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The Charge Density and Magnetic Moments of the Nucleons*

MURRAY SLOTNICK
Columbia University, New York, New York,

AND

WALTER HEITLER
Dublin Institute for Advanced Studies, Dublin, Eire
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Using pseudoscalar meson theory, the behavior of a single-nucleon system in an electric field is studied. From the electrostatic interaction, the interaction between neutrons and electrons is computed. From the spin-orbit interaction, the magnetic moments of neutron and proton are computed. The calculation is carried out relativistically, with nucleon and pseudoscalar meson fields both subjected to second quantization. The Hamiltonian of the nucleon-meson system is diagonalized to second order in the coupling parameters by two canonical transformations, and transitions induced by the electric field between single-nucleon states are investigated. For pseudoscalar coupling of 282 e.m. mesons, estimating the coupling constants for charge-symmetric theory from the observed singlet neutron-proton scattering length (using static nuclear forces), it is found that the volume integral of the neutron-electron interaction potential is about -14 kev $\times(4\pi/3)(e^2/mc^2)^3$, and that μ_P and μ_N are about 1.7 and -5 nuclear magnetons respectively. Results are also given for pure charged theory. Pseudovector coupling is compared to pseudoscalar coupling, and is found to give the same magnetic moments, but a logarithmically divergent neutron-electron interaction. The influence of the contact interaction term is discussed.

The results are compared with those of "non-relativistic" methods and with experimental values, and the neutron-electron interaction is computed approximately for several types of mesons with an improved non-relativistic method. Charge renormalization and the approximate distribution of the charge cloud around a neutron are discussed. The pseudoscalar meson contribution to the Lamb Shift of the $2S$ hydrogen level is estimated as $+0.08$ Mc.

I. INTRODUCTION

A MESON field coupled to nucleons modifies the behavior of the nucleons in electromagnetic fields. It had thus been hoped that meson theory would provide an explanation of the anomalous magnetic moments and other properties of neutrons and protons, as it does indeed qualitatively.[1] Unfortunately, quantitative investigation of the magnetic moments[2] and charge clouds[3] led to divergent results with the ordinary weak-coupling theory that necessitated the introduction of ad hoc cut-offs, while other theories[4] led to incorrect results. This, together with the failure of meson theories to yield nuclear forces free from objectionable singularities was considered to limit severely the value of the theory.

However, in the calculations with weak-coupling theory cited above,[2,3] the divergences may have been introduced by the following assumptions made there:

(1) The nucleon was taken to be infinitely heavy. If nucleon recoil is taken into account, certain energy denominators occurring in the calculations will be large for high momentum virtual states and may decrease the degree of divergence.

(2) Hole theory was not used, and it is conceivable that negative-energy processes might cancel part of the divergences of positive-energy processes, as is the case for electron self-energy, etc.

Moreover, recent advances in quantum electrodynamics suggest how divergences due to mass effects can be separated from true divergences in relativistic calculations. It was thus thought worth while to reinvestigate the divergent results relativistically.

On the other hand, experimental evidence has very recently been obtained by Havens, Rabi, and Rainwater,[5] and Fermi and Marshall[6] that indicates a weak attraction between neutrons and electrons that is not a spin effect. Improved experiments by Rainwater, Rabi, and Havens are in progress;[7] the earlier experiment[5] suggesting a value of minus several kev for the quantity $U_0\equiv$Volume integral of interaction $\div(4\pi/3)(e^2/mc^2)^3$, which is called the "neutron-electron interaction." This is of the sign and order of magnitude expected qualitatively from meson theory,[6] so that a detailed investigation is of considerable interest.

The two forms of weakly coupled meson theory that have proved most useful are the pseudoscalar and vector theories, and we will restrict our attention principally to the former. We will investigate the behavior of a single nucleon in an electric field. The interaction between the neutron charge cloud and the electric field of an electron will be used to

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[1] W. Heitler, Nature **148**, 680 (1941); H. Fröhlich and W. Heitler, Nature **141**, 37 (1938).

[2] Fröhlich, Heitler, and Kemmer, Proc. Roy. Soc. **A166**, 127 (1938).

[3] Fröhlich, Heitler, and Kahn, Proc. Roy. Soc. **A177**, 269 (1939); W. E. Lamb, Jr. and L. I. Schiff, Phys. Rev. **53**, 651 (1938).

[4] See G. Wentzel, Rev. Mod. Phys. **19**, 1 (1947).

[5] Havens, Rabi, and Rainwater, Phys. Rev. **72**, 634 (1947).

[6] Fermi and Marshall, Phys. Rev. **72**, 1139 (1947).

[7] See Rainwater, Rabi, and Havens, Phys. Rev. **75**, 1295 (1949).

compute the neutron-electron interaction, while the magnetic moments can be computed from the spin-orbit interaction of the nucleon in an electric field.[8] By dealing only with electric fields, we avoid use of the more complicated vector potentials and current density operators.

It should also be stressed that exact quantitative agreement with experiment can hardly be expected from pseudoscalar meson theory alone, since it fails in other respects[4] (high singularity of tensor force), and because the solution will be carried only to second order in the coupling parameter. It is also likely that mesons of mass 700–1000 e.m. will add contributions.

II. METHOD OF COMPUTATION

Lagrangian and Hamiltonian

The nucleons will be described by a doubly quantized Fermi-Dirac field, with 8-component field operators $\psi=\begin{pmatrix}\psi_P\\ \psi_N\end{pmatrix}$, where ψ_P and ψ_N are 4-component Dirac operators for proton and neutron respectively. We define the charge operators

$$\tau=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \tau^\dagger=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \tau^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$
$$\tau_P=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \tau_N=\begin{pmatrix}0&0\\0&1\end{pmatrix},\qquad (1)$$

where τ changes neutrons to protons, $\tau^\dagger$ protons to neutrons, and τ_P and τ_N respectively single out the proton and neutron parts of ψ. ψ^* is the Hermitian conjugate of ψ.

The mesons are described by a doubly quantized Einstein-Bose field with pseudoscalar field operators Ψ and its Hermitian conjugate $\Psi^\dagger$ for charged mesons and $\Psi^0=\Psi^{0\dagger}$ for neutral mesons. $\hbar$ and c will be taken as unity throughout and the summation convention will be used. We use the Schrödinger representation with time-independent operators.

The Lorentz- and gauge-invariant Lagrangian density of the system of mesons and nucleons in the external field is:

$$\mathfrak{L}=-\{d_\nu{}^\dagger\Psi^\dagger d_\nu\Psi+\mu^2\Psi^\dagger\Psi\}-\frac{1}{2}\left\{\frac{\partial\Psi^0}{\partial x_\nu}\frac{\partial\Psi^0}{\partial x_\nu}+\mu^2\Psi^{02}\right\}+i\psi^\dagger\left\{\gamma_\nu\left(\frac{\partial}{\partial x_\nu}-ie\tau_PA_\nu\right)+M\right\}\psi+(4\pi)^{\frac{1}{2}}g(B\Psi+B^\dagger\Psi^\dagger)$$
$$+(4\pi)^{\frac{1}{2}}g_0B^0\Psi^0-\frac{(4\pi)^{\frac{1}{2}}}{\mu}f(B_\nu d_\nu\Psi+B_\nu{}^\dagger d_\nu{}^\dagger\Psi^\dagger)-\frac{(4\pi)^{\frac{1}{2}}}{\mu}f_0B_\nu{}^0\frac{\partial\Psi^0}{\partial x_\nu}+s\frac{4\pi}{\mu^2}f^2B_\nu{}^\dagger B_\nu+\frac{t}{2}\frac{4\pi}{\mu^2}f_0{}^2B_\nu{}^0B_\nu{}^0\qquad (2)$$

where the following notation is employed:

$\psi^\dagger=i\psi^*\beta$ is the "adjoint" operator of ψ;

$$d_\nu=\frac{\partial}{\partial x_\nu}-ieA_\nu;\quad d_\nu{}^\dagger=\frac{\partial}{\partial x_\nu}+ieA_\nu;$$

$A_\nu=(\mathbf{A}(\mathbf{r}),\ iV(\mathbf{r}))=$4-potential of external electromagnetic field;

$\mu=$meson mass (assumed equal for charged and neutral mesons);

$M=$neutron mass$=$proton mass (the mass difference is neglected);

$$B=(\psi^\dagger\gamma_5\tau\psi);\ B^\dagger=(\psi^\dagger\gamma_5\tau^\dagger\psi);\ B^0=(\psi^\dagger\gamma_5\tau^0\psi);$$
$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4;$$
$$B_\nu=(\psi^\dagger\gamma_5\gamma_\nu\tau\psi);\ B_\nu{}^\dagger=(\psi^\dagger\gamma_5\gamma_\nu\tau^\dagger\psi);$$
$$B_\nu{}^0=(\psi^\dagger\gamma_5\gamma_\nu\tau^0\psi);$$

s and t are arbitrary constants to be discussed later;

g, g_0 are "pseudoscalar coupling" constants for charged and neutral mesons, respectively;

f, f_0 are "pseudovector coupling" constants for charged, neutral mesons.

The Hamiltonian density is given by

$$\mathfrak{H}(\mathbf{r})=\Pi^\dagger\Pi+\nabla\Psi^\dagger\cdot\nabla\Psi+\mu^2\Psi^\dagger\Psi+\tfrac{1}{2}\{\Pi^{02}+(\nabla\Psi^0)^2+\mu^2\Psi^{02}\}+\psi^*\left[\boldsymbol{\alpha}\cdot\frac{\nabla}{i}+\beta M\right]\psi-(4\pi)^{\frac{1}{2}}g(B\Psi+B^\dagger\Psi^\dagger)$$
$$-(4\pi)^{\frac{1}{2}}g_0B^0\Psi^0+\frac{(4\pi)^{\frac{1}{2}}}{\mu}f(\mathbf{B}\cdot\nabla\Psi+\mathbf{B}^\dagger\cdot\nabla\Psi^\dagger)+\frac{(4\pi)^{\frac{1}{2}}}{\mu}f(\Pi^\dagger B_0+\Pi B_0{}^\dagger)+\frac{(4\pi)^{\frac{1}{2}}}{\mu}f_0(\mathbf{B}^0\cdot\nabla\Psi^0)$$
$$+\frac{(4\pi)^{\frac{1}{2}}}{\mu}f_0\Pi^0B_0{}^0-s\frac{4\pi f^2}{\mu^2}\mathbf{B}^\dagger\cdot\mathbf{B}+(1+s)\frac{4\pi f^2}{\mu^2}B_0{}^\dagger B_0-\frac{t}{2}\frac{4\pi f_0{}^2}{\mu^2}\mathbf{B}^{02}+\frac{(1+t)}{2}\frac{4\pi f_0{}^2}{\mu^2}B_0{}^{02}$$
$$+ie\{V(\Pi^\dagger\Psi^\dagger-\Pi\Psi)+\Psi^\dagger\mathbf{A}\cdot\nabla\Psi-\Psi\mathbf{A}\cdot\nabla\Psi^\dagger\}+e^2\mathbf{A}^2\Psi^\dagger\Psi+eV\psi^*\tau_P\psi-e\mathbf{A}\cdot(\psi^*\boldsymbol{\alpha}\tau_P\psi)$$
$$+ie\frac{(4\pi)^{\frac{1}{2}}}{\mu}f\{\Psi^\dagger\mathbf{A}\cdot\mathbf{B}^\dagger-\Psi\mathbf{A}\cdot\mathbf{B}\}\qquad (3)$$

[8] We are indebted to Professor H. A. Bethe for pointing out this fact.

where Π, $\Pi^\dagger$, and Π^0 are the momenta conjugate to Ψ, $\Psi^\dagger$, and Ψ^0, and

$$B=(\psi^*\alpha_1\alpha_2\alpha_3\beta\tau\psi),\qquad B^\dagger=(\psi^*\alpha_1\alpha_2\alpha_3\beta\tau^\dagger\psi),\qquad B^0=(\psi^*\alpha_1\alpha_2\alpha_3\beta\tau^0\psi),$$

$$\mathbf{B}=(\psi^*\boldsymbol{\sigma}\tau\psi),\qquad \mathbf{B}^\dagger=(\psi^*\boldsymbol{\sigma}\tau^\dagger\psi),\qquad \mathbf{B}^0=(\psi^*\boldsymbol{\sigma}\tau^0\psi),$$

$$B_0=-i(\psi^*\alpha_1\alpha_2\alpha_3\tau\psi),\quad B_0{}^\dagger=-i(\psi^*\alpha_1\alpha_2\alpha_3\tau^\dagger\psi),\quad B_0{}^0=-i(\psi^*\alpha_1\alpha_2\alpha_3\tau^0\psi).$$

In (3), the first line refers to the pure meson and nucleon fields, the second to the meson-nucleon interaction, the third to a self-energy and a contact or "δ-function" interaction between two nucleons that will be discussed later, and the fourth line to the interaction between mesons and nucleons and the electromagnetic field. Putting $f_0=g_0=0$ gives pure charged theory, while $g_0=g/\sqrt{2}$, $f_0=f/\sqrt{2}$ gives symmetrical meson theory. More generally, the neutral meson need not have the mass μ and τ^0 can be any diagonal charge-operator, but we will deal only with pure charged and symmetrical theory.

We transform to momentum representation by

$$\Psi(\mathbf{r})=\textstyle\sum_{\mathbf{k}}(2\epsilon_kL^3)^{-\frac{1}{2}}\{q_{+\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})+q_{-\mathbf{k}}{}^\dagger\exp(-i\mathbf{k}\cdot\mathbf{r})\}$$

$$\Psi^0(\mathbf{r})=\textstyle\sum_{\mathbf{k}}(2\epsilon_kL^3)^{-\frac{1}{2}}\{q_{0\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})+q_{0\mathbf{k}}{}^\dagger\exp(-i\mathbf{k}\cdot\mathbf{r})\}$$

$$\Pi(\mathbf{r})=\textstyle\sum_{\mathbf{k}}i(\epsilon_k/2L^3)^{\frac{1}{2}}\{q_{+\mathbf{k}}{}^\dagger\exp(-i\mathbf{k}\cdot\mathbf{r})-q_{-\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})\}$$

$$\Pi^0(\mathbf{r})=\textstyle\sum_{\mathbf{k}}i(\epsilon_k/2L^3)^{\frac{1}{2}}\{q_{0\mathbf{k}}{}^\dagger\exp(-i\mathbf{k}\cdot\mathbf{r})-q_{0\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r})\}$$

$$\psi(\mathbf{r})=\textstyle\sum_{\mathbf{p}}L^{-\frac{3}{2}}a_{\mathbf{p}}u_{\mathbf{p}}\exp(i\mathbf{p}\cdot\mathbf{r}),\tag{4}$$

and the conjugate equations. Here, L is the length of the cubic enclosure in which we imagine our system, $\epsilon_k=(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}$, and $u_{\mathbf{p}}$ is a normalized 8-component Dirac spinor for which we have omitted subscripts for spin direction, sign of energy, and charge. The commutation relations are in this representation:

$$[q_{+\mathbf{k}}{}^\dagger,q_{+\mathbf{k}'}]=[q_{-\mathbf{k}}{}^\dagger,q_{-\mathbf{k}'}]=[q_{0\mathbf{k}}{}^\dagger,q_{0\mathbf{k}'}]=\delta_{\mathbf{k}\mathbf{k}'};\quad [a_{\mathbf{p}}{}^\dagger,a_{\mathbf{p}'}]=\delta_{\mathbf{p}\mathbf{p}'}.\tag{5}$$

Other commutators involving q's and anticommutators involving a's are zero.

