The Meson Fields and the Equation of Motion of a Spinning Particle

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Following Riesz, the meson fields generated by a point dipole at a point outside the world line have been defined in terms of the derivatives of an integral which is an analytical function of a parameter, α, and converges for α lying between $2<\alpha<3$. It is then shown that by the process of analytical continuation of the parameter to the value $\alpha=2$ for which the Riesz fields satisfy the fundamental meson field equations, we not only obtain the classical Liénard-Wiechert-Bhabha fields at a point outside the world line but also a field at a point on the world line itself which is finite and represents the reaction of the field correctly. The equations for rotational and translational motions follow immediately from the usual electrodynamical equations of a dipole given by Frenkel. The equations thus obtained are free from singularities and do not contain any arbitrary constant except the mass and spin angular momentum of the dipole.

INTRODUCTION

It is now well known that the meson field differs fundamentally from the electromagnetic field in its interaction with the nuclear particles, neutron and proton. Whereas the interaction of the electron of charge e with the electromagnetic field is completely described in classical electrodynamics by the four potentials of the field, the interaction of the nucleon with the meson field requires for its description not only a charge g_1 but also a dipole moment g_2 which has an explicit spin interaction with the meson field. It is this dipole type interaction which is absent in the electron case although it could always be introduced mathematically[1] and it leads to a scattering of mesons which increases as the square of their energy. Following Lorentz, Heisenberg[2] first derived an equation of motion for an extended dipole in a meson field and showed that the effect of the reaction of the proper field of the nucleon gives rise to an inertia of the motion of its spin which considerably reduces the scattering at high energies. Heisenberg's theory was, however, not relativistically invariant in that he assumed a dipole having finite extension. A relativistically invariant theory of the motion of a point dipole in a meson-field was first developed by Bhabha and Corben[3] and by Bhabha[4] following a method first initiated by Dirac.[5] The method of Lorentz had to be abandoned as the equations of motion of a particle involve the field at its world line which is infinite for a point particle in the usual classical theory. It is therefore necessary to calculate first the flow of energy, momentum and the angular momentum out of a portion of a thin tube surrounding the world line of the point particle, the radius of which is ultimately made to tend to zero. The equations of motion are then obtained from the principle of conservation of energy and momentum according to which this inflow must depend only on the conditions at the two ends of the tube. The infinities which appear in the expressions for the flow of energy, momentum and angular momentum when the tube shrinks to the world line are perfect differentials and are therefore subtracted away. The field which finally contributes fundamentally to the equations of motion is found to be the *so-called* radiation field which is defined as half the retarded minus the advanced field, and is finite on the world line. It has already been shown by Wentzel[6] that by introducing a time like four vector λ it is possible to form the limiting values of the field at the world point of the particle by finally putting $\lambda=0$ (λ-limiting process), this limiting value representing the radiation field. Harish-Chandra[7]

[1] W. Pauli, *Handbuch der Physik* (Springer, Verlag-Berlin, 1933), Quantentheorie Band 24/1.
[2] W. Heisenberg, Zeits. f. physik. **113**, 61 (1939).
[3] H. J. Bhabha and H. C. Corben, Proc. Roy. Soc. **A178**, 273 (1941).
[4] H. J. Bhabha, Proc. Roy. Soc. **A178**, 314 (1941).
[5] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).
[6] G. Wentzel, Zeits. f. physik **86**, 479 (1933).
[7] Harish-Chandra, Proc. Nat. Acad. of Sci. (India) **14**, 195 (1945). (Just published.)

has also been able to construct an effective field expressed as an integral of the actual field at a world point, the integral being taken over a small closed three dimensional surface. The effective field reduces to the radiation field on the world line of the particle. The final equations obtained by Bhabha and Corben and by Bhabha can, therefore, be taken to represent the fundamental equations of motion of a point particle. However, there is an element of arbitrariness in the subtraction of the singular terms and the introduction of radiation field. Further, the equations are extremely complicated and involve a number of arbitrary constants which are not determined uniquely.

