \delta x_1}. \tag{49}$$

Now, over a finite time, it is always possible to choose φ_0 so small that $k\delta x_1$ is also small; hence $\exp ik\ Re\ \delta x_1$ can be expanded, and we get

$$\frac{dV}{dt}=\frac{d}{dt}(\delta V_1)-Re\frac{\epsilon k^2\varphi_0}{m}e^{i(kx_0-\omega t_0)}e^{i(kV_0-\omega)(t-t_0)}Re\ \delta x_1,$$

where δV_1 is the first-order correction to V, given in Eq. (3). To obtain δx_1, one integrates Eq. (3) for δV_1, obtaining

$$\delta x_1=\int_{t_0}^{t}\delta V_1 dt=\frac{\epsilon k\varphi_0}{m}\int_{t_0}^{t}e^{i(kx_0-\omega t_0)}\times\left(\frac{e^{i(kV_0-\omega)(t-t_0)}-1}{kV_0-\omega}\right)dt. \tag{50}$$

We wish to average δV_2 over x_0, and then over $(t-t_0)$ with the weighting factor $\exp-(t-t_0)/\tau$. This can be accomplished by first averaging $d\delta V_2/dt$ over x_0 then integrating the latter to obtain δV_2, and then averaging the latter over t_0. To average over x_0, we use the theorem that if f and g are complex numbers proportional to $\exp ikx_0$, then $\langle Re\ f\ Re\ g\rangle_{\text{Av}}=\langle Re\ (f^*g/2)\rangle_{\text{Av}}$. We then get

$$\left\langle\frac{d}{dt}\delta V_2\right\rangle_{\text{Av}}=-\frac{\epsilon^2\varphi_0^2k^3}{2m^2(kV_0-\omega)}\times Re\left[\frac{1-e^{-i(kV_0-\omega)(t-t_0)}}{i(kV_0-\omega)}-(t-t_0)e^{-i(kV_0-\omega)(t-t_0)}\right]. \tag{51}$$

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[6] D. F. Martyn, Nature **159**, 27 (1947).
[7] A. V. Haeff, Phys. Rev. **74**, 1532 (1948).
[8] L. Spitzer, Phys. Rev. **70**, 777 (1947).

The next step is to average over $(t-t_0)$. To do this, we use the result that

$$\int_{t-t_0=0}^{\infty} \frac{e^{-(t-t_0)/\tau}}{\tau} \delta V_2(t-t_0) d(t-t_0) = \int_{t-t_0=0}^{\infty} e^{-(t-t_0)/\tau} \frac{d\delta V_2}{dt} d(t-t_0), \quad (52)$$

which can be obtained by integration by parts, and noting that $\delta V(t=t_0)$ vanishes by definition. Thus, we obtain for average over both x_0 and $t-t_0$,

$$\langle V_2 \rangle_{\mathrm{Av}} = -\frac{\epsilon^2 \varphi_0^2 k^3}{2m^2(kV_0-\omega)} \times Re \int_0^{\infty} d\eta e^{-\eta/\tau} \left[\frac{1-e^{-i(kV_0-\omega)\eta}}{i(kV_0-\omega)} - \eta e^{-i(kV_0-\omega)\eta} \right], \quad (53)$$

where we have replaced $(t-t_0)$ by η. This can be integrated to yield

$$\langle \delta V_2 \rangle_{\mathrm{Av}} = -\frac{\epsilon^2 \varphi_0^2 k^3}{m^2} \frac{(kV_0-\omega)}{[(\omega-kV_0)^2+1/\tau^2]^2}. \quad (54)$$

Let us now average the term δV_1^2 over x_0 (see Eq. (20)). From (3), we obtain

$$\langle \delta V_1^2 \rangle_{\mathrm{Av}} = \langle \tfrac{1}{2} |\delta V_1|^2 \rangle_{\mathrm{Av}} = \frac{\epsilon^2 \varphi_0^2 k^2}{m^2(\omega-kV_0)^2} \times (1-\cos(kV_0-\omega)(t-t_0)).$$

When this is multiplied by $\exp-(t-t_0)/\tau$, and integrated over $(t-t_0)$, we get

$$\langle \delta V_1^2 \rangle_{\mathrm{Av}} = \frac{\epsilon^2 \varphi_0^2 k^2}{m^2(\omega-kV_0)^2} \left[1 - Re \frac{1}{1+i(\omega-kV_0)\tau} \right] = \frac{\epsilon^2 \varphi_0^2 k^2}{m^2} \frac{1}{(\omega-kV_0)^2+1/\tau^2}. \quad (55)$$

We now evaluate the term $\delta N(x_0, t_0)\delta V_1$. From Poisson's equation,

$$\nabla^2 \varphi = +4\pi\epsilon\delta N$$

or

$$\delta N(x_0, t_0) = -\frac{k^2\varphi_0}{4\pi\epsilon} e^{i(kx_0-\omega t_0)}.$$

Averaging over x_0, we get

$$\langle \delta N \delta V_1 \rangle_{\mathrm{Av}} = -\frac{1}{2} \frac{k^3 \varphi_0^2}{4\pi m(kV_0-\omega)} Re(e^{i(kV_0-\omega)(t-t_0)}-1).$$

Averaging over $(t-t_0)$ yields

$$\langle \delta N \delta V_1 \rangle_{\mathrm{Av}} = \frac{1}{2} \frac{k^3 \varphi_0^2}{4\pi m} \frac{(kV_0-\omega)}{(kV_0-\omega)^2+1/\tau^2}. \quad (56)$$

The complete expression for ΔW is then, according to Eq. (22),

$$\langle dW_1/dt \rangle_{\mathrm{Av}} = -\frac{mf(V_0)}{\tau} \times [n_0 V_0 \delta V_2 + V_0 \delta N \delta V_1 + \tfrac{1}{2} n_0 \delta V_1^2] = \frac{n_0 f(V_0)}{\tau} \frac{\epsilon^2 \varphi_0^2 k^2}{m} \left[\frac{kV_0(kV_0-\omega)}{[(kV_0-\omega)^2+1/\tau^2]^2} - \frac{kV_0}{2\omega_P^2} \frac{(kV_0-\omega)}{(kV_0-\omega)^2+1/\tau^2} - \frac{1}{2} \frac{1}{(kV_0-\omega)^2+1/\tau^2} \right]. \quad (57)$$

For large values of τ, the first term is clearly the main one, and one obtains

$$\langle dW_1/dt \rangle_{\mathrm{Av}} \cong \frac{n_0 f(V_0)}{\tau} \frac{\epsilon^2 \varphi_0^2 k^4}{m} \frac{V_0(V_0-\omega/k)}{[(kV_0-\omega)^2+1/\tau^2]^2}. \quad (58)$$