The Hamiltonian $H=\int\mathcal{H}d\tau$ then becomes (we restrict ourselves to the case $\mathbf{A}(\mathbf{r})=0$):

$$H=H^m+H^n+H^i+H^e+H^\delta,\tag{6}$$

where

$$H^m=\textstyle\sum_{\mathbf{k}}\epsilon_k(N_{+\mathbf{k}}+N_{-\mathbf{k}}+N_{0\mathbf{k}}),\quad H^n=\sum_{\mathbf{p}}E_{\mathbf{p}}N_{\mathbf{p}},$$

$$\begin{aligned}H^i=\sum_{\mathbf{p}\mathbf{p}'\mathbf{k}}a_{\mathbf{p}'}{}^\dagger a_{\mathbf{p}}(2\pi/\epsilon_kL^3)^{\frac{1}{2}}\Bigg(u_{\mathbf{p}'}{}^*\Bigg\{\Bigg[&(q_{+\mathbf{k}}\tau+q_{-\mathbf{k}}\tau^\dagger)\left(-g\alpha_1\alpha_2\alpha_3\beta-\frac{f}{\mu}\epsilon_k\alpha_1\alpha_2\alpha_3+\frac{if}{\mu}\boldsymbol{\sigma}\cdot\mathbf{k}\right)\\&+q_{0\mathbf{k}}\tau^0\left(-g_0\alpha_1\alpha_2\alpha_3\beta-\frac{f_0}{\mu}\epsilon_k\alpha_1\alpha_2\alpha_3+\frac{if_0}{\mu}\boldsymbol{\sigma}\cdot\mathbf{k}\right)\Bigg]\delta_{\mathbf{p}',\mathbf{p}+\mathbf{k}}\\&+\Bigg[(q_{+\mathbf{k}}{}^\dagger\tau^\dagger+q_{-\mathbf{k}}{}^\dagger\tau)\left(-g\alpha_1\alpha_2\alpha_3\beta+\frac{f}{\mu}\epsilon_k\alpha_1\alpha_2\alpha_3-\frac{if}{\mu}\boldsymbol{\sigma}\cdot\mathbf{k}\right)\\&+q_{0\mathbf{k}}{}^\dagger\tau^0\left(-g_0\alpha_1\alpha_2\alpha_3\beta+\frac{f_0}{\mu}\epsilon_k\alpha_1\alpha_2\alpha_3-\frac{if_0}{\mu}\boldsymbol{\sigma}\cdot\mathbf{k}\right)\Bigg]\delta_{\mathbf{p}'+\mathbf{k},\mathbf{p}}\Bigg\}u_{\mathbf{p}}\Bigg)\end{aligned}$$

$$\begin{aligned}H^e&=H^{em}+H^{en}\\&=\frac{e}{2L^3}\int d\tau V(\mathbf{r})\sum_{\mathbf{k}\mathbf{k}'}(\epsilon_k\epsilon_{k'})^{-\frac{1}{2}}\{(q_{+\mathbf{k}'}{}^\dagger q_{+\mathbf{k}}-q_{-\mathbf{k}'}{}^\dagger q_{-\mathbf{k}})(\epsilon_k+\epsilon_{k'})\exp(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r})\\&\qquad+q_{-\mathbf{k}'}q_{+\mathbf{k}}(\epsilon_k-\epsilon_{k'})\exp(i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r})-q_{+\mathbf{k}'}{}^\dagger q_{-\mathbf{k}}{}^\dagger(\epsilon_k-\epsilon_{k'})\exp(-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r})\}\\&\qquad+\frac{e}{L^3}\int d\tau V(\mathbf{r})\sum_{\mathbf{p}\mathbf{p}'}a_{\mathbf{p}'}{}^\dagger a_{\mathbf{p}}(u_{\mathbf{p}'}{}^*\tau_Pu_{\mathbf{p}})\exp(i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})\end{aligned}$$

$$\begin{aligned}H^\delta=&-\left(\frac{4\pi f^2}{\mu^2L^3}\right)\sum_{\mathbf{p}\mathbf{p}'\mathbf{p}_0\mathbf{p}_0'}\delta_{\mathbf{p}+\mathbf{p}',\mathbf{p}_0+\mathbf{p}_0'}a_{\mathbf{p}}{}^\dagger a_{\mathbf{p}_0'}a_{\mathbf{p}'}{}^\dagger a_{\mathbf{p}_0}\{(1+s)(u_{\mathbf{p}}{}^*\alpha_1\alpha_2\alpha_3\tau^\dagger u_{\mathbf{p}_0})(u_{\mathbf{p}'}{}^*\alpha_1\alpha_2\alpha_3\tau u_{\mathbf{p}_0})\\&\qquad+s(u_{\mathbf{p}}{}^*\boldsymbol{\sigma}\tau^\dagger u_{\mathbf{p}_0'})\cdot(u_{\mathbf{p}'}{}^*\boldsymbol{\sigma}\tau u_{\mathbf{p}_0})\}\\&-\left(\frac{4\pi f_0{}^2}{2\mu^2L^3}\right)\sum_{\mathbf{p}\mathbf{p}'\mathbf{p}_0\mathbf{p}_0'}\delta_{\mathbf{p}+\mathbf{p}',\mathbf{p}_0+\mathbf{p}_0'}a_{\mathbf{p}}{}^\dagger a_{\mathbf{p}_0'}a_{\mathbf{p}'}{}^\dagger a_{\mathbf{p}_0}\{(1+t)(u_{\mathbf{p}}{}^*\alpha_1\alpha_2\alpha_3\tau^0u_{\mathbf{p}_0'})(u_{\mathbf{p}'}{}^*\alpha_1\alpha_2\alpha_3\tau^0u_{\mathbf{p}_0})\\&\qquad+t(u_{\mathbf{p}}{}^*\boldsymbol{\sigma}\tau^0u_{\mathbf{p}_0'})\cdot(u_{\mathbf{p}'}{}^*\boldsymbol{\sigma}\tau u_{\mathbf{p}_0})\}.\end{aligned}\tag{7}$$

Here, $E_\mathbf{p}=(u_\mathbf{p}^*[\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M]u_\mathbf{p})$, and is $\pm(\mathbf{p}^2+M^2)^{\frac{1}{2}}$ while $N_\mathbf{p}=a_\mathbf{p}^\dagger a_{\mathbf{p}0}$, $N_{+\mathbf{k}}=q_{+\mathbf{k}}^\dagger q_{+\mathbf{k}}$, $N_{-\mathbf{k}}=q_{-\mathbf{k}}^\dagger q_{-\mathbf{k}}$, $N_{0\mathbf{k}}=q_{0\mathbf{k}}^\dagger q_{0\mathbf{k}}$.

It should be mentioned that the nucleon charge density operator used above, namely, $e\psi^*\tau_P\psi$ (see (3)), gives a negative infinite expectation value for the total charge in the vacuum state. This can be avoided by using Heisenberg's symmetrized charge density $e/2(\psi^*\tau_P\psi-\psi\tau_P\psi^*)$. However, only off-diagonal elements of the charge density will be encountered, and by virtue of the commutation relations, the two definitions give identical results for such elements.

The Canonical Transformations

The representation in which $N_\mathbf{p}$, $N_{+\mathbf{k}}$, $N_{-\mathbf{k}}$, $N_{0\mathbf{k}}$, and hence $H^0=H^m+H^n$ is diagonal will be used as the unperturbed representation. The state in which all N's are zero, except that the $N_\mathbf{p}$'s for negative-energy states are unity, is the vacuum state. A single-nucleon state of momentum $\mathbf{p}$ is one that differs from the vacuum state only in that for this one positive-energy state, $N_\mathbf{p}=1$.

If a canonical transformation $\exp(-iS)$ is applied to H to diagonalize the part $H^0+H^i+H^\delta$ not containing the electric field, the unperturbed single-nucleon states are transformed into the true single-nucleon states in zero field. The transformed electrical part $\exp(-iS)H^e\exp(iS)$ has matrix elements between the zero-field single-nucleon states that determine the properties of single nucleons in external fields. Actually, however, the diagonalization will only be carried to second order in the coupling parameters, and an unknown error will thus be introduced into the final results.

Before proceeding, account must be taken of the self-energies, which have the effect of causing a mass change δM for the nucleons and $\delta\mu$ for the mesons. (δM diverges logarithmically for pseudoscalar and quadratically for pseudovector coupling.[9]) We write the Hamiltonian as follows:

$$H=(H^0+\delta H)+(H^i+H^\delta-\delta H)+H^e$$

where

$$\delta H=\int d\tau\{\delta M\psi^*\beta\psi+\delta\mu^2\Psi^\dagger\Psi+\tfrac{1}{2}\delta\mu_0{}^2\Psi^{02}\}.$$

The term in the first parenthesis in H is simply H^0 for the nucleon mass $M+\delta M$, charged meson mass $\mu+\delta\mu$, and neutral meson mass $\mu+\delta\mu_0$. These, however, are the experimentally measured masses, and in a correct theory would presumably differ only by a second-order quantity from the inertial masses. The Hamiltonian could then be written, at least to second order, in the form

$$H=H^0+(H^i+H^\delta-\delta H)+H^e, \quad (8)$$

where H^0, etc., are all given by (7), but with M and μ standing for the experimental masses, and δH chosen so that it just cancels the self-energy parts of H^i+H^δ. The form (8) will be used hereafter despite the fact that it is not strictly applicable. It is very reasonable to suppose that any convergent results so obtained will agree to second order with those of the "correct" theory. Divergences, however, should really be treated with refined methods, such as Feynman's relativistic cut-off[10] before it can decisively be stated that they are not really "mass effects."

We note the formula:

$$\exp(-iS)H\exp(iS)=H+(-i)[S,H]+\frac{(-i)^2}{2!}[S,[S,H]]+\cdots. \quad (9)$$

Then, to second order, we have from (8):

$$\begin{aligned}H_I\equiv&\exp(-iS)H\exp(iS)\\=&H^0-i[S,H^0]-\tfrac{1}{2}[S,[S,H^0]]\\&+H^i-i[S,H^i]+H^\delta-\delta H+H^e\\&-i[S,H^e]-\tfrac{1}{2}[S,[S,H^e]]. \quad (10)\end{aligned}$$

To eliminate first-order terms, put $-i[S,H^0]+H^i=0$, so that

$$\begin{aligned}H_I=&H^0-(i/2)[S,H^i]+H^\delta-\delta H+H^e\\&-i[S,H^e]-\tfrac{1}{2}[S,[S,H^e]] \quad (11)\end{aligned}$$

with the elements of S between states e and a of energies $E(e)$ and $E(a)$, given by

$$S_{ea}=iH_{ea}{}^i/(E(e)-E(a)). \quad (12)$$

We next apply a transformation $\exp(-iT)$ to eliminate second-order terms from H_I. Then

$$\begin{aligned}H_{II}\equiv&\exp(-iT)H_I\exp(iT)\\=&H^0-i[T,H^0]-(i/2)[S,H^i]\\&+H^\delta-\delta H+H^e-i[T,H^e]\\&-i[S,H^e]-\tfrac{1}{2}[S,[S,H^e]]. \quad (13)\end{aligned}$$

"Self-energy transitions" are those for which either initial and final states are identical or (because of the β in δH) those where a nucleon has merely changed the sign of its energy. For such transitions, the matrix elements of $(-(i/2)[S,H^i]+H^\delta-\delta H)$ are zero by our choice of δH. Then

$$T_{ea}=0 \quad \text{(self-energy transitions)}. \quad (14a)$$

Otherwise, we put $T=T^i+T^\delta$, with

$$T_{ea}{}^i=\frac{iH_{ez}{}^iH_{za}{}^i}{2(E(e)-E(a))}\left[\frac{1}{E(e)-E(z)}-\frac{1}{E(z)-E(a)}\right];$$

$$T_{ea}{}^\delta=\frac{iH_{ea}{}^\delta}{E(e)-E(a)}$$

$$\text{(non-self-energy transitions)} \quad (14b)$$

[9] N. Kemmer, Proc. Roy. Soc. **A166**, 127 (1938).

[10] R. P. Feynman, Phys. Rev. **74**, 1430 (1948).

giving

$$H_{II}=H^0+H^e-i[S,H^e]-i[T,H^e]-\tfrac{1}{2}[S,[S,H^e]]. \quad (15)$$

In (15), H^0 has no off-diagonal matrix elements, all transitions being induced by the field. The H^e term has the same matrix elements between single-nucleon states as it would have had if there were no mesons; zero for neutrons and the customary Dirac value for protons. The $[S, H^e]$ term is not diagonal in the number of mesons, and hence has no elements between single-nucleon states. The operator of interest is thus $K=K^i+K^\delta$, where

$$K^i=-i[T^i,H^e]-\tfrac{1}{2}[S,[S,H^e]],\quad K^\delta=-i[T^\delta,H^e]. \quad (16)$$

Then, from (12) and (14), we have

$$K_{ea}{}^i=\frac{H_{ew}{}^iH_{wz}{}^iH_{za}{}^e}{(E(e)-E(w))(E(e)-E(z))}+\frac{H_{ew}{}^eH_{wz}{}^iH_{za}{}^i}{(E(w)-E(a))(E(z)-E(a))}-\frac{H_{ew}{}^iH_{wz}{}^eH_{za}{}^i}{(E(e)-E(w))(E(z)-E(a))},$$

$$K_{ea}{}^\delta=-\frac{H_{ew}{}^\delta H_{wa}{}^e}{E(w)-E(e)}+\frac{H_{ew}{}^eH_{wa}{}^\delta}{E(a)-E(w)},$$

(non-self-energy transitions), (17a)

except that in case the transition $e\rightarrow z$ in the first term of K^i or $w\rightarrow a$ in the second term are self-energy transitions, we must make respectively the substitutions

$$\frac{H_{ew}{}^iH_{wz}{}^iH_{za}{}^e}{(E(e)-E(w))(E(e)-E(z))}\rightarrow\frac{H_{ew}{}^iH_{wz}{}^iH_{za}{}^e}{2(E(e)-E(w))(E(w)-E(z))};$$

$$\frac{H_{ew}{}^eH_{wz}{}^iH_{za}{}^i}{(E(w)-E(a))(E(z)-E(a))}\rightarrow\frac{H_{ew}{}^eH_{wz}{}^iH_{za}{}^i}{2(E(w)-E(z))(E(z)-E(a))}. \quad (17b)$$

In K^δ, only non-self-energy elements of H^δ should be taken. Equation (17) is our basic formula.