Recently Riesz[8] has developed an elegant method for solving the differential equations of the hyperbolic type, in which the well known divergent difficulties, first studied by Hadamard, are eliminated by a process of analytical continuation of an integral which is an analytical function of a parameter α. The method admits of a calculation of the meson fields generated by the point nucleon not only at a point outside the world line but also at a point on the world line occupied by the nucleon itself. The field quantities are first defined for values of α large enough for the integrals to be convergent. Then by analytical continuation with regard to the parameter α the potentials and the fields are determined. Riesz's method has been recently applied by Gustafson[9] to the problem of the self-energy of a point particle and by Fremberg[10] in obtaining the classical equation of a charged particle in a meson field. In the present paper we shall show that by defining a Riesz potential for the dipole which is convergent for a parameter α lying between $2<\alpha<5$ and by analytical continuation to $\alpha=2$ for which the field quantities are found to satisfy the fundamental meson equations, we can obtain not only the field quantities generated by a point dipole at a point outside the world line, but also finite values on the world line itself. The latter fields describe the reaction of the field correctly. The equations of motion of a point dipole then follow immediately from the usual equation for the dipole given by Frenkel.[11] The equations are free from singularities and do not contain any arbitrary constant except those which appear as mass and spin angular momentum of the dipole.

[8] M. Riesz, Conférence de la Réunion internat. des. math., tenue à Paris en Juillet 1937 (Paris) 1939.

[9] T. Gustafson, Nature **157**, 734 (1946); Nature, **158**, 273 (1946).

[10] N. E. Fremberg, Medd. Fran. Lunds. Uni. Mat. Sem. Band 7 (1946); Proc. Roy. Soc. **A188**, 18 (1946).

[11] J. Frenkel Zeits. f. physik. **37**, 243 (1926).

1. THE MESON FIELD OF A DIPOLE

We shall use the ordinary tensor notation throughout and take the fundamental metric tensor $g_{\mu\nu}$ in the form $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$ with all other components vanishing. We shall put the velocity of light c equal to unity for convenience. The particle shall be treated as a point, its co-ordinates being z_μ which are functions of the proper time, τ, measured from an arbitrary point on the world line of the particle. The spin of the particle will be described by an antisymmetrical tensor, $S_{\mu\nu}$, which will also be considered as a function of τ and which together with its derivatives up to a certain order is continuous and bounded for $\tau\rightarrow-\infty$. The meson field generated by the spinning particle is described by potentials, φ_μ, and by field strengths, $G_{\mu\nu}$, which satisfy the following fundamental equations

$$\frac{\partial G_{\mu\nu}}{\partial x_\mu}+\chi^2\varphi_\nu=4\pi g_2\frac{\partial S_{\mu\nu}}{\partial x_\mu} \tag{1}$$

where

$$G_{\mu\nu}=\frac{\partial\varphi_\nu}{\partial x^\mu}-\frac{\partial\varphi_\mu}{\partial x^\nu}. \tag{2}$$

Now assuming

$$\partial\varphi_\nu/\partial x_\nu=0, \tag{3}$$

we obtain

$$\frac{\partial^2\varphi_\nu}{\partial x_\rho\partial x^\rho}+\chi^2\varphi_\nu=4\pi g_2\frac{\partial S_{\mu\nu}}{\partial x_\mu} \tag{4}$$

and

$$\frac{\partial G_{\mu\nu}}{\partial x^\lambda}+\frac{\partial G_{\nu\lambda}}{\partial x^\mu}+\frac{\partial G_{\lambda\mu}}{\partial x^\nu}=0. \tag{5}$$

We define the Riesz potential for the dipole at a point $P(x^\mu)$ outside the world line in the form

$$\varphi_\nu{}^\alpha(P)=\frac{g_2\chi^{4-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\frac{\partial}{\partial x_\mu}\int_{-\infty}^{\tau_0}S_{\mu\nu}(\tau)\times(\chi s)^{(\alpha-4)/2}J_{(\alpha-4)/2}(\chi s)d\tau \tag{6}$$

where $J_n(x)$ is the Bessel function of order n and s the four dimensional distance given by

$$s^2=s_\mu s^\mu,\quad s_\mu=x_\mu-z_\mu. \tag{7}$$

τ_0 is the retarded proper time determined by $s^2=0$. The integral in (6) is convergent for $2<\alpha<3$.

We assume that the world line has a time-like asymptote for $\tau\rightarrow-\infty$ and that for a large value of τ,

$$z^\mu(\tau)=v_{-\infty}{}^\mu\tau+\sum_{\kappa=0}^{\infty}C_\kappa{}^\mu\tau^{-\kappa}\quad(v_{-\infty}{}^\mu=\lim_{\tau\rightarrow-\infty}v^\mu).\quad(8)$$

As the integrand in (6) is singular at $\tau=\tau_0$ care should be taken to carry out differentiation under the integration sign. We introduce s as an integration variable and consider s and x_μ as independent variables. z_μ and consequently τ then become functions of s and x_μ. We thus obtain on differentiation and changing the variable again to τ

$$\varphi_\nu{}^\alpha(P)=\frac{g_2\chi^{4-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\frac{\partial}{\partial\tau}\left(\frac{s^\mu S_{\mu\nu}}{\kappa}\right)\times(\chi s)^{(\alpha-4)/2}J_{(\alpha-4)/2}(\chi s)d\tau\quad(9)$$

where we have used

$$\frac{ds}{d\tau}=-\frac{\kappa}{s},\quad\frac{\partial\tau}{\partial x^\mu}=\frac{s_\mu}{\kappa};\quad\kappa=s_\mu v^\mu.\quad(10)$$

The integral (9) converges for $2<\alpha<5$.