III. NEUTRON IN ELECTRIC FIELD: CONVERGENCE INVESTIGATION

The Matrix Element

The matrix element due to the electric field for a transition from a single-neutron state of momentum $\mathbf{p}_0$ to a single-neutron state of momentum $\mathbf{p}$ (not equal to $\mathbf{p}_0$), $K_{\mathbf{pp}_0}$ will now be examined. In the following table of processes contributing to $K_{\mathbf{pp}_0}$, which will be seen to be exhaustive, μ^+ and μ^- stand for positive and negative mesons, N^+ and N^- for positive- and negative-energy neutrons, and P^+ and P^- for protons. The momentum of each particle is written immediately after the symbol for the particle, and the term in the Hamiltonian causing the transition indicated by arrows is written in square parentheses.

1. Processes contributing to $K_{\mathbf{pp}_0}{}^i$ due to H^{em}:

(a) $N^+(\mathbf{p}_0)$ destroyed before $N^+(\mathbf{p})$ created:

(α) $N^+(\mathbf{p}_0)[H^i]\rightarrow P^+(-\mathbf{k})+\mu^-(\mathbf{b})$; $\mu^-(\mathbf{b})[H^e]\rightarrow\mu^-(\mathbf{a})$; $P^+(-\mathbf{k})+\mu^-(\mathbf{a})[H^i]\rightarrow N^+(\mathbf{p})$
(β) $[H^e]\rightarrow\mu^+(-\mathbf{b})+\mu^-(\mathbf{a})$; $N^+(\mathbf{p}_0)+\mu^+(-\mathbf{b})[H^i]\rightarrow P^+(-\mathbf{k})$; $P^+(-\mathbf{k})+\mu^-(\mathbf{a})[H^i]\rightarrow N^+(\mathbf{p})$
(γ) $N^+(\mathbf{p}_0)[H^i]\rightarrow P^+(-\mathbf{k})+\mu^-(\mathbf{b})$; $P^+(-\mathbf{k})[H^i]\rightarrow N^+(\mathbf{p})+\mu^+(-\mathbf{a})$; $\mu^-(\mathbf{b})+\mu^+(-\mathbf{a})[H^e]\rightarrow$.

(b) $N^+(\mathbf{p})$ created before $N^+(\mathbf{p}_0)$ destroyed:

(α) $P^-(-\mathbf{k})[H^i]\rightarrow N^+(\mathbf{p})+\mu^+(-\mathbf{a})$; $\mu^+(-\mathbf{a})[H^e]\rightarrow\mu^+(-\mathbf{b})$; $N^+(\mathbf{p}_0)+\mu^+(-\mathbf{b})[H^i]\rightarrow P^-(-\mathbf{k})$
(β) $P^-(-\mathbf{k})[H^i]\rightarrow N^+(\mathbf{p})+\mu^+(-\mathbf{a})$; $N^+(\mathbf{p}_0)[H^i]\rightarrow P^-(-\mathbf{k})+\mu^-(\mathbf{b})$; $\mu^+(-\mathbf{a})+\mu^-(\mathbf{b})[H^e]\rightarrow$
(γ) $[H^e]\rightarrow\mu^+(-\mathbf{b})+\mu^-(\mathbf{a})$; $P^-(-\mathbf{k})+\mu^-(\mathbf{a})[H^i]\rightarrow N^+(\mathbf{p})$; $N^+(\mathbf{p}_0)+\mu^+(-\mathbf{b})[H^i]\rightarrow P^-(-\mathbf{k})$.

2. Processes contributing to $K_{\mathbf{pp}_0}{}^i$ due to H^{en}:

(a) $N^+(\mathbf{p}_0)$ destroyed before $N^+(\mathbf{p})$ created:

(α) $N^+(\mathbf{p}_0)[H^i]\rightarrow P^+(\mathbf{b})+\mu^-(-\mathbf{k})$; $P^+(\mathbf{b})[H^e]\rightarrow P^+(\mathbf{a})$; $P^+(\mathbf{a})+\mu^-(-\mathbf{k})[H^i]\rightarrow N^+(\mathbf{p})$
(β) $P^-(\mathbf{b})[H^e]\rightarrow P^+(\mathbf{a})$; $N^+(\mathbf{p}_0)[H^i]\rightarrow P^-(\mathbf{b})+\mu^-(-\mathbf{k})$; $P^+(\mathbf{a})+\mu^-(-\mathbf{k})[H^i]\rightarrow N^+(\mathbf{p})$
(γ) $N^+(\mathbf{p}_0)[H^i]\rightarrow P^+(\mathbf{b})+\mu^-(-\mathbf{k})$; $P^-(\mathbf{a})+\mu^+(-\mathbf{k})[H^i]\rightarrow N^+(\mathbf{p})$; $P^+(\mathbf{b})[H^e]\rightarrow P^-(\mathbf{a})$.

(b) $N^+(\mathbf{p})$ created before $N^+(\mathbf{p}_0)$ destroyed:

(α) $P^-(\mathbf{a})[H^i]\rightarrow N^+(\mathbf{p})+\mu^+(\mathbf{k})$; $P^-(\mathbf{b})[H^e]\rightarrow P^-(\mathbf{a})$; $N^+(\mathbf{p}_0)+\mu^+(\mathbf{k})[H^i]\rightarrow P^-(\mathbf{b})$
(β) $P^-(\mathbf{a})[H^i]\rightarrow N^+(\mathbf{p})+\mu^+(\mathbf{k})$; $N^+(\mathbf{p}_0)+\mu^+(\mathbf{k})[H^i]\rightarrow P^+(\mathbf{b})$; $P^+(\mathbf{b})[H^e]\rightarrow P^-(\mathbf{a})$
(γ) $P^-(\mathbf{b})[H^e]\rightarrow P^+(\mathbf{a})$; $P^+(\mathbf{a})[H^i]\rightarrow N^+(\mathbf{p})+\mu^+(\mathbf{k})$; $N^+(\mathbf{p}_0)+\mu^+(\mathbf{k})[H^i]\rightarrow P^-(\mathbf{b})$.

3. Processes contributing to $K_{\mathbf{pp}_0}{}^{\delta}$ due to H^{en}:

(a) $P^-(\mathbf{b})[H^e]\to P^+(\mathbf{a})$; $N^+(\mathbf{p}_0)+P^+(\mathbf{a})[H^\delta]\to N^+(\mathbf{p})+P^-(\mathbf{b})$
(b) $P^-(\mathbf{a})+N^+(\mathbf{p}_0)[H^\delta]\to N^+(\mathbf{p})+P^+(\mathbf{k})$; $P^+(\mathbf{b})[H^e]\to P^-(\mathbf{a})$.

Here,

$$\mathbf{a}=\mathbf{k}+\mathbf{p},\quad \mathbf{b}=\mathbf{k}+\mathbf{p}_0, \tag{18}$$

and we note that none of these processes are self-energy processes (since $\mathbf{p}\neq\mathbf{p}_0$), so that (17a) should be used.

Using (5), (7), (8), and (17a), as well as properties of Dirac matrices and the usual projection operator tricks,[11] one obtains:

$$K_{\mathbf{pp}_0}=a_{\mathbf{p}}{}^{\dagger}a_{\mathbf{p}0}(-\pi e/2L^6)\left\{\int d\tau\, V(\mathbf{r})\exp(i(\mathbf{p}_0-\mathbf{p})\cdot\mathbf{r})\right\}(u_{\mathbf{p}}{}^*\mathfrak{M}u_{\mathbf{p}0}), \tag{19}$$

where for general pseudovector and pseudoscalar coupling:

$$\mathfrak{M}=\frac{f^2}{\mu^2}F,\quad F\equiv F^i+F^\delta\equiv\sum_{\mathbf{k}}(F_{\mathbf{k}}{}^i+F_{\mathbf{k}}{}^\delta),\quad F^i\equiv F^{im}+F^{in}\equiv\sum_{\mathbf{k}}(F_{\mathbf{k}}{}^{im}+F_{\mathbf{k}}{}^{in}) \tag{20a}$$

$$\begin{aligned}F_{\mathbf{k}}{}^{im}=&(\epsilon_a+\epsilon_b)\frac{(1+\mathfrak{A})(1+K')(1+\mathfrak{B})}{(\epsilon_a-E_p+E)(\epsilon_b-E_{p0}+E)}-(\epsilon_a-\epsilon_b)\frac{(1+\mathfrak{A})(1+K')(1-\mathfrak{B})}{(\epsilon_a-E_p+E)(\epsilon_a+\epsilon_b-E_p+E_{p0})}\\&+(\epsilon_a-\epsilon_b)\frac{(1-\mathfrak{A})(1+K')(1+\mathfrak{B})}{(\epsilon_a+\epsilon_b-E_{p0}+E_p)(\epsilon_b-E_{p0}+E)}+(\epsilon_a+\epsilon_b)\frac{(1-\mathfrak{A})(1-K')(1-\mathfrak{B})}{(\epsilon_a+E_p+E)(\epsilon_b+E_{p0}+E)}\\&-(\epsilon_a-\epsilon_b)\frac{(1-\mathfrak{A})(1-K')(1+\mathfrak{B})}{(\epsilon_a+E_p+E)(\epsilon_a+\epsilon_b+E_p-E_{p0})}+(\epsilon_a-\epsilon_b)\frac{(1+\mathfrak{A})(1-K')(1-\mathfrak{B})}{(\epsilon_a+\epsilon_b+E_{p0}-E_p)(\epsilon_b+E_{p0}+E)}\end{aligned} \tag{20m}$$

$$\begin{aligned}F_{\mathbf{k}}{}^{in}=&-\epsilon(1+\mathcal{K})\left\{\frac{(1+A)(1+B)}{(E_a-E_p+\epsilon)(E_b-E_{p0}+\epsilon)}-\frac{(1+A)(1-B)}{(E_a-E_p+\epsilon)(E_a+E_b-E_p+E_{p0})}\right.\\&\left.-\frac{(1-A)(1+B)}{(E_a+E_b+E_p-E_{p0})(E_b-E_{p0}+\epsilon)}\right\}(1+\mathcal{K})\\&-\epsilon(1-\mathcal{K})\left\{\frac{(1-A)(1-B)}{(E_a+E_p+\epsilon)(E_b+E_{p0}+\epsilon)}-\frac{(1-A)(1+B)}{(E_a+E_p+\epsilon)(E_a+E_b+E_p-E_{p0})}\right.\\&\left.-\frac{(1+A)(1-B)}{(E_a+E_b-E_p+E_{p0})(E_b+E_{p0}+\epsilon)}\right\}(1-\mathcal{K})\end{aligned} \tag{20n}$$

$$\begin{aligned}F_{\mathbf{k}}{}^\delta=&-2\frac{(1+s)(1+A)(1-B)-s\boldsymbol{\alpha}(1+A)\cdot(1-B)\boldsymbol{\alpha}}{E_a+E_b-E_p+E_{p0}}-2\frac{(1+s)(1-A)(1+B)-s\boldsymbol{\alpha}(1-A)\cdot(1+B)\boldsymbol{\alpha}}{E_a+E_b+E_p-E_{p0}}\\&+\left(\frac{2f_0{}^2}{f^2}\right)\left(\frac{-4ti\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{b}}{E_aE_b}\right)\left(\frac{1}{E_a+E_b-E_p+E_{p0}}+\frac{1}{E_a+E_b+E_p-E_{p0}}\right).\end{aligned} \tag{20δ}$$

Here,

$$\begin{aligned}&E_a=+(\mathbf{a}^2+M^2)^{\frac12},\quad \epsilon_a=+(\mathbf{a}^2+\mu^2)^{\frac12},\ \text{etc.},\quad (E\equiv E_k,\ \epsilon\equiv\epsilon_k);\quad \rho=\mu g/f;\\&A=(\boldsymbol{\alpha}\cdot\mathbf{a}-\beta M)/E_a,\qquad B=(\boldsymbol{\alpha}\cdot\mathbf{b}-\beta M)/E_b,\qquad K=(\boldsymbol{\alpha}\cdot\mathbf{k}-\beta M)/E;\\&A'=-(\boldsymbol{\alpha}\cdot\mathbf{a}+\beta M)/E_a,\quad B'=-(\boldsymbol{\alpha}\cdot\mathbf{b}+\beta M)/E_b,\quad K'=-(\boldsymbol{\alpha}\cdot\mathbf{k}+\beta M)/E;\\&\mathfrak{A}=-(\boldsymbol{\alpha}\cdot\mathbf{a}+\beta\rho)/\epsilon_a,\qquad \mathfrak{B}=-(\boldsymbol{\alpha}\cdot\mathbf{b}+\beta\rho)/\epsilon_b,\qquad \mathcal{K}=+(\boldsymbol{\alpha}\cdot\mathbf{k}-\beta\rho)/\epsilon.\end{aligned} \tag{21}$$

For pseudoscalar coupling alone ($f=0$),

$$\mathfrak{M}=g^2G,\quad G=G^m+G^n=\sum_{\mathbf{k}}(G_{\mathbf{k}}{}^m+G_{\mathbf{k}}{}^n) \tag{22a}$$

[11] See W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1947), p. 150.

$$G_{\mathbf{k}}^{m}=\frac{(1+K)}{\epsilon_a\epsilon_b}\left\{\frac{\epsilon_a+\epsilon_b}{(\epsilon_a-E_p+E)(\epsilon_b-E_{p0}+E)}+\frac{\epsilon_a-\epsilon_b}{(\epsilon_a-E_p+E)(\epsilon_a+\epsilon_b-E_p+E_{p0})}-\frac{\epsilon_a-\epsilon_b}{(\epsilon_a+\epsilon_b+E_p-E_{p0})(\epsilon_b-E_{p0}+E)}\right\}+\frac{(1-K)}{\epsilon_a\epsilon_b}\left\{\frac{\epsilon_a+\epsilon_b}{(\epsilon_a+E_p+E)(\epsilon_b+E_{p0}+E)}+\frac{\epsilon_a-\epsilon_b}{(\epsilon_a+E_p+E)(\epsilon_a+\epsilon_b+E_p-E_{p0})}-\frac{\epsilon_a-\epsilon_b}{(\epsilon_a+\epsilon_b-E_p+E_{p0})(\epsilon_b+E_{p0}+E)}\right\} \quad (22m)$$

$$G_{\mathbf{k}}^{n}=-\frac{1}{\epsilon}\left\{\frac{(1+A')(1+B')}{(E_a-E_p+\epsilon)(E_b-E_{p0}+\epsilon)}-\frac{(1+A')(1-B')}{(E_a-E_p+\epsilon)(E_a+E_b-E_p+E_{p0})}-\frac{(1-A')(1+B')}{(E_a+E_b+E_p-E_{p0})(E_b-E_{p0}+\epsilon)}\right\}-\frac{1}{\epsilon}\left\{\frac{(1-A')(1-B')}{(E_a+E_p+\epsilon)(E_b+E_{p0}+\epsilon)}-\frac{(1-A')(1+B')}{(E_a+E_p+\epsilon)(E_a+E_b+E_p-E_{p0})}-\frac{(1+A')(1-B')}{(E_a+E_b-E_p+E_{p0})(E_b+E_{p0}+\epsilon)}\right\}. \quad (22n)$$

In (20) and (22), the terms are listed in the same order as in the table of processes, so that for instance 2 (a β) contributes the second term of (20n). (In (20δ), the first and second lines come from the charged and neutral mesons, respectively.) We note too that the contribution of every (b) process is the same as that of the analogous (a) process, except that the signs of α, β, $E_{\mathbf{p}}$, and $E_{\mathbf{p}0}$ are reversed. The $a_{\mathbf{p}}{}^{\dagger}a_{\mathbf{p}0}$ in (19) stands for the matrix element of this operator between the two neutron states in question, and is $+1$ or -1, depending on the ordering of the states. It is not important physically (when $K_{\mathbf{pp}0}$ is squared to compute transition probabilities, it has no effect at all) and will not be carried further.

Charge Conservation and Ambiguity of the Integrals

Since $\sum_{\mathbf{k}}\sim(L/2\pi)^3\int dk_xdk_ydk_z$, and the operators (21) are all of order of magnitude unity (or less) for large $\mathbf{k}$, F can diverge no worse than quadratically; G no worse than logarithmically. In fact, the quadratically divergent terms of $\sum_{\mathbf{k}}F_{\mathbf{k}}$ are from processes 1 (a α) and 1 (b α), which contribute $\sim+2/k$ each to $F_{\mathbf{k}}{}^{im}$, and 2 (a α) and 2 (b α), which contribute $\sim-2/k$ each to $F_{\mathbf{k}}{}^{in}$. Thus, in both the meson and nucleon contributions, positive- and negative-energy processes contribute terms of the same sign that diverge with equal strength. (For $\sum_{\mathbf{k}}G_{\mathbf{k}}$, the situation is the same: 1 (a α) and 1 (b α) contribute $\sim(1/2k^3)$ each, while 2 (a α) and 2 (b α) contribute $\sim-1/(2k^3)$.) This contradicts the conjecture made initially that negative-energy processes would help the convergence.

The effect of negative-energy processes found here is to be expected intuitively. If, for instance, the total meson charge in a single-neutron state is considered, it is seen that this differs from the meson charge in vacuum for two reasons:

(1) The single neutron can make transitions to a proton-negative meson state, causing a net negative meson charge.

(2) Certain transitions (from negative-energy protons to positive meson+the neutron in question) that would take place in vacuum are now excluded, decreasing the positive meson charge and effectively providing a further negative meson charge. Thus, instead of canceling most of the negative meson charge, the negative-energy processes increase it.

The case with nucleon charge is exactly similar: (1) causes a positive nucleon charge while (2), preventing "dissociation" of negative-energy protons, further increases the positive nucleon charge.

A convergent matrix element is thus possible only if the meson and nucleon charge clouds, which are of opposite sign, give contributions that cancel to a sufficient extent. In fact, it will now be shown that charge-conservation implies that (19) is zero for $\mathbf{p}=\mathbf{p}_0$. Thus, the leading divergences of $F_{\mathbf{k}}$ and $G_{\mathbf{k}}$, which are independent of $\mathbf{p}$ and $\mathbf{p}_0$, must cancel. (This already guarantees the convergence of G.)

The formal statement of charge conservation is that the Hamiltonian and contact transformations are all diagonal in the total charge; i.e., they commute with the total charge operator

$$Q=e\sum_{\mathbf{k}}(N_{+\mathbf{k}}-N_{-\mathbf{k}})+e\sum_{\substack{\text{Proton}\\\text{states only}}}N_{\mathbf{p}}. \quad (23)$$

(Note that since we will consider only changes of total charge, use of the simple charge operator rather than Heisenberg's symmetrized operator is immaterial.) By (7), H^e becomes identical to Q if $V(\mathbf{r})=1$, so that $K_{\mathbf{pp}0}$ for this $V(\mathbf{r})$ gives the change in charge due to the canonical transformations, which must be set equal to zero. Now if $\mathbf{p}=\mathbf{p}_0$, the (b) processes in the tabulation cannot occur, and (17b) must be used for processes where H^e occurs first or last—and these give a zero contribution, as may be seen from the numerators of the corresponding terms in (20). The result is that

$$\sum_{\mathbf{k}}\left(u_{p0}^*\left[2\epsilon_b\frac{(1+\mathcal{B})(1+K')(1+\mathcal{B})}{(\epsilon_b-E_{p0}+E)^2}-\epsilon\frac{(1+\mathcal{K})(1+B)(1+B)(1+\mathcal{K})}{(E_b-E_{p0}+\epsilon)^2}\right]u_{p0}\right)=0. \quad (24)$$

Since the same equation holds true if $-\boldsymbol{\alpha}$, $-\boldsymbol{\beta}$, $-E_{p0}$ are substituted for $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, E_{p0} in (24) (see the Appendix), it is seen that (19) is zero as asserted for $\mathbf{p}=\mathbf{p}_0$.

It is very difficult to prove (24) by direct calculation. This is connected with the naming of the momenta that has heretofore been used, which was chosen so that the electric field scatters momenta of magnitude $|\mathbf{a}|$ and $|\mathbf{b}|$ in both nucleon and meson cases, which suggests itself rather naturally. If, however, we substitute *in* $F_{\mathbf{k}}^{im}$ *and* $G_{\mathbf{k}}^{n}$ *only*

$$-(\mathbf{k}+\tfrac{1}{2}\boldsymbol{\Delta}) \text{ for } \mathbf{a};\quad -(\mathbf{k}-\tfrac{1}{2}\boldsymbol{\Delta}) \text{ for } \mathbf{b};$$

$$-\left(\mathbf{k}+\frac{\mathbf{s}}{2}\right)=-\left(\mathbf{a}+\frac{\boldsymbol{\Delta}}{2}\right)=-\left(\mathbf{b}-\frac{\boldsymbol{\Delta}}{2}\right) \text{ for } \mathbf{k}, \quad (25)$$

where

$$\mathbf{s}=\mathbf{p}_0+\mathbf{p},\quad \boldsymbol{\Delta}=\mathbf{p}_0-\mathbf{p} \quad (25a)$$

it can immediately be seen for $\mathbf{p}_0=\mathbf{p}$, that not only is (24) satisfied, but that $F_{\mathbf{k}}$ and $G_{\mathbf{k}}$ are identically zero. If the meson and nucleon integrals converged individually, such changes of variable would certainly be admissible. However, especially for a quadratic divergence, such a change can alter the results essentially, so that both choices of variable should be examined to verify that the results are not ambiguous. (If they were, the results obtained using the substitution (25) would be preferred, since it is known that charge conservation would then be satisfied practically as well as formally.)

We note that the substitution (25) could just as well have been made in the nucleon terms only, but this will be seen to give the same results.

Degree of Divergence and Effect of Contact Interaction

To determine whether K_{pp0} converges, (20) must be expanded in a power series in k^{-1}. Only the three leading terms (in k^{-1}, k^{-2}, and k^{-3}) give divergent contributions to F, so that all quantities appearing in (20) need be expanded to second order only. One then obtains for finite, but not necessarily small, $\mathbf{p}$ and $\mathbf{p}_0$, after averaging over the possible angles of the vector $\mathbf{k}$,

$$F_{\mathbf{k}}^{i}\sim[2\boldsymbol{\Delta}^2-\tfrac{2}{3}(E_{p0}-E_p)^2+2i\boldsymbol{\sigma}\cdot\mathbf{p}_0\times\mathbf{p}]\cdot k^{-3}+\text{terms in } k^{-4} \text{ etc.}$$

$$F_{\mathbf{k}}^{\delta}\sim\left[-\left(\frac{5-4s}{3}\right)\boldsymbol{\Delta}^2+\left(\frac{1-4s}{3}\right)(E_{p0}-E_p)^2-2i\boldsymbol{\sigma}\cdot\mathbf{p}_0\times\mathbf{p}\right]\cdot k^{-3}+\text{terms in } k^{-4} \text{ etc.}$$

$$F_{\mathbf{k}}\sim\left(\frac{1+4s}{3}\right)[\boldsymbol{\Delta}^2-(E_{p0}-E_p)^2]\cdot k^{-3}+\text{terms in } k^{-4} \text{ etc.} \quad (26)$$

The lengthy algebra used to derive (26), which made use of the substitutions listed in the Appendix, will not be reproduced here.

If the substitution (25) is made in (20), precisely the same result (26) is obtained, so that any ambiguity is unlikely. In fact, if $F_{\mathbf{k}}^{im}$ is expanded in the original notation, and then $-(\mathbf{k}+\tfrac{1}{2}\mathbf{s})$ is substituted for $\mathbf{k}$ (before the averaging over the angles of $\mathbf{k}$), further expanded, and then averaged over angles, only the terms in $\mathbf{s}^2$ and $\boldsymbol{\sigma}$ can be changed. The former is, however, zero by charge conservation or direct computation, while the spin term is found to be actually unchanged. Moreover, it is immaterial whether (25) is used on the meson or nucleon contributions to $F_{\mathbf{k}}$.

The absence of a term in ρ^2 (or even ρ) in (26) again verifies the convergence of the pseudoscalar coupling contribution. Moreover, since the divergence of G^m and G^n is only logarithmic, a change of variables such as (25) is very unlikely to change the value of the matrix element.

We note that $F_{\mathbf{k}}$ has a relativistically invariant form, which would seem to indicate that relativistic invariance has not been lost due to divergences. It is important that if $F_{\mathbf{k}}^{\delta}$ were neglected, the result would not have an invariant form. Also, the spin-dependent term, which will be seen in Section IV to represent the magnetic moment, diverges if $F_{\mathbf{k}}^{\delta}$ is neglected. However, if the contact term is consistently handled, the magnetic moment is seen to converge for any value of s, i.e., regardless of whether or not any invariant contact term is added to the Lagrangian. This convergence has been noticed for the special choice $s=-1$ by Case.[12]

However, the matrix element F will diverge unless we choose $s=-\tfrac{1}{4}$. On the other hand, the physical significance of the contact terms is rather dubious, so that results critically dependent on them, including the convergence of the magnetic

[12] K. M. Case, thesis, Harvard University (1948); Phys. Rev. 74, 1884 (1948).

moments, must be accepted with great caution. In this connection, note that the neutral meson part of (20δ) sums to zero, at least if $E_p=E_{p0}$, as is most easily seen by using (25). It should also be mentioned that the choice $s=-\frac{1}{3}$ has sometimes been recommended to remove contact terms from the static nuclear potentials, but this is a very questionable procedure.

Comparison of Pseudoscalar and Pseudovector Couplings

Dyson has shown rather generally[13] that the pseudoscalar and pseudovector couplings give equivalent results, at least to first order. This may be seen directly from (7), which gives for the matrix element for a transition from a nucleon of momentum $\mathbf{p}_0$ to a meson of momentum $\mathbf{k}$ and nucleon of momentum $\mathbf{p}=\mathbf{p}_0-\mathbf{k}$ the value

$$(2\pi/\epsilon L^3)^{\frac{1}{2}}\{-g(u_{\mathbf{p}}{}^*\alpha_1\alpha_2\alpha_3\beta u_{\mathbf{p}0}) +(f/\mu)(u_p{}^*[\epsilon\alpha_1\alpha_2\alpha_3-i\boldsymbol{\sigma}\cdot\mathbf{k}]u_{\mathbf{p}0})\}$$

and similarly for absorption. If both nucleons have positive energy, it follows from the Appendix that this is just

$$\left(\frac{2\pi}{\epsilon L^3}\right)^{\frac{1}{2}}\left\{\left(-\frac{g}{2M}\right)\left(u_{\mathbf{p}}{}^*i\boldsymbol{\sigma}\left[-\mathbf{k}+\left(\frac{E_{p0}-E_p}{E_{p0}+E_p}\right)(\mathbf{p}+\mathbf{p}_0)\right]u_{p0}\right)+\left(\frac{f}{\mu}\right)\left(u_{\mathbf{p}}{}^*i\boldsymbol{\sigma}\cdot\left[-\mathbf{k}+\frac{\epsilon}{E_{p0}+E_p}(\mathbf{p}+\mathbf{p}_0)\right]u_{\mathbf{p}0}\right)\right\}. \quad (27)$$

Thus, apart from the second term in each square bracket, which is negligible for low momentum processes, the couplings behave identically; a pseudovector coupling of strength f behaving like a pseudoscalar coupling

$$g=-(2M/\mu)f. \quad (28)$$

In many calculations (such as of static nuclear forces, where only positive-energy processes are considered and only in non-relativistic approximation) the two couplings give rise to the same result. Although this has frequently been overlooked, one need only put $(f-\mu g/2M)$ for f to rectify the omission of pseudoscalar coupling. (Note that from (28) a g^2 of 30 is no more "strongly coupled" than an f^2 of about 0.2.)

It may thus seem surprising that pseudovector coupling can give a divergent interaction while pseudoscalar does not. However, a closer inspection of (27) for high momentum processes ($|\mathbf{k}|\gg M$ or $|\mathbf{p}_0|$) shows that the square bracket of the g term approaches $-\mathbf{k}+\mathbf{k}$, while that of the f term approaches $-\mathbf{k}-\mathbf{k}$, so that the pseudoscalar coupling matrix elements are less divergent. Moreover, many matrix elements between nucleons of different energy signs occur in the calculations, and here the couplings differ considerably.

[13] F. J. Dyson, Phys. Rev. **73**, 929 (1948).