Correspondingly, we define the Riesz field of the dipole by the equation

$$G_{\mu\nu}{}^\alpha=(\partial\varphi_\nu{}^\alpha/\partial x^\mu)-(\partial\varphi_\mu{}^\alpha/\partial x^\nu).\quad(11)$$

Differentiation of the potentials (9) in the manner described above, taking s and x_μ as independent variables, gives

$$G_{\mu\nu}{}^\alpha=\frac{g_2\chi^{4-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\left[2\frac{\partial}{\partial\tau}\left(\frac{S_{\mu\nu}}{\kappa}\right)+\frac{\partial}{\partial\tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s_\mu s^\rho S_{\rho\nu}-s_\nu s^\rho S_{\rho\mu}}{\kappa}\right)\right\}\right]\times(\chi s)^{(\alpha-4)/2}J_{(\alpha-4)/2}(\chi s)d\tau\quad(12)$$

which converges for $2<\alpha<7$. It is to be observed that formulae (9) and (12) are exactly what we would have obtained by carrying out directly formal differentiation in (6) and (9) under the sign of integration.

Now it follows from (9) that outside the world line

$$\operatorname{div}\varphi^\alpha=g^{\mu\nu}(\partial\varphi_\nu{}^\alpha/\partial x^\mu)=0,\quad(13)$$

and

$$\Box\varphi_\mu{}^\alpha+\chi^2\varphi_\mu{}^\alpha=\varphi_\mu{}^{\alpha-2}.\quad(14)$$

Further, defining Riesz current density by

$$(\partial S_{\mu\nu}{}^\alpha/\partial x_\mu)=(1/4\pi)\varphi_\nu{}^{\alpha-2}(P),\quad(15)$$

it can be proved that the introduced field quantities satisfy the fundamental meson equations (3) to (5). In particular, by analytical continuation to $\alpha=2$, we have $\varphi_\mu{}^{(0)}=0$ outside the world line and consequently

$$\Box\varphi_\mu{}^{(2)}+\chi^2\varphi_\mu{}^{(2)}=0.\quad(16)$$

We shall frequently use the following formulae

$$J_{n-1}(x)=(2n/x)J_n(x)-J_{n+1}(x),\quad(17)$$

$$(d/dx)(x^{-n}J_n(x))=-x^{-n}J_{n+1}(x),\quad(18)$$

$$(d/dx)(x^nJ_n(x))=x^nJ_{n-1}(x),\quad(19)$$

$$\lim_{x\rightarrow0}(J_n(x)/x^n)=1/(2^n\cdot n!).\quad(20)$$

Using formula (17), the potentials (9) may be expressed as

$$\varphi_\nu{}^\alpha(P)=\frac{g_2\chi^{4-\alpha}(\alpha-2)}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\frac{\partial}{\partial\tau}\left(\frac{s^\mu S_{\mu\nu}}{\kappa}\right)\times(\chi s)^{(\alpha-6)/2}J_{(\alpha-2)/2}(\chi s)d\tau-\frac{g_2\chi^{4-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\frac{\partial}{\partial\tau}\left(\frac{s^\mu S_{\mu\nu}}{\kappa}\right)\times(\chi s)^{(\alpha-4)/2}J_{\alpha/2}(\chi s)d\tau\quad(21)$$

where the first integral converges for $2<\alpha<7$ and the second for $0<\alpha<5$. We introduce s as the integration variable and perform analytical continuation to $\alpha=2$ with the help of the lemma

$$\lim_{\beta\rightarrow+0}\beta\int_0^\infty f(t,\beta)t^{\beta-1}dt=f(0,0),\quad(22)$$

which is valid if the integral converges for $\beta>0$ and $f(t,\beta)$ is continuous in $t=\beta=0$. The potential at a point $P(x^\mu)$ lying outside the world line is thus given by

$$\varphi_\nu{}^{\alpha=2}(P)=\varphi_\nu{}^{\text{ret}}(P)=g_2\left[\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s^\rho S_{\rho\nu}}{\kappa}\right)\right]_0-g_2\chi\int_{-\infty}^{\tau_0}\frac{\partial}{\partial\tau}\left(\frac{s^\rho S_{\rho\nu}}{\kappa}\right)\frac{J_1(\chi s)}{s}d\tau$$
$$=g_2\left[\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s^\rho S_{\rho\nu}}{\kappa}\right)\right]_0-\tfrac{1}{2}g_2\chi^2\left[\frac{s^\rho S_{\rho\nu}}{\kappa}\right]_0+g_2\chi^2\int_{-\infty}^{\tau_0}s^\rho S_{\rho\nu}\frac{J_2(\chi s)}{s^2}d\tau,\quad(23)$$

where the index 0 implies that the value of the function is to be taken at the point retarded with respect to P.

Similarly, expressing the field (12) in the form

$$G_{\mu\nu}{}^{\alpha}=\frac{g_2\chi^{4-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}(\alpha-2)\int_{-\infty}^{\tau_0}\left[2\frac{\partial}{\partial\tau}\left(\frac{S_{\mu\nu}}{\kappa}\right)+\frac{\partial}{\partial\tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s_\mu s^\rho S_{\rho\nu}-s_\nu s^\rho S_{\rho\mu}}{\kappa}\right)\right\}\right]\times(\chi s)^{(\alpha-6)/2}J_{(\alpha-2)/2}(\chi s)d\tau$$
$$-\frac{g_2\chi^{4-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\left[2\frac{\partial}{\partial\tau}\left(\frac{S_{\mu\nu}}{\kappa}\right)+\frac{\partial}{\partial\tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s_\mu s^\rho S_{\rho\nu}-s_\nu s^\rho S_{\rho\mu}}{\kappa}\right)\right\}\right]\times(\chi s)^{(\alpha-4)/2}J_{\alpha/2}(\chi s)d\tau, \quad (24)$$

analytical continuation to $\alpha=2$ and repeated partial integrations give the field at a point $P(x^\mu)$ outside the world line:

$$G_{\mu\nu}{}^{\alpha=2}(P)=G_{\mu\nu}{}^{\mathrm{ret}}(P)=g_2\left\{2\left[\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{S_{\mu\nu}}{\kappa}\right)\right]_0+\left[\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{\mathrm{K}_{\mu\nu}}{\kappa}\right)\right)\right]_0\right\}-g_2\chi^2\left[\frac{S_{\mu\nu}}{\kappa}\right]_0$$
$$-\frac{g_2\chi^2}{2}\left[\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{\mathrm{K}_{\mu\nu}}{\kappa}\right)\right]_0+2g_2\chi^2\int_{-\infty}^{\tau_0}S_{\mu\nu}\frac{J_2(\chi s)}{s^2}d\tau+\frac{g_2\chi^4}{8}\left[\frac{\mathrm{K}_{\mu\nu}}{\kappa}\right]_0-g_2\chi^3\int_{-\infty}^{\tau_0}\mathrm{K}_{\mu\nu}\frac{J_3(\chi s)}{s^3}d\tau, \quad (25)$$

where

$$\mathrm{K}_{\mu\nu}=(s_\mu s^\rho S_{\rho\nu}-s_\nu s^\rho S_{\rho\mu}). \quad (26)$$

Expressions (23) and (25) were first obtained by Bhabha, and by Bhabha and Corben.

2. REACTIONS OF THE FIELD

We now proceed to calculate the fields at a point P on the world line. They will obviously represent the reaction of the field on the motion of the particle itself. We consider $\varphi_\nu{}^\alpha(P)$ as given by the expression (21) of which the first integral converges for $2<\alpha<7$ and the second for $0<\alpha<5$ even when P is on the world line. We divide the first integral into two parts, one in the interval $-\infty$ to $\tau_0-\epsilon$ and the other from $\tau_0-\epsilon$ to τ_0. The first part converges for $\alpha<7$ and therefore vanishes, especially for $\alpha=2$ on account of the factor $(\alpha-2)$. For the second part we expand the integrands in power series of $(\tau-\tau_0)$. We take $\tau_0=0$ for simplicity and have

$$s_\mu=-\tau[v_\mu+(1/2)\dot{v}_\mu\tau+(1/6)v_\mu{}^{\mathrm{ii}}\tau^2+(1/24)v_\mu{}^{\mathrm{iii}}\tau^3+(1/120)v_\mu{}^{\mathrm{iv}}\tau^4+O(\tau^5)],$$
$$s^2=\tau^2[1-(1/12)\dot{v}^2\tau^2-(1/12)(\dot{v}v^{\mathrm{ii}})\tau^3-\{(1/40)(\dot{v}v^{\mathrm{iii}})+(1/45)v^{\mathrm{ii}2}\}\tau^4+O(\tau^5)], \quad (27)$$
$$\kappa=-\tau[1-(1/6)\dot{v}^2\tau^2-(5/24)(\dot{v}v^{\mathrm{ii}})\tau^3-\{(3/40)(\dot{v}v^{\mathrm{iii}})+(1/15)v^{\mathrm{ii}2}\}\tau^4+O(\tau^5)].$$