It is very instructive to extend Dyson's proof to our problem. We define H_f and H_g to be the Hamiltonians (6) with $g=0$ and $f=0$ respectively. We also omit the neutral meson field—which we have seen makes no difference in the neutron scattering element—and set $s=0$ for simplicity. If a unitary transformation $\exp(iR)$ of H_f is carried out, with $R=(4\pi)^{\frac{1}{2}}(f/\mu)\int d\tau(B_0\Psi+B_0{}^\dagger\Psi^\dagger)$, we obtain to second order for the transformed H_f, after considerable reduction, $\exp(iR)H_f\exp(-iR)=H_g+H'$, where

$$H'=(4\pi f^2/\mu^2)\int d\tau\left\{\tfrac{1}{2}(B_0{}^\dagger B_0-B_0B_0{}^\dagger)+\frac{i}{2}[(\psi^*\tau^0\psi)(\Pi\Psi-\Pi^\dagger\Psi^\dagger)+(\psi^*\boldsymbol{\alpha}\tau^0\psi)\cdot(\Psi\nabla\Psi^\dagger-\Psi^\dagger\nabla\Psi)]-(\psi^*\beta M\psi)(\Psi^\dagger\Psi+\Psi\Psi^\dagger)\right\},$$

and in H_g, g is given by (28). (It is convenient to perform the reduction in coordinate representation.)

In diagonalizing the non-electrical part of H_f, we may first carry out the transformation $\exp(iR)$, and we then obtain for the neutron scattering element precisely the value for pseudoscalar coupling with an additional term K_{ea}' given by (cf. (17a))

$$K_{ea}'=\frac{H_{ew}'H_{wa}{}^e}{E(e)-E(w)}+\frac{H_{ew}{}^eH_{wa}'}{E(a)-E(w)}.$$

Only two transition schemes give non-zero contributions:

1. $[H^e]\rightarrow\mu^+(\mathbf{a})+\mu^-(-\mathbf{b})$; $\mu^+(\mathbf{a})+\mu^-(-\mathbf{b})+N^+(\mathbf{p}_0)[H']\rightarrow N^+(\mathbf{p})$.
2. $N^+(\mathbf{p}_0)[H']\rightarrow N^+(\mathbf{p})+\mu^+(-\mathbf{a})+\mu^-(\mathbf{b})$; $\mu^+(-\mathbf{a})+\mu^-(\mathbf{b})[H^e]\rightarrow$.

One obtains finally for the contribution to $F_{\mathbf{k}}$ of (20a):

$$F_{\mathbf{k}}'=\left(\frac{\epsilon_b-\epsilon_a}{\epsilon_a\epsilon_b}\right)\left[\frac{(\epsilon_b-\epsilon_a)+\boldsymbol{\alpha}\cdot(\mathbf{a}+\mathbf{b})}{\epsilon_a+\epsilon_b+(E_{p0}-E_p)}+\frac{(\epsilon_b-\epsilon_a)-\boldsymbol{\alpha}\cdot(\mathbf{a}+\mathbf{b})}{\epsilon_a+\epsilon_b-(E_{p0}-E_p)}\right].$$

If $E_p=E_{p0}$, $F_{\mathbf{k}}'$ becomes $2(\epsilon_b-\epsilon_a)^2/[\epsilon_a\epsilon_b(\epsilon_a+\epsilon_b)]$, which is rigorously spin-independent. Thus H' gives zero contribution to the magnetic moment, so that —to second order at least—pseudoscalar and pseudovector couplings give exactly the same neutron magnetic moment. Investigation of the term proportional to s in (20) shows it to be spin-independent, so this equality of the magnetic moments holds even for non-zero s. The arguments may also be extended to the case of the proton moment.

However, $\sum_{\mathbf{k}} F_{\mathbf{k}}'$ itself diverges logarithmically, since it is easily verified that asymptotically, $F_{\mathbf{k}}' \sim \frac{1}{3}[\boldsymbol{\Delta}^2-(E_{p0}-E_p)^2]k^{-3}$. Since H_g gives no divergent contribution, $F_{\mathbf{k}}'$ is the only term giving rise to a divergence, and $F_{\mathbf{k}}$ itself $\sim\frac{1}{3}[\boldsymbol{\Delta}^2-(E_{p0}-E_p)^2]k^{-3}$. This agrees exactly with (26) (since we have here put $s=0$), which was computed entirely independently. We note that H' has self-energy terms corresponding to an infinite mass renormalization, but as is to be expected (cf. Section VI), it contains no "charge renormalization" terms.

It thus seems that the pseudovector element diverges genuinely (except for the special choice of s mentioned). On the other hand, Case, using the newer methods, has recently reported that pseudovector and pseudoscalar couplings are exactly equivalent to second order insofar as scattering in an electric field is concerned.[14] The reason for this discrepancy is not known to us at present. If it is not spurious, it may prove to be of great interest.

In any case, the greater simplicity of the pseudoscalar coupling, its freedom from questionable contact interactions, its apparently stronger convergence, and its equivalence to pseudovector coupling in the theory of nuclear forces, are arguments for its exclusive use in the present preliminary stage of the theory. Only pseudoscalar coupling will be considered in Sections IV and V.

IV. THE NEUTRON-ELECTRON INTERACTION AND MAGNETIC MOMENT OF THE NEUTRON

Determination from Matrix Element

In a static electric field, only energy-conserving processes are of interest; i.e., we may take $E_p=E_{p0}$. In this case, G^m, etc., may be written, after using the Appendix, in terms of the following scalars:

$$\mathbf{p}^2=\mathbf{p}_0^2,\quad \mathbf{p}\cdot\mathbf{p}_0,\quad \boldsymbol{\sigma}\cdot\mathbf{p},\quad \boldsymbol{\sigma}\cdot\mathbf{p}_0,\quad \boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{p}_0,$$

which (apart from the masses) are the only independent ones available. As may easily be verified, $\boldsymbol{\sigma}$ enters only in triple products involving a cross-product of two other vectors, which either vanishes when averaged over angles, or takes the form $\boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{p}_0$. Thus the expressions will take the form (using charge conservation):

$$\begin{aligned}
G^m/L^3 &= C_0+(\mathbf{s}^2/M^2)C_s+(\boldsymbol{\Delta}^2/M^2)C_\Delta{}^m \\
&\quad +(i\boldsymbol{\sigma}\cdot\mathbf{p}_0\times\mathbf{p}/M^2)C_\sigma{}^m \\
G^n/L^3 &= -C_0-(\mathbf{s}^2/M^2)C_s+(\boldsymbol{\Delta}^2/M^2)C_\Delta{}^n \\
&\quad +(i\boldsymbol{\sigma}\cdot\mathbf{p}_0\times\mathbf{p}/M^2)C_\sigma{}^n \\
G/L^3 &= (\boldsymbol{\Delta}^2/M^2)(C_\Delta{}^m+C_\Delta{}^n) \\
&\quad +(i\boldsymbol{\sigma}\cdot\mathbf{p}_0\times\mathbf{p}/M^2)(C_\sigma{}^m+C_\sigma{}^n)
\end{aligned}\tag{29}$$

where $C_0(\mu/M)$ is a logarithmically divergent integral and C_s, $C_\Delta{}^m$, $C_\Delta{}^n$, $C_\sigma{}^m$, $C_\sigma{}^n$ are dimensionless functions of (μ/M), $(\mathbf{s}^2/M^2)$, and $(\boldsymbol{\Delta}^2/M^2)$. (Since $\mathbf{p}$ and $\mathbf{p}_0$ of interest are $\ll M$, the C's will be computed for $\boldsymbol{\Delta}^2/M^2=\mathbf{s}^2/M^2=0$; i.e., only the first non-vanishing terms of an expansion in $\mathbf{p}$ and $\mathbf{p}_0$ will be used.) The matrix element for a transition between the two single-neutron states in the electric field is then, by (19),

$$K_{\mathbf{pp}_0}=\left(-\frac{\pi e g^2}{2L^3}\right)\left\{\int d\tau\, V(\mathbf{r})\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})\right\}\times\left(u_{\mathbf{p}}^*\left[\frac{\boldsymbol{\Delta}^2}{M^2}(C_\Delta{}^m+C_\Delta{}^n)+\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{p}_0}{M^2}(C_\sigma{}^m+C_\sigma{}^n)\right]u_{\mathbf{p}0}\right).\tag{30}$$

On the other hand, a magnetic dipole moment $\boldsymbol{\mu}$ is associated with the spin-orbit interaction energy $-\boldsymbol{\mu}\cdot(-\nabla V)\times[-(i/M)\nabla]$ $(-i\nabla$ is the momentum operator), which has a matrix element for the transition in question of

$$\int(L^{-\frac{3}{2}}u_{\mathbf{p}}^*\exp(-i\mathbf{p}\cdot\mathbf{r}))\left(-\boldsymbol{\mu}\cdot\nabla V\times\frac{i}{M}\nabla\right)\times(L^{-\frac{3}{2}}u_{\mathbf{p}0}\exp(i\mathbf{p}_0\cdot\mathbf{r}))d\tau.$$

This is seen to be, apart from a neglected surface integral,

$$(\mu_N e/2M^2L^3)\left\{\int d\tau\, V(\mathbf{r})\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})\right\}(u_{\mathbf{p}}^*i\boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{p}_0u_{\mathbf{p}0})$$

where we have put $\boldsymbol{\mu}=\mu_N(e/2M)\boldsymbol{\sigma}$; μ_N being the neutron moment in nuclear magnetons. Comparing the above expression with (30), we obtain for the neutron moment

$$\mu_N=-\pi g^2(C_\sigma{}^m+C_\sigma{}^n).\tag{31}$$

(Note that as in the Schwinger calculation of radiative corrections to the electron spin-orbit interaction, no Thomas factor must be used.)

In the neutron scattering experiments from which the neutron-electron interaction is determined,[5,7] spin-dependent effects are not detected because of lack of polarization of both neutron beam and

[14] K. M. Case, Phys. Rev. 75, 1306 (1949).

scattering sample, which averages out interference between spin-dependent neutron-electron and neutron-nucleus scattering. Thus only the C_Δ part of (30) is to be compared with these experiments, which may be done, for instance, by inserting the effective atomic for $V(\mathbf{r})$ in (30).

Alternately, the experimental result may be given in terms of a neutron-electron potential $U(\mathbf{r})$. This would give, for an electron bound to a fixed position (taken as the origin), a matrix element between the two neutron states of

$$\int d\tau (L^{-\frac{3}{2}} u_{\mathbf{p}}^* \exp(-i\mathbf{p}\cdot\mathbf{r}))\, U(\mathbf{r})\, (L^{-\frac{3}{2}} u_{\mathbf{p}_0} \exp(i\mathbf{p}_0\cdot\mathbf{r})).$$

If $U(\mathbf{r})$ has a range short compared to the neutron wave-lengths—a condition well satisfied—this reduces to $\{L^{-3}\int d\tau U(\mathbf{r})\}(u_p^* u_{p_0})$. Comparing this to (30) after noting that $\nabla^2 \exp(i\mathbf{\Delta}\cdot\mathbf{r}) = -\mathbf{\Delta}^2 \exp(i\mathbf{\Delta}\cdot\mathbf{r})$, using Green's theorem and then discarding the surface integral, and using $\nabla^2 V = -4\pi\rho(\mathbf{r}) = +4\pi e\delta(\mathbf{r})$ for an electron fixed at the origin, the expression

$$\int U(r)d\tau \equiv U_0 V_e = (2\pi^2 e^2 g^2/M^2)(C_\Delta{}^m + C_\Delta{}^n) \quad (32)$$

is obtained, where by convention $V_e = 4\pi/3(e^2/mc^2)^3$.

Computation for Pseudoscalar Coupling

Using the Appendix, (22) can be written as a multiple of the unit operator plus the scalar product of $\boldsymbol{\sigma}$ and a vector. The spin-dependent term, which contains the magnetic moment, converges even before averaging over angles, so transformations, as (25), do not change the magnetic moment.

If the part of $G_\mathbf{k}$ not involving $\boldsymbol{\sigma}$ is expanded in a series in $\mathbf{p}$ and $\mathbf{p}_0$ (or $\mathbf{s}$ and $\mathbf{\Delta}$), one obtains a logarithmically divergent term in $\mathbf{k}$ alone, plus converging terms. If (25) is used, the term in $\mathbf{\Delta}^2$ is not influenced at all, while the $\mathbf{s}^2$ term, which is of no interest, is changed in form but integrates to zero in either case.

The results are:

$$C_\sigma{}^m = \int_0^\infty \frac{dk}{2\pi^2}\frac{4k^2}{R^2}\left[\frac{1}{E}+\frac{1}{\epsilon}-\frac{2k^2}{3\epsilon^2}\left(\frac{2}{E}+\frac{1}{2\epsilon}+\frac{2}{ER}\right)\right] = \int_0^\infty \frac{dk}{2\pi^2}\frac{16k^4(E+\epsilon)^2}{3E^2\epsilon R^3}\left(1+\frac{\epsilon R}{4E(E+\epsilon)^2}\right) \quad (33\text{m})$$

$$C_\sigma{}^n = \int_0^\infty \frac{dk}{2\pi^2}\frac{4k^2}{R^2E^4}\left[2E+\epsilon+\frac{4E}{3R}+\frac{R}{2E}\right] \quad (33\text{n})$$

$$C_\Delta{}^m = -\int_0^\infty \frac{dk}{2\pi^2}\frac{k^2}{R^2}\left\{\frac{2}{E}+\frac{1}{2\epsilon}+\frac{k^2}{2\epsilon^3}\left(2-\frac{k^2}{3\epsilon^2}\right)+\left(\frac{\epsilon^2+k^2}{\epsilon}+\frac{2k^2}{E}\right)\left[\frac{2E}{\epsilon R}\left(1-\frac{k^2}{3\epsilon^2}\right)-\frac{4k^2(E+\epsilon)^2}{3\epsilon^2R^2}\right]\right\} \quad (34\text{m})$$

$$C_\Delta{}^n = -\int_0^\infty \frac{dk}{2\pi^2}\frac{k^2}{R^2}\left\{\left(\frac{R}{\epsilon}-\frac{2}{E}\right)\left[\frac{4k^2(E+\epsilon)^2}{3E^2R^2}-\frac{2\epsilon}{ER}\left(1-\frac{k^2}{3E^2}\right)\right] + \frac{k^2(2E+\epsilon)}{3E^3\epsilon}\left(\frac{R}{E^2}-1\right)-\frac{k^2}{E^3}\left(1+\frac{1}{3E^2}\right)+\frac{2k^2}{3E^4\epsilon}\right\} \quad (34\text{n})$$

where

$R=(E+\epsilon)^2-1$ and $E=(k^2+1)^{\frac{1}{2}}$, $\epsilon=[k^2+(\mu/M)^2]^{\frac{1}{2}}$,

which differs slightly from previous notation. The second form of $C_\sigma{}^m$ is obtained with (25), and was used as a check.