In order to perform the analytical continuation of the potential $\varphi_\nu{}^\alpha$ to $\alpha=2$ we expand the quantities in the first of the integrals of (21) with the help of the expansions (27) and directly put $\alpha=2$ in the second integral. We thus obtain for the potentials on the world line:

$$\varphi_\nu{}^{\alpha=2}=\varphi_\nu{}^{\mathrm{ret}}=-g_2\{\tfrac{1}{3}\dot{v}^2v^\rho S_{\rho\nu}+\tfrac{1}{3}v^{\mathrm{ii}\rho}S_{\rho\nu}+\dot{v}^\rho\dot{S}_{\rho\nu}+v^\rho S_{\rho\nu}{}^{\mathrm{ii}}\}-(g_2\chi^2/2)v^\rho S_{\rho\nu}+g_2\chi^2\int_{-\infty}^{\tau_0}s^\rho S_{\rho\nu}\frac{J_2(\chi s)}{s^2}d\tau. \quad (28)$$

For the calculation of the field, $F_{\mu\nu}{}^\alpha(P)$, we consider (24) of which the first integral converges for $4<\alpha<9$ and the second for $2<\alpha<7$ when P is on the world line. By partial integration and using (18) we can express the second integral in the form

$$-\frac{g_2\chi^{6-\alpha}(\alpha-2)}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\left\{2S_{\mu\nu}+\frac{\partial}{\partial\tau}\left(\frac{\mathrm{K}_{\mu\nu}}{\kappa}\right)\right\}\times(\chi s)^{(\alpha-8)/2}J_{\alpha/2}(\chi s)d\tau$$
$$+\frac{g_2\chi^{6-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\left\{2S_{\mu\nu}+\frac{\partial}{\partial\tau}\left(\frac{\mathrm{K}_{\mu\nu}}{\kappa}\right)\right\}\times(\chi s)^{(\alpha-6)/2}J_{(\alpha+2)/2}(\chi s)d\tau, \quad (29)$$

where the first integral converges for $2<\alpha<7$ and the second for $0<\alpha<5$. Analytical continuation to $\alpha=2$, carried out as before, gives for the fields on the world line

$$G_{\mu\nu}{}^{\alpha=2}=G_{\mu\nu}{}^{\mathrm{ret}}=g_2\{\tfrac{2}{3}S_{\mu\nu}(\dot{v}v^{\mathrm{ii}})+\tfrac{2}{3}\dot{S}_{\mu\nu}\dot{v}^2+\tfrac{2}{3}S_{\mu\nu}{}^{\mathrm{iii}}+[(4/3)v_\mu S_\nu{}'''+2v_\mu S_\nu{}'\dot{v}^2+\tfrac{1}{3}v_\mu{}^{\mathrm{iii}}\$_\nu-\tfrac{2}{3}v_\mu{}^{\mathrm{ii}}\dot{v}^\rho S_{\rho\nu}+(4/3)v_\mu{}^{\mathrm{ii}}S_\nu{}'-(4/3)v_\mu v^{\mathrm{ii}\rho}\dot{S}_{\rho\nu}-\tfrac{2}{3}\dot{v}_\mu v^{\mathrm{ii}\rho}S_{\rho\nu}-\tfrac{1}{3}v_\mu v^{\mathrm{iii}\rho}S_{\rho\nu}-2v_\mu\dot{v}^\rho S_{\rho\nu}{}^{\mathrm{ii}}+2\dot{v}_\mu S_\nu{}''-2\dot{v}_\mu\dot{v}^\rho\dot{S}_{\rho\nu}+2v_\mu\$_\nu(\dot{v}v^{\mathrm{ii}})-v_\mu\dot{v}^\rho S_{\rho\nu}\dot{v}^2+\dot{v}_\mu\$_\nu\dot{v}^2]_-+\chi^2[\tfrac{1}{2}\dot{S}_{\mu\nu}+v_\mu S_\nu{}'-\tfrac{1}{2}v_\mu\dot{v}^\rho S_{\rho\nu}+\tfrac{1}{2}\dot{v}_\mu\$_\nu]_-\}+2g_2\chi^2\int_{-\infty}^{\tau_0}S_{\mu\nu}\frac{J_2(\chi s)}{s^2}d\tau-g_2\chi^3\int_{-\infty}^{\tau_0}[s_\mu s^\rho S_{\rho\nu}]_-\frac{J_3(\chi s)}{s^3}d\tau,\quad(30)$$

where

$$\$_\nu=S_{\nu\rho}v^\rho,\quad S_\nu{}'=\dot{S}_{\nu\rho}v^\rho,\quad S_\nu{}''=S_{\nu\rho}{}^{\mathrm{ii}}v^\rho,\quad S_\nu{}'''=S_{\nu\rho}{}^{\mathrm{iii}}v^\rho,\quad(31)$$

and the minus sign as subscript denotes that the terms inside the brackets are to be subtracted with interchanges of μ and ν.

It is to be noted that the field given by (30) is the retarded field at the point τ_0 on the world line. This field agrees with the *so-called* radiation field, i.e. $\frac{1}{2}$ (retarded − advanced) field, as calculated by Bhabha and Corben, and by Bhabha provided we take $k_1=\frac{1}{3}$, $k_2=-(7/15)$, $k_3=\frac{1}{3}$; $k_4=(4/3)$, $k_6=0$ in expression (140) of B&C and $k=2$ in expression (10) of Bhabha. There is, however, disagreement in regard to terms with constant k_5 as has also been noticed by Harish Chandra[12] whose calculation of the radiation field by retaining only the finite and unambiguous terms of the field given by Bhabha agrees with our result (30).

3. THE EQUATION OF MOTION

A. The equation of rotational motion is now obtained directly from the equation of a dipole given by Frenkel

$$I\dot{S}_{\lambda\mu}=g_2[S\cdot G^{\mathrm{ext}}]_{\lambda\mu}+g_2[S\cdot G^{\mathrm{react}}]_{\lambda\mu},\quad(32)$$

where I is the spin angular momentum. The field $G_{\mu\nu}{}^{\mathrm{react}}$ is given by (30) and it represents the reaction of the field.

We now derive the equations of motion of the spinning particle in which the dipole always remains a pure magnetic dipole in the system in which the particle is at rest. This is expressed as

$$S_{\mu\nu}v^\nu=0\quad(33)$$

and is one which is of physical interest. The rotational equation (32) is now to be modified to

$$I\{\dot{S}_{\lambda\mu}+v_\lambda S_\mu{}'-v_\mu S_\lambda{}'\}=g_2[S_\lambda{}^\sigma\{G_{\sigma\mu}{}^{\mathrm{ext}}-(G_{\sigma\nu}{}^{\mathrm{ext}}v^\nu v_\mu-G_{\mu\rho}{}^{\mathrm{ext}}v^\rho v_\sigma)\}]_-+g_2[S_\lambda{}^\sigma D_{\sigma\mu}{}^{\mathrm{react}}]_-\quad(34)$$

with

$$D_{\sigma\mu}{}^{\mathrm{react}}=G_{\sigma\mu}{}^{\mathrm{react}}-(G_{\sigma\nu}{}^{\mathrm{react}}v^\nu v_\mu-G_{\mu\nu}{}^{\mathrm{react}}v^\nu v_\sigma)$$

where $G_{\sigma\mu}{}^{\mathrm{react}}$ is given by (30). Thus

$$D_{\sigma\mu}{}^{\mathrm{react}}=g_2\{\tfrac{2}{3}S_{\sigma\mu}(\dot{v}v^{\mathrm{ii}})+\tfrac{2}{3}S_{\sigma\mu}{}^{\mathrm{iii}}+\tfrac{2}{3}\dot{S}_{\sigma\mu}\dot{v}^2-[\tfrac{2}{3}S_\sigma{}'''v_\mu+(4/3)S_\sigma{}'v_\mu\dot{v}^2+\tfrac{2}{3}S_\sigma{}'v_\mu{}^{\mathrm{ii}}+(4/3)S_\sigma{}''\dot{v}_\mu+\tfrac{2}{3}\dot{S}_{\sigma\rho}\dot{v}^\rho\dot{v}_\mu]_-+\chi^2[\tfrac{1}{2}\dot{S}_{\sigma\mu}-S_\sigma{}'v_\mu]_-\}+\tilde{G}_{\sigma\mu}-(\tilde{G}_{\sigma\rho}v^\rho v_\mu-\tilde{G}_{\mu\rho}v^\rho v_\sigma),\quad(35)$$

$$\tilde{G}_{\sigma\mu}=2g_2\chi^2\int_{-\infty}^{\tau_0}\frac{S_{\sigma\mu}J_2(\chi s)}{s^2}d\tau+g_2\chi^3\int_{-\infty}^{\tau_0}[s_\sigma S_{\mu\rho}s^\rho]_-\frac{J_3(\chi s)}{s^3}d\tau.\quad(36)$$

Equation (34) agrees with the corresponding equation given by Bhabha and Corben provided we take $d=-\frac{2}{3}$.