We note that (33m) and (33n), taken together with (43c) and (43s) below, agree exactly with the results of a recent calculation of the magnetic moments by J. M. Luttinger, who used an entirely different method.[15]

[15] J. M. Luttinger, Helv. Phys. Acta **21**, 483 (1948), especially Eqs. (12), (23), and (25), after correcting some misprints. Note that his μ_p and g^2 correspond to our μ_p-1 and $g^2/2$. Unfortunately, Luttinger substituted for his g^2 twice the experimental value for symmetrical theory and four times the experimental value for pure charged theory, and made some additional numerical errors in his final substitution. He points out that the integrals corresponding to our (33) are elementary and may be brought to familiar form by introducing the variable $Z=(E+\epsilon)^2$. This is also true for (34).

Equations (33) and (34), integrated graphically for $\mu/M=0.154$ corresponding to a meson mass of 282 e.m., give

$$2\pi^2 C_\sigma{}^m = 0.343; \quad 2\pi^2 C_\sigma{}^n = 0.477;$$
$$2\pi^2 C_\Delta{}^m = -0.520; \quad 2\pi^2 C_\Delta{}^n = -0.001. \quad (35)$$

Using (31) and (32), it follows that

$$\mu_N = -0.128g^2; \quad U_0 = -0.340g^2 \text{ kev.} \quad (36)$$

Further discussion of the behavior of the integrands at low momenta, variation with μ/M, etc., will be left to Section VI.*

V. THE MAGNETIC MOMENT OF THE PROTON

The matrix element for the transition from a single-proton state of momentum $\mathbf{p}_0$ to one of mo-

* For $\mu/M=0.50$, (918 e.m.), $2\pi^2C_\Delta{}^m=-0.118$, $2\pi^2C_\Delta{}^n=-0.010$, $U_0=-0.0834g^2$ kev.

mentum $\mathbf{p}$ in an electric field is given by the ordinary Dirac term plus $K_{\mathbf{pp}_0}$ as defined by (17). The Dirac term, as is well known, gives a spin-orbit interaction corresponding to one nuclear magneton (with a Thomas factor of $\frac{1}{2}$) when a reduction to two-component wave functions is made. The anomalous part of the proton moment, μ_P-1 nuclear magnetons, is given in analogy to (31) by

$$\mu_P-1=-\pi g^2(D_\sigma{}^m+D_\sigma{}^n). \quad (37)$$

Here, $D_\sigma{}^m$ and $D_\sigma{}^n$ are the coefficients, exactly analogous to $C_\sigma{}^m$ and $C_\sigma{}^n$ in (29), in the expansion of the operators G^{+m} and G^{+n}, where $\mathfrak{M}$ in (19) is given by

$$\mathfrak{M}\equiv g^2G^+\equiv g^2(G^{+m}+G^{+n}). \quad (37a)$$

Discussions and interpretations will be considered in Section VI.

Charged Meson Contribution

The processes that contribute to the meson part G^{+m} of G^+ are precisely those under (1) in the tabulation of Section III, except that the P's and N's must be interchanged, as well as the signs of all the meson charges. It is easily seen from (7) that this change does not alter any of the matrix elements, except those involving H^e, where it always reverses the sign. Thus, as was to be expected intuitively, we obtain

$$G_{\mathbf{k}}{}^{+m}=-G_{\mathbf{k}}{}^m;\quad D_\sigma{}^m=-C_\sigma{}^m. \quad (38)$$

For the nucleon contribution, none of the processes (2) of Section III, or slight modifications, can be used. For instance, if the initial proton state changes to a virtual neutron plus positive meson state, the neutron cannot interact with the field. However, the following processes are now possible:

1. $P^+(\mathbf{p}_0)[H^i]\to N^+(\mathbf{b})+\mu^+(-\mathbf{k})$; $P^-(\mathbf{p}_0)[H^e]\to P^+(\mathbf{p})$; $N^+(\mathbf{b})+\mu^+(-\mathbf{k})[H^i]\to P^-(\mathbf{p}_0)$.
2. $P^-(\mathbf{p}_0)[H^i]\to N^+(\mathbf{a})+\mu^+(-\mathbf{k})$; $P^+(\mathbf{p}_0)[H^e]\to P^-(\mathbf{p})$; $N^+(\mathbf{a})+\mu^+(-\mathbf{k})[H^i]\to P^+(\mathbf{p})$.
3. (a) $P^+(\mathbf{p}_0)[H^e]\to P^+(\mathbf{p})$; $P^+(\mathbf{p})[H^i]\to N^+(\mathbf{a})+\mu^+(\mathbf{k})$; $N^+(\mathbf{a})+\mu^+(-\mathbf{k})[H^i]\to P^+(\mathbf{p})$.
   (b) $P^+(\mathbf{p}_0)[H^e]\to P^+(\mathbf{p})$; $N^-(\mathbf{b})[H^i]\to P^+(\mathbf{p}_0)+\mu^-(\mathbf{k})$; $P^+(\mathbf{p}_0)+\mu^-(\mathbf{k})[H^i]\to N^-(\mathbf{b})$.
4. (a) $P^-(\mathbf{p}_0)[H^e]\to P^+(\mathbf{p})$; $P^+(\mathbf{p}_0)[H^i]\to N^+(\mathbf{b})+\mu^+(-\mathbf{k})$; $N^+(\mathbf{b})+\mu^+(-\mathbf{k})[H^i]\to P^-(\mathbf{p}_0)$.
   (b) $P^-(\mathbf{p}_0)[H^e]\to P^+(\mathbf{p})$; $N^-(\mathbf{b})[H^i]\to P^-(\mathbf{p}_0)+\mu^-(\mathbf{k})$; $P^+(\mathbf{p}_0)+\mu^-(\mathbf{k})[H^i]\to N^-(\mathbf{b})$.
5. (a) $P^+(\mathbf{p}_0)[H^i]\to N^+(\mathbf{b})+\mu^+(-\mathbf{k})$; $N^+(\mathbf{b})+\mu^+(-\mathbf{k})[H^i]\to P^+(\mathbf{p}_0)$; $P^+(\mathbf{p}_0)[H^e]\to P^+(\mathbf{p})$.
   (b) $N^-(\mathbf{a})[H^i]\to P^+(\mathbf{p})+\mu^-(\mathbf{k})$; $P^+(\mathbf{p})+\mu^-(\mathbf{k})[H^i]\to N^-(\mathbf{a})$; $P^+(\mathbf{p}_0)[H^e]\to P^+(\mathbf{p})$.
6. (a) $P^-(\mathbf{p})[H^i]\to N^+(\mathbf{a})+\mu^+(-\mathbf{k})$; $N^+(\mathbf{a})+\mu^+(-\mathbf{k})[H^i]\to P^+(\mathbf{p})$; $P^+(\mathbf{p}_0)[H^e]\to P^-(\mathbf{p})$.
   (b) $N^-(\mathbf{a})[H^i]\to P^+(\mathbf{p})+\mu^-(\mathbf{k})$; $P^-(\mathbf{p})+\mu^-(\mathbf{k})[H^i]\to N^-(\mathbf{a})$; $P^+(\mathbf{p}_0)[H^e]\to P^-(\mathbf{p})$.

Note that

(*i*) All of the above processes in which an H^i transition follows an H^i transition are self-energy processes, so that the substitutions (17b) must be made in (17a).

(*ii*) The above list of processes is exhaustive, except that processes in which neither $P^+(\mathbf{p})$ nor $P^+(\mathbf{p}_0)$ are involved in the H^i transitions can also occur. These, however, cancel since the contribution of such a process to the third term of (17a) just compensates the contribution to the first two (for which (17b) must be used). These processes correspond to irrelevant vacuum fluctuations.

(*iii*) 4 (*a*) cancels half the contribution of 1; 6 (*a*) cancels half of 2.

In the usual manner, the above processes are found to result in a total contribution:

$$G_{\mathbf{k}}{}^{+n}=\frac{1}{2\epsilon}\left\{\frac{1+A'}{(E_a+\epsilon-E_p)^2}+\frac{1-A'}{(E_a+\epsilon+E_p)^2}\right\}\times\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M}{E_p}\right)+\frac{1}{2\epsilon}\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}_0+\beta M}{E_{p0}}\right)\times\left\{\frac{1+B}{(E_b+\epsilon-E_{p0})^2}+\frac{1-B'}{(E_b+\epsilon+E_{p0})^2}\right\}. \quad (39)$$

It is significant that the highest term in a power series expansion in k^{-1} is $+(1/k^3)$. Since the highest term of $G_{\mathbf{k}}{}^{+m}$ (see (38) and Section III) is $-(1/k^3)$, these two terms, which give rise separately to logarithmically divergent contributions, just cancel, so that $G^+=G^{+m}+G^{+n}$ converges as in the neutron case. One could then compute the correction to Rutherford scattering of protons and electrons, but this will not be done here.

The spin-dependent term, not only the first term of the power series expansion, is found from (39), by a brief calculation, to be zero exactly. Thus

$$D_\sigma{}^n=0\quad\text{(charged mesons)}. \quad (40)$$

This result is physically clear, since all the processes in the above table are just contributions from the renormalization of the nucleon wave function, which is evidently a pure numerical factor independent of spin. Renormalization of the nucleon wave function might change the Dirac moment, but this would also only appear on reduction to two-component wave functions (see Section VI).

Neutral Mesons; Symmetrical Theory

Although neutral mesons contribute to neither neutron moment nor neutron-electron interaction, they contribute to proton processes. There is no meson contribution due to neutral mesons, and the nucleon contribution arises from the following two sets of processes (again, canceling vacuum fluctuations are neglected):

(a) Every transition listed above under charged mesons, except that N must be changed to P, and μ^+, μ^- to μ^0. Since for neutral mesons the matrix element for emission or absorption of μ^0 by P is given by $-(2\pi/\epsilon L^3)^{\frac{1}{2}}g_0\alpha_1\alpha_2\alpha_3\beta$ (see (7)) as opposed to $-(2\pi/\epsilon L^3)^{\frac{1}{2}}g\alpha_1\alpha_2\alpha_3\beta$ for emission or absorption of charged mesons by neutrons or protons, this gives a contribution to G^+ of precisely $(g_0/g)^2$ times (39), or

$$G_{\mathbf{k}}^{0a}=(g_0/g)^2G_{\mathbf{k}}^{+n}=\tfrac{1}{2}G_{\mathbf{k}}^{+n};\quad D_\sigma^{0a}=0 \tag{41}$$

since $g_0=g/\sqrt{2}$ for symmetrical theory. By (40), this gives rise to no spin-dependent term, i.e., to no contribution to the neutron moment.

(b) Every transition listed under (2) in the tabulation of Section III is possible, except that N must be changed to P and μ^+, μ^- to μ^0. Just as above, this gives precisely $G_{\mathbf{k}}^n$, apart from a factor $(g_0/g)^2$ or $\frac{1}{2}$. Thus,

$$G_{\mathbf{k}}^{0b}=(g_0/g)^2G_{\mathbf{k}}^{n}=\tfrac{1}{2}G_{\mathbf{k}}^{n};\quad D_\sigma^{0b}=\tfrac{1}{2}C_\sigma^n. \tag{42}$$

Note that the highest term in the expansion of $G_{\mathbf{k}}^{0a}$ is $+(g_0/g)^2(1/k^3)$, while that of $G_{\mathbf{k}}^{0b}$ is $-(g_0/g)^2(1/k^3)$. The sum $G^0=G^{0a}+G^{0b}$ thus converges.

The final result for the magnetic moments in nuclear magnetons is thus:

$$\mu_N=-\pi g^2(C_\sigma^m+C_\sigma^n);\quad \mu_P-1=+\pi g^2C_\sigma^m \quad \text{(pure charged)} \tag{43c}$$

and

$$\mu_N=-\pi g^2(C_\sigma^m+C_\sigma^n);\quad \mu_P-1=+\pi g^2(C_\sigma^m-\tfrac{1}{2}C_\sigma^n) \quad \text{(symmetrical)} \tag{43s}$$

where C_σ^m and C_σ^n are given by (33). This agrees with results obtained recently by Case, as well as with the results of Luttinger.

For $(\mu/M)=0.154$, (43) becomes

$$\mu_N=-0.128g^2;\quad \mu_P-1=+0.0535g^2 \quad \text{(pure charged)} \tag{44c}$$

$$\mu_N=-0.128g^2;\quad \mu_P-1=+0.0164g^2 \quad \text{(symmetrical).} \tag{44s}$$

VI. NON-RELATIVISTIC METHODS, THE CHARGE CLOUD, AND COMPARISON WITH EXPERIMENT

In this section, the previous results, especially those pertaining to the neutron-electron interaction, will be discussed in a more qualitative and pictorial manner, and the connection with "non-relativistic" methods and experimental results will be given.

The Non-Relativistic Approximation. Comparison with Present Method

In the usual "non-relativistic" method (more properly, non-relativistic nucleon method), it is assumed that the nucleon is so heavy that it can be well localized, that negative-energy processes, which involve energy changes $\sim 2M$, contribute negligibly, and that the nucleon recoil is negligibly small. In fact, M is usually set equal to infinity at the outset and the nucleon is taken as localized at a point. This procedure gives zero for calculations with pseudoscalar coupling, as is evident from (27) or (28), and a slightly modified procedure will be discussed below in connection with the charge cloud. First, however, we will consider a different method that is less graphic, but simpler and more direct.

We will assume that the nucleon mass M, though finite, is so large that all quantities can be expanded in a series in M^{-1}, and only the term of lowest order kept. The scattering element $K_{\mathbf{pp}_0}$, as well as (29), (31), and (32), may still be used, though matrix elements and energy denominators will be replaced by the simpler "non-relativistic" expressions.

In general, processes involving negative-energy nucleons are thus excluded because of the large energy denominators, so only processes 1 (a α), 1 (a β), 1 (a γ), and 2 (a α) of Section III are involved. For pseudoscalar coupling, their contribution can easily be computed using the simpler expressions from the start, or, since we have already done the work, directly from the appropriate terms in the exact expression (22). In either case we easily obtain

$$G_{\mathbf{k}}^{m1}=\frac{2}{M^2}\frac{\mathbf{a}\cdot\mathbf{b}+i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{b}}{\epsilon_a\epsilon_b(\epsilon_a+\epsilon_b)};\quad G_{\mathbf{k}}^{n1}=-\frac{1}{M^2}\frac{k^2}{\epsilon^3}. \tag{45}$$

Note that G^{m1} and G^{n1} individually diverge quadratically, and the sum G^1 diverges logarithmically. However, since it is known that the exact expressions for G^1 converge, we can compute it approximately by integrating only up to a finite cut-off momentum. Since the integrand is large only in the neighborhood of $|\mathbf{k}|\sim\mu$, and the integral varies quite slowly with the cut-off momentum if it is taken several times as large as μ, this procedure will give a result close to the true value.

At this point, a phenomenon peculiar to pseudoscalar coupling must be mentioned. Matrix elements of H^i between positive-energy states are proportional to M^{-1}, while matrix elements joining positive- and negative-energy states are seen to be of order unity. Thus even though the energy denominators are large, the total contribution of the processes 1 (b) and 2 (b α) of Section III are seen to be of the same order in M as (45). In fact, these processes contribute

$$G_{\mathbf{k}}^{m2}=\frac{2}{M^2}\frac{1}{\epsilon_a+\epsilon_b};\quad G_{\mathbf{k}}^{n2}=-\frac{1}{M^2}\frac{1}{\epsilon} \tag{46}$$

to $G_{\mathbf{k}}^m$ and $G_{\mathbf{k}}^n$ respectively. These terms too give rise to a logarithmically divergent contribution that must be cut off.

If (45) and (46) are expanded as in (29), we find readily that

$$C_\Delta^{m1} = -\int_0^K \frac{dk}{2\pi^2}\left(\frac{k^2}{4\epsilon^3}\right)\left\{\frac{5}{3}+\frac{\mu^2}{6\epsilon^2}-\frac{5\mu^4}{6\epsilon^4}\right\} \qquad C_\Delta^{n1}=0$$

$$C_\Delta^{m2} = -\int_0^K \frac{dk}{2\pi^2}\left(\frac{k^2}{4\epsilon^3}\right)\left\{\frac{1}{3}+\frac{\mu^2}{6\epsilon^2}\right\} \qquad C_\Delta^{n2}=0$$

$$C_\Delta^{m} = -\int_0^K \frac{dk}{2\pi^2}\left(\frac{k^2}{2\epsilon^3}\right)\left\{1+\frac{1}{6}\frac{\mu^2}{\epsilon^2}-\frac{5}{12}\frac{\mu^4}{\epsilon^4}\right\}$$

$$C_\Delta^{n}=0 \quad (47)$$

where K is the cut-off momentum and ϵ as usual is $(k^2+\mu^2)^{\frac{1}{2}}$. Note that the above expressions for C_Δ^m and C_Δ^n (as well as C_σ^m and C_σ^n to be discussed below) are easily obtained from (34) (and (33)) by keeping only the first term in M. If K is much greater than μ, (47) is given very closely by

$$2\pi^2 C_\Delta^{m1} = -\frac{1}{4}\left\{\frac{5}{3}\log\frac{2K}{\mu}-\frac{3}{2}\right\};$$

$$2\pi^2 C_\Delta^{m2} = -\frac{1}{4}\left\{\frac{1}{3}\log\frac{2K}{\mu}-\frac{1}{2}\right\};$$

$$2\pi^2 C_\Delta^{m} = -\frac{1}{2}\left(\log\frac{2K}{\mu}-1\right); \quad C_\Delta^n=0. \quad (48)$$

K must be taken large enough to include the region in which the integrands are large, but not so large that the incorrect asymptotic behavior that causes the logarithmic divergence affects the integral appreciably. A value of K midway between μ and M, or about $M/2$, seems plausible. (We see incidentally from (48) that the C_Δ's are not very sensitive to changes of the meson mass.)

For $K=M/2$ and $\mu/M=0.154$, we obtain

$$2\pi^2 C_\Delta^{m1}\approx -0.37; \quad 2\pi^2 C_\Delta^{m2}\approx -0.025;$$
$$2\pi^2 C_\Delta^{m}\approx -0.40; \quad C_\Delta^n=0. \quad (49)$$

Comparing (49) and (35), we see that errors of about 20 or 30 percent arise from this approximation. We note too that negative-energy processes have little effect even for the singular case of pseudoscalar coupling. In Fig. 1, the exact integrands of C_Δ^m and C_Δ^n are compared with the non-relativistic expressions, for $\mu/M=0.154$. It is seen that even for this mass ratio, which is not very small, the agreement is fair for low momenta.

The spin-orbit interaction cannot be used to compute the magnetic moments by the above non-relativistic approximation, essentially because the spin-orbit interaction is a higher order effect. At first sight, this seems false, since (45) contains a spin-dependent term that converges and is easily evaluated. Moreover, it gives a value of the neutron moment in better agreement with experiment than the exact method, because it gives a zero nucleon contribution. (The exact method gives a nucleon contribution that will be seen below to be much too high.)

However, this agreement is illusory, since the substitution (25) makes (45) give zero for both meson and nucleon contributions to μ_N (cf. the second form of (33m)). Moreover, this substitution, which has no effect on the C_Δ's, is to be insisted upon, since otherwise, when (45) is expanded, it is seen to contain a spurious term in $\mathbf{s}^2$, which violates charge conservation. A "non-relativistic" method for computing the moments by considering interaction with a magnetic field is possible, but will not be attempted here. It is interesting to note that many phenomena noted above have counterparts in the calculation of the Lamb effect and the radiative corrections to the electron moment.

The Neutron-Electron Interaction for Other Types of Mesons

Since pseudoscalar and pseudovector couplings are equivalent (using (28)) for positive-energy processes in non-relativistic approximation, and to this order, as may be verified from (20), no negative-energy processes (including the contact interaction) contribute, we obtain for pseudovector coupling,

$$U_0V_e=(8\pi^2e^2f^2/\mu^2)(C_\Delta^{m1}+C_\Delta^{n1}). \quad (50)$$

We have used (28), (32), and (47). However, we have seen that cutting off is here unjustified except for the special choice of contact term mentioned above. Note that the C_Δ^{m2} term for pseudoscalar coupling is small, so that (28) may be applied to (36) without very great error to get the pseudovector value.

The non-relativistic calculation may very easily be carried through for the case of vector mesons. Here too, two couplings occur, characterized by constants f_v and g_v, which, however, are not equivalent. Moreover, both couplings have matrix elements of order unity for positive-energy transitions, so that no negative-energy processes need be investigated.

For charged vector mesons we have:

$$H^{em}=(e/2L^3)\int d\tau\, V(\mathbf{r})\sum_{\mathbf{kk'jj'}}(\epsilon_k\epsilon_{k'})^{-\frac{1}{2}}(\mathbf{j}\cdot\mathbf{j'})$$
$$\times\{(q_{+\mathbf{j'k'}}{}^\dagger q_{+\mathbf{jk}}-q_{-\mathbf{j'k'}}{}^\dagger q_{-\mathbf{jk}})(\epsilon_k+\epsilon_{k'})$$
$$\times\exp(i(\mathbf{k}-\mathbf{k'})\cdot\mathbf{r})+q_{-\mathbf{j'k'}}q_{+\mathbf{jk}}(\epsilon_k-\epsilon_{k'})$$
$$\times\exp(i(\mathbf{k}+\mathbf{k'})\cdot\mathbf{r})-q_{+\mathbf{j'k'}}{}^\dagger q_{-\mathbf{jk}}{}^\dagger(\epsilon_k-\epsilon_{k'})$$
$$\times\exp(-i(\mathbf{k}+\mathbf{k'})\cdot\mathbf{r})\}, \quad (51)$$

where $\mathbf{j}$ is the polarization unit vector and can have three possible values for each $\mathbf{k}$: one along $\mathbf{k}$ (longitudinal mesons), and two others perpendicular to $\mathbf{k}$ and to each other (transverse mesons). In non-relativistic approximation, the matrix elements for emission and absorption of a meson of momentum $\mathbf{k}$ are, disregarding q, a, and τ factors as well as the $u_{\mathbf{p}}$'s,*

$$(2\pi/\epsilon L^3)^{\frac{1}{2}}\{\pm i(g_v/\mu_v)k\} \qquad (52\parallel)$$

for longitudinal mesons and

$$(2\pi/\epsilon L^3)^{\frac{1}{2}}\{\pm i(f_v/\mu_v)\boldsymbol{\sigma}\cdot\mathbf{j}\times\mathbf{k}\} \qquad (52\perp)$$

for transverse, where the upper signs are for absorption.

Then from (51), (52), and (17a), we have immediately for the relevant processes of Section III the contributions:

$$\mathfrak{M}=\frac{g_v^2}{\mu_v^2}(G_{\parallel}{}^m+G_{\parallel}{}^n)+\frac{f_v^2}{\mu_v^2}(F_{\perp}{}^m+F_{\perp}{}^n); \qquad (53)$$

$$G_{\parallel}{}^m=\sum_{\mathbf{k}}\frac{2}{\epsilon_a\epsilon_b}\left\{\frac{\mathbf{a}\cdot\mathbf{b}}{\epsilon_a\epsilon_b}(\epsilon_a+\epsilon_b)-\frac{(\epsilon_a-\epsilon_b)(-\mathbf{a}\cdot\mathbf{b})}{\epsilon_a(\epsilon_a+\epsilon_b)}+\frac{(\epsilon_a-\epsilon_b)(-\mathbf{a}\cdot\mathbf{b})}{\epsilon_b(\epsilon_a+\epsilon_b)}\right\}=\sum_{\mathbf{k}}\frac{8\mathbf{a}\cdot\mathbf{b}}{\epsilon_a\epsilon_b(\epsilon_a+\epsilon_b)}$$

$$G_{\parallel}{}^n=\sum_{\mathbf{k}}\frac{2}{\epsilon^2}\left\{-\frac{2\epsilon\mathbf{k}\cdot\mathbf{k}}{\epsilon^2}\right\}=\sum_{\mathbf{k}}-\frac{4k^2}{\epsilon^3}, \qquad (54\parallel)$$

and similarly

$$F_{\perp}{}^m=\sum_{\mathbf{k}}\frac{8}{\epsilon_a\epsilon_b(\epsilon_a+\epsilon_b)}\sum_{\mathbf{j}_a\mathbf{j}_b}(\boldsymbol{\sigma}\cdot\mathbf{j}_a\times\mathbf{a})(\mathbf{j}_a\cdot\mathbf{j}_b)(\boldsymbol{\sigma}\cdot\mathbf{j}_b\times\mathbf{b})$$

$$=\sum_{\mathbf{k}}\frac{8}{\epsilon_a\epsilon_b(\epsilon_a+\epsilon_b)}[2\mathbf{a}\cdot\mathbf{b}-i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{b}]$$

$$F_{\perp}{}^n=\sum_{\mathbf{k}}\left(-\frac{4}{\epsilon^3}\right)\sum_{\mathbf{j}}(\boldsymbol{\sigma}\cdot\mathbf{j}\times\mathbf{k})(\mathbf{j}\cdot\mathbf{j})(\boldsymbol{\sigma}\cdot\mathbf{j}\times\mathbf{k})$$

$$=\sum_{\mathbf{k}}-\frac{4}{\epsilon^3}[2k^2]. \qquad (54\perp)$$

Comparing (53) and (54) with (45) and (47), we see that

$$U_0V_e=(8\pi e^2/\mu_v^2)(C_\Delta{}^{m1}+C_\Delta{}^{n1})(g_v^2+2f_v^2). \qquad (55)$$

Note that here, the legitimacy of cutting off has not been investigated. Assuming it possible, we have for an arbitrary mixture of pseudoscalar and vector mesons (neglecting $C_\Delta{}^{m2}$ and other small effects, as well as the change in the C_Δ's due to the difference between μ and μ_v):

$$U_0V_e=8\pi e^2(C_\Delta{}^{m1}+C_\Delta{}^{n1})\times\left[\frac{1}{\mu_v^2}(g_v^2+2f_v^2)+\frac{1}{\mu^2}\left(f-\frac{\mu g}{2M}\right)^2\right]. \qquad (56)$$

In fact, if we adjust (56) to agree with the exact expression (36) for $f=f_v=g_v=0$ and $\mu/M=0.154$, we obtain

$$U_0=57.3\left[\left(f-\frac{\mu g}{2M}\right)^2+\left(\frac{\mu}{\mu_v}\right)^2(g_v^2+2f_v^2)\right]\text{ kev.} \qquad (57)$$

* The u's for the intermediate states may be omitted, since they are effectively two-component wave functions and are summed over both spin indices. Thus $\sum_{p'\text{ spins}}(u_{\mathbf{p}}{}^*\mathcal{O}_1u_{\mathbf{p}'})\times(u_{\mathbf{p}'}{}^*\mathcal{O}_2u_{\mathbf{p}_0})=(u_{\mathbf{p}}{}^*\mathcal{O}_1\mathcal{O}_2u_{\mathbf{p}_0})$ for any operators $\mathcal{O}_1$ and $\mathcal{O}_2$ involving $\boldsymbol{\sigma}$ and unity. Of course, the $u_{\mathbf{p}}{}^*$ and $u_{\mathbf{p}_0}$ must be inserted.

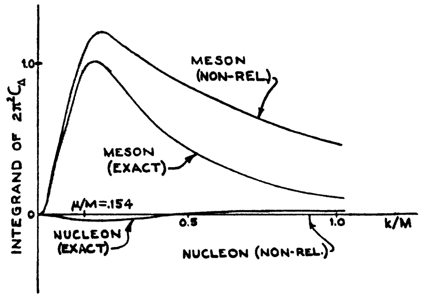

FIG. 1. Comparison of exact (Eq. (34)) and non-relativistic (Eq. (47)) integrands of $2\pi^2C_\Delta$. Note that minus the integrand is drawn. For $C_\Delta{}^n$, multiply abscissas by 2 and divide ordinates by 2.

The Charge Cloud. Charge Renormalization

If the spin-independent parts of the scattering matrix element $K_{\mathbf{pp}_0}$ can be interpreted as resulting from the interaction of a charge cloud $\rho(|\mathbf{r}-\mathbf{r}_0|)$ around a nucleon at $\mathbf{r}_0$, and an electric field $V(\mathbf{r})$, we would have

$$K_{\mathbf{pp}_0}=\int d\tau_0(L^{-\frac{3}{2}}u_p{}^*\exp(-i\mathbf{p}\cdot\mathbf{r}_0))\times\left\{\int d\tau V(\mathbf{r})\rho(|\mathbf{r}-\mathbf{r}_0|)\right\}\times(L^{-\frac{3}{2}}u_{\mathbf{p}_0}\exp(i\mathbf{p}_0\cdot\mathbf{r}_0)). \qquad (58)$$

Comparing this with (19), we see that we must have

$$\int\rho(|\mathbf{r}-\mathbf{r}_0|)\exp(i\boldsymbol{\Delta}\cdot\mathbf{r}_0)d\tau_0=(-e\pi/2L^3)\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})\mathfrak{M}, \qquad (58a)$$

where $\mathfrak{M}$ is a function of $\mathbf{p}_0$ and $\boldsymbol{\Delta}$ only. By Fourier's integral theorem:

$$\rho(\mathbf{r}-\mathbf{r}_0)=\int\exp(i\boldsymbol{\Delta}\cdot(\mathbf{r}-\mathbf{r}_0))\frac{d^3\Delta}{(2\pi)^3}\left(-\frac{e\pi}{2L^3}\right)\mathfrak{M}$$

$$=\left(-\frac{e\pi}{2L^6}\right)\sum\nolimits_\Delta\exp(i\boldsymbol{\Delta}\cdot(\mathbf{r}-\mathbf{r}_0))\mathfrak{M}(\boldsymbol{\Delta},\mathbf{p}_0), \qquad (59)$$

so that ρ is essentially the Fourier transform of $\mathfrak{M}$.

We therefore see immediately that a constant term in $\mathfrak{M}$ corresponds to a δ-function charge distribution; i.e., a point charge at the position of the nucleon. It follows, for instance, that the constant term in G^m (Eq. (29)) corresponds to an alteration of the nucleon charge by g^2 times a logarithmically divergent integral. Since G^n gives an equal and opposite alteration, the nucleon charge is entirely unchanged. The divergent point charges are thus not to be looked on in any sense as a renormalization of nucleon charge due to the meson field. This circumstance, guaranteed by charge conservation, is necessary if proton and electron charges are to be of the same magnitude. However, the fact that G^m, etc., give rise to a (divergent) point charge plus a convergent continuous distribution shows that it is sensible to separate meson and nucleon contributions. (It might have been objected that since G^m and G^n individually diverge, only their sum has any significance and a separation into $C_\Delta{}^m$ and $C_\Delta{}^n$ is inadmissible.)

Equation (59) can be rewritten, with $\mathbf{R}=\mathbf{r}-\mathbf{r}_0$, as

$$\rho(R)=\left(-\frac{e\pi}{2L^6}\right)\sum_\Delta\sum_\mathbf{k} M_\mathbf{k}\exp(i\boldsymbol{\Delta}\cdot\mathbf{R})$$

$$=\left(-\frac{e\pi}{2L^6}\right)\sum_\mathbf{a}\sum_\mathbf{b} M_\mathbf{k}\exp(i(\mathbf{b}-\mathbf{a})\cdot\mathbf{R}). \quad (60)$$

The usual non-relativistic method of obtaining the charge density is to evaluate the expectation value of the charge density operator $-ie(\Pi\Psi-\Pi^\dagger\Psi^\dagger)$ (see Eq. (3)) for an infinitely heavy nucleon at rest at the origin. Using perturbation theory, precisely the value (60) is obtained if the non-relativistic $\mathfrak{M}$ is used. The double-summation in (60) is there due to the neglect of nucleon recoil, and hence momentum conservation, so that $\mathbf{a}$ and $\mathbf{b}$ are independent. If this method is to be applied to pseudoscalar coupling, one need only remember that g/M is finite, but that for other purposes M may be taken infinite, and that a contribution from negative-energy processes is being omitted. (59) may also be looked on as a transformation back to coordinate representation, or as resulting from a summation of plane-wave matrix elements $K_{\mathbf{pp}_0}$ to give the matrix element for a localized neutron.

The charge density will now be evaluated in the non-relativistic approximation for a mixture of pseudoscalar and vector mesons (neglecting the small negative-energy contribution for pseudoscalar coupling). We have then from (60), (45), and (54):

$$\rho(R)=\left(-\frac{e\pi}{2L^6\mu^2}\right)\left[\left(f-\frac{\mu g}{2M}\right)^2+(g_v^2+2f_v^2)\right]$$

$$\times\sum_\mathbf{k}\sum_\Delta\left\{\frac{8\mathbf{a}\cdot\mathbf{b}}{\epsilon_a\epsilon_b(\epsilon_a+\epsilon_b)}-\frac{4k^2}{\epsilon^3}\right\}\exp(i\boldsymbol{\Delta}\cdot\mathbf{R}). \quad (61)$$

The second term in this summation—the nucleon contribution—is independent of $\boldsymbol{\Delta}$ and corresponds entirely to a δ-function charge. This is due to the fact that the nucleon charge is smeared out over approximately the neutron Compton wave-length (divided by 2π), which is here assumed zero (see Fig. 1). We will accordingly omit this term, but remember that it cancels the δ-functions arising from the meson portion and causes the total charge to be zero. The integrals in the meson contribution may with this understanding be computed in the sense of summability, which has the effect of neglecting such singularities. We then obtain, integrating first over the angles of $\mathbf{a}$ and $\mathbf{b}$:

$$\rho(R)=\frac{-e}{\pi^3\mu^2}\left[\left(f-\frac{\mu g}{2M}\right)^2+(g_v^2+2f_v^2)\right]$$

$$\times\int_0^\infty\int_0^\infty dadba^2b^2\left(\frac{\cos aR}{R}-\frac{\sin aR}{aR^2}\right)$$

$$\times\left(\frac{\cos bR}{R}-\frac{\sin bR}{bR^2}\right)\frac{1}{\epsilon_a\epsilon_b(\epsilon_a+\epsilon_b)}$$

$$=-\frac{2e}{\pi^3\mu^2}\left[\quad\right]$$

$$\times\int_0^\infty\int_0^\infty dadba^2b^2\left(\frac{\cos aR}{R}-\frac{\sin aR}{aR^2}\right)$$

$$\times\left(\frac{\cos bR}{R}-\frac{\sin bR}{bR^2}\right)\frac{1}{\epsilon_b(a^2-b^2)}$$

$$=\frac{e}{\pi^2\mu^2}\left[\quad\right]\frac{1}{R^3}\int_0^\infty\frac{b^2db}{(\mu^2+b^2)^{\frac{1}{2}}}$$

$$\times\left[\cos 2bR+\frac{1}{2}\left(bR-\frac{1}{bR}\right)\sin 2bR\right]$$

$$=\frac{e}{2\pi}\left[\quad\right]\frac{1}{R^3}\left\{\frac{5i}{4}H_0^{(1)}(2i\mu R)\right.$$

$$\left.-\frac{5}{4\mu R}H_1^{(1)}(2i\mu R)-\frac{\mu R}{2}H_1^{(1)}(2i\mu R)\right\}. \quad (62)$$

Though this converges at any finite R, it has a $1/R^5$ singularity, and should be cut off at a value of R that gives a value for the neutron-electron interaction near that given by (57). For the total meson charge outside the cut-off radius R_c, we obtain

$$\int_{>R_c}\rho(R)d\tau=-\frac{e}{2}\left[\quad\right]$$

$$\times\left\{\frac{-5H_1^{(1)}(2iR_c\mu)}{2R_c\mu}+iH_0^{(1)}(2iR_c\mu)\right\} \quad (63)$$

which diverges $\sim 1/R_c^2$ as $R_c\rightarrow 0$.

The volume integral of the electrostatic potential U_e due to a charge distributed with density (62) outside of R_c, and with a charge at the origin sufficient to make the total charge zero is

$$\int U_e d\tau = -(4\pi)^2\int_{R_c}^{\infty} RdR\int_{R}^{\infty} r^2\rho dr - (4\pi)^2\int_{0}^{R_c} RdR\int_{R_c}^{\infty} r^2\rho dr + \frac{(4\pi)^2}{3}\int_{R_c}^{\infty}\rho R^4 dR$$

$$= -(4\pi/6)\int_{R_c}^{\infty}\rho R^4 dR$$

$$= e\pi/6\mu^2[\quad]\{5iH_0^{(1)}(2iR_c\mu) + \tfrac{1}{2}(2R_c\mu)^2 iH_0^{(1)}(2iR_c\mu) - 7/2(2R_c\mu)H_1^{(1)}(2iR_c\mu)\}. \quad (64)$$

(The three terms in the first line are due respectively to charges inside R for $R>R_c$, to the bare nucleon for $R<R_c$, and to charges outside R for all R.) The volume integral of the potential of a neutron in the field of an electron is given by $U_0V_e=-e\int U_e d\tau$. For agreement with (56) or (57) the expression in curly brackets in (64) must be about unity, so that $2R_c\mu$ must be slightly greater than 1. The total dissociated meson charge given by (63) is correspondingly about $\frac{1}{10}$ or $\frac{1}{5}$ of an electronic charge, using a value of about 0.2 for the factor in square brackets, as will be discussed in the next paragraph. The range of this interaction is about R_c, or a little more than $\sim 1/(2\mu)$, or about $\frac{1}{4}$ of the classical electron radius for $\mu=282$ e.m. Figure 2 is an attempt to depict very roughly the meson charge cloud and neutron-electron potential as a function of the distance from the neutron. In it, the cut-off distributions have been smoothed out to simulate the result that would have been obtained from an exact calculation.

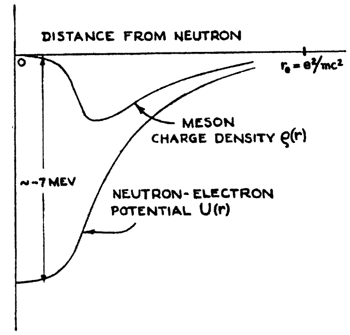

FIG. 2. Neutron-electron potential and meson charge cloud around neutron (very rough).

VII. COMPARISON WITH EXPERIMENT AND CONCLUSIONS

To compare the results (44), (36), and (57) with experiment, the coupling constants will be evaluated by comparison with the known nuclear forces. To avoid questions associated with the highly singular tensor force, the ^1S neutron-proton potential will be examined, since it does not involve the tensor force. For a mixture of pseudoscalar and vector mesons of mass μ, the theory gives a singlet potential

$$-\left[\left(f-\frac{\mu g}{2M}\right)^2+2f_v^2-g_v^2\right]r^{-1}\exp(-\mu r)$$

for symmetric theory

and

$$-2\left[\left(f-\frac{\mu g}{2M}\right)^2+2f_v^2-g_v^2\right]r^{-1}\exp(-\mu r)$$

for pure charged theory.

All non-static terms have been neglected in these expressions, so an unknown—perhaps quite large—error has been introduced. On the other hand, the experimental data can be fitted to a potential $-0.239r^{-1}\exp(-\mu r)$ calculated for a singlet scattering length of $-2.375\cdot 10^{-10}$ cm and $\mu=282$ e.m. from Eq. (35), of Rosenfeld's book.[16]

We first assume $f=f_v=g_v=0$. Then from the above equations, $(\mu g/2M)^2=0.239$ or $g^2=40.3$ for symmetrical, and similarly $g^2=20.1$ for pure charged theory. We then have from (36) and (44)

$$U_0=-13.7\text{ kev};\ \mu_N=-5.15;\ \mu_P-1=+0.66 \quad (65s)$$

for symmetric theory, and

$$U_0=-6.9\text{ kev};\quad \mu_N=-2.56;\quad \mu_P-1=1.08 \quad (65c)$$

for pure charged theory, as compared to the experimental values

$$U_0\sim-\text{several kev};\quad \mu_N=-1.91;\quad \mu_P-1=1.79. \quad (65e)$$

It should be remembered that pure charged theory gives unsatisfactory nuclear forces even apart from the tensor force difficulty, so that (65c) is not to be taken too seriously.

The most important conclusion to be drawn from (65) is that pseudoscalar-coupled pseudoscalar meson theory gives results that are convergent and correct in sign and order of magnitude. Quantitatively, however, the magnetic moments depart con-

[16] L. Rosenfeld, *Nuclear Forces I* (Interscience Publishers, Inc., New York, 1948), p. 88.

siderably from the experimental values. This is not due merely to the coupling constant, since for the ratio $-(\mu_P-1)/\mu_N$, which is independent of g^2, we have

$$-(\mu_P-1)/\mu_N=0.128 \text{ (sym.)};\ 0.418 \text{ (charged)};\ 0.936 \text{ (exp.)} \quad (66)$$

respectively for symmetrical theory, pure charged theory, and experiment. As mentioned before, quantitative disagreements due to higher order terms, other meson masses, etc., are to be expected, but the disagreement of $-(\mu_P-1)/\mu_N$ may indicate a more serious fault. The difficulty stems from the fact that if equations such as (43s) or (43c) are to be fitted to experiment, the nucleon contribution to the moments must be very small, while in fact, (35) shows it to be 40 percent greater than the meson contribution to the neutron moment. If the nucleon contribution is omitted altogether, we find for symmetrical theory $\mu_N=-2.16$, $\mu_P-1=2.16$, in good agreement with experiment even for the present coupling constant.

The large size of the nucleon contribution may also seem surprising in the light of simple qualitative considerations.[1] For instance, if a neutron of $S_z=+\frac{1}{2}$ is dissociated into a negative meson of $L=1$ and $L_z=+1$ and a proton of $S_z=-\frac{1}{2}$ for about $\frac{1}{10}$ or $\frac{1}{5}$ of the time (in accord with the total meson charge estimated in the last paragraph) we would expect a negative contribution of about $\frac{1}{10}$ to $\frac{1}{5}$ from the proton, and a further negative contribution from the meson about M/μ times as large (since the meson magneton is M/μ times the proton magneton). Instead, we find a nucleon contribution even larger than the meson contribution (and negative, so it does not arise from transitions involving mesons of $L=0$) for the neutron moment, and none at all for the proton moment.

To understand these results, we note that the nucleon contribution, at least to the spin-orbit interaction, does not seem to arise in the manner suggested above. In fact, the spin-orbit interaction in the ordinary Dirac case arises because the electrical matrix element

$$(q/L^3)\int d\tau\, V(\mathbf{r})\exp(i\mathbf{\Delta}\cdot\mathbf{r})(u_{\mathbf{p}}^*u_{\mathbf{p}0}),$$

becomes

$$\sim(q/L^3)\left\{\int d\tau\, V(\mathbf{r})\exp(i\mathbf{\Delta}\cdot\mathbf{r})\right\}\times\left(u_{\mathbf{p}}^{1*}\left[1+\frac{\mathbf{p}\cdot\mathbf{p}_0}{4M^2}+\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{p}_0}{4M^2}\right]u_{\mathbf{p}0}^{1}\right)$$

when reduction to 2-component wave functions $u_{\mathbf{p}}^1$ etc., is made. Here q is the charge of the particle. Thus, since the charge of the proton is reduced (by an infinite amount!) due to "dissociation," the expected decrease of the Dirac moment does appear on going to the 2-component functions. However, the meson charge results in a term in

$$\int d\tau\, V(\mathbf{r})\exp(i\mathbf{\Delta}\cdot\mathbf{r})(u_{\mathbf{p}}^*u_{\mathbf{p}0})$$

in the Hamiltonian that just compensates the reduction of proton charge. The case of the neutron is exactly similar. Charge conservation thus ensures that the spin-orbit interaction does not change on reduction to 2-component functions. The nucleon contribution is thus to be thought of as arising from sharing of the orbital motion by the nucleon, which also explains the zero result for the proton moment (orbital motion of the neutron gives no contribution). Though this too may be thought to be small, we have seen that high momentum ($\sim M$) contributions are important, and here relativistic mass increases makes the sharing large. It should also be remembered that higher radiative corrections, which, judging from the "dissociation time," are $\frac{1}{10}$ to $\frac{1}{5}$ or even larger, have been entirely neglected. For vector mesons, which possess an intrinsic