It must be noticed that the equation (34) ensures the condition

$$S_{\mu\nu}S^{\mu\nu}=\mathrm{constant}.\quad(37)$$

B. The equation for the translational motion of the dipole can also be taken over directly from the corresponding equation given by Frenkel

$$m\dot{v}_\mu+\frac{d}{d\tau}\{IS_\mu{}'-\tfrac{1}{2}g_2v_\mu S^{\lambda\sigma}G_{\lambda\sigma}{}^{\mathrm{ext}}-g_2S_\mu{}^\lambda G_{\lambda\rho}{}^{\mathrm{ext}}v^\rho\}=T_\mu{}^{\mathrm{self}}-\tfrac{1}{2}g_2S^{\lambda\sigma}\frac{\partial G_{\lambda\sigma}{}^{\mathrm{ext}}}{\partial x^\mu},\quad(38)$$

[12] Harish-Chandra, Proc. Roy. Soc. **A185**, 269 (1946).

where

$$T_\mu{}^{\text{self}} = -\frac{1}{2}g_2 S^{\lambda\sigma}\frac{\partial G_{\lambda\sigma}{}^{\text{react}}}{\partial x^\mu} + \frac{1}{2}\frac{d}{d\tau}(g_2 v_\mu S^{\lambda\sigma} G_{\lambda\sigma}{}^{\text{react}}) + \frac{d}{d\tau}(g_2 S_\mu{}^\lambda G_{\lambda\rho}{}^{\text{react}} v^\rho). \quad (39)$$

It is to be noted that $\partial G_{\lambda\sigma}{}^{\text{react}}/\partial x^\mu$ cannot be obtained from (30) by direct differentiation. It is to be found out by evaluating the value of $\partial G_{\lambda\sigma}{}^{\alpha}/\partial x^\mu$ for $\alpha=2$ on the world line. For this we first differentiate $G_{\lambda\sigma}{}^\alpha$ as given by (24) and get

$$\frac{\partial G_{\lambda\sigma}{}^\alpha}{\partial x^\mu} = \frac{g_2\chi^{4-\alpha}}{2^{(\alpha-2)/2}\Gamma(\alpha/2)}\int_{-\infty}^{\tau_0}\left[2\frac{\partial}{\partial\tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s_\mu S_{\lambda\sigma}}{\kappa}\right)\right\} + \frac{\partial}{\partial\tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s_\lambda S_{\mu\sigma}-s_\sigma S_{\mu\lambda}}{\kappa}\right)\right\} + \frac{\partial}{\partial\tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial\tau}(g_{\lambda\mu}s^\rho S_{\rho\sigma}-g_{\sigma\mu}s^\rho S_{\rho\lambda})\right\} + \frac{\partial}{\partial\tau}\left\{\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{1}{\kappa}\frac{\partial}{\partial\tau}\left(\frac{s_\lambda s_\mu s^\rho S_{\rho\sigma}-s_\sigma s_\mu s^\rho S_{\rho\lambda}}{\kappa}\right)\right)\right\}\right] \times(\chi s)^{(\alpha-4)/2}J_{(\alpha-4)/2}(\chi s)d\tau. \quad (40)$$

Then by analytical continuation to $\alpha=2$ we evaluate $\partial G_{\lambda\sigma}{}^{\alpha=2}/\partial x^\mu$ on the world line. The calculations are cumbersome and lengthy and we simply give the final result

$$T_\mu{}^{\text{self}} = T_\mu{}^{\text{self},0} + T_\mu{}^{\text{self},\chi}, \quad (41)$$

where

$$\begin{aligned} T_\mu{}^{\text{self},0} = g_2{}^2[&v_\mu\{(1/3)(\dot{S}S^{\text{iii}})] \\ &-(2/3)S''^2-(8/3)(S''\dot{S}\dot{v})-2(S'S''') \\ &-2(S'S^{\text{ii}}\dot{v})+(2/3)(\dot{v}\dot{S}\dot{S}\dot{v}) \\ &-2(S'\dot{S}v^{\text{ii}})-(1/5)S^2v^{\text{ii}2} \\ &-(4/15)S^2(\dot{v}v^{\text{iii}})+\dot{S}^2\dot{v}^2-4S'^2\dot{v}^2 \\ &-(1/3)S^2\dot{v}^4\}+\dot{v}_\mu\{(\dot{S}S^{\text{ii}})-4(S'S'') \\ &-4(S'\dot{S}\dot{v})-(2/3)S^2(\dot{v}v^{\text{ii}})\} \\ &+v_\mu{}^{\text{ii}}\{(2/3)\dot{S}^2-2S'^2-\tfrac{1}{3}S^2\dot{v}^2\} \\ &-(1/15)v_\mu{}^{\text{iv}}S^2-(2/3)S_{\mu\sigma}{}^{\text{iii}}S'^\sigma \\ &-(4/3)S_{\mu\sigma}{}^{\text{ii}}\dot{S}^{\sigma\rho}\dot{v}_\rho-(2/3)S_{\mu\sigma}{}^{\text{ii}}S''^\sigma \\ &-(2/3)\dot{S}_{\mu\sigma}S'''^\sigma-2\dot{S}_{\mu\sigma}S^{\text{ii}\sigma\rho}\dot{v}_\rho \\ &-(8/15)S_{\mu\sigma}\dot{S}^{\sigma\rho}v_\rho{}^{\text{iii}}-(2/3)\dot{S}_{\mu\sigma}S'^\sigma\dot{v}^2 \\ &-(4/3)\dot{S}_{\mu\sigma}\dot{S}^{\sigma\rho}v_\rho{}^{\text{ii}}-(2/15)S_{\mu\sigma}S''''^\sigma \\ &-(6/5)S_{\mu\sigma}S^{\text{iii}\sigma\rho}\dot{v}_\rho-(22/15)S_{\mu\sigma}S^{\text{ii}\sigma\rho}v_\rho{}^{\text{ii}} \\ &+\tfrac{2}{3}S_{\mu\sigma}S''^\sigma\dot{v}^2+(\tfrac{1}{3}-\tfrac{1}{3})S_{\mu\sigma}\dot{S}^{\sigma\rho}\dot{v}_\rho\dot{v}^2 \\ &+\tfrac{2}{3}S_{\mu\sigma}S'^\sigma(\dot{v}v^{\text{ii}})], \end{aligned} \quad (42)$$

the invariants formed by any combination of tensors and vectors having been written in the usual matrix notations; as for example

$$(\dot{v}S^{\text{ii}}\dot{S}v) = \dot{v}^\mu S_{\mu\rho}{}^{\text{ii}}\dot{S}^{\rho\nu}v_\nu,$$

and

$$\begin{aligned} T_\mu{}^{\text{self},\chi} = g_2{}^2\chi^2\{&v_\mu(\tfrac{1}{2}\dot{S}^2-S'^2-\tfrac{1}{6}\dot{v}^2S^2) \\ &-\tfrac{1}{6}v_\mu{}^{\text{ii}}S^2+\tfrac{2}{3}S_{\mu\sigma}S''^\sigma+\tfrac{1}{3}S_{\mu\sigma}\dot{S}^{\sigma\rho}\dot{v}^\rho\} \\ &-\tfrac{1}{8}g_2{}^2\chi^4v_\mu S^2-\tfrac{1}{2}g_2S^{\lambda\sigma}\tilde{G}_{\lambda\sigma,\mu} \\ &+\frac{1}{2}\frac{d}{d\tau}(g_2v_\mu S^{\lambda\sigma}\tilde{G}_{\lambda\sigma})+\frac{d}{d\tau}(g_2S_\mu{}^\lambda\tilde{G}_{\lambda\rho}v^\rho), \end{aligned} \quad (43)$$

where

$$\tilde{G}_{\lambda\sigma,\mu} = -g_2\chi^3\int_{-\infty}^{\tau_0}\{2S_{\lambda\sigma}s_\mu+(g_{\mu\lambda}s^\rho S_{\rho\sigma}-g_{\mu\sigma}s^\rho S_{\rho\lambda}) + (s_\lambda S_{\mu\sigma}-s_\sigma S_{\mu\lambda})\}\frac{J_3(\chi s)}{s^3}d\tau + g_2\chi^4\int_{-\infty}^{\tau_0}s_\mu(s_\lambda s^\rho S_{\rho\sigma}-s_\sigma s^\rho S_{\rho\lambda})\frac{J_4(\chi s)}{s^4}d\tau, \quad (44)$$

and $\tilde{G}_{\lambda\sigma}$ is given by (36). The results agree with those of Bhabha[13] if we put his constant $d=-\frac{2}{3}$. The disagreement in the coefficients of $\dot{v}_\mu(S'\dot{S}\dot{v})$ may, however, be noted.

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[13] H. J. Bhabha, Proc. India. Acad. Sci. **A11**, 247 (1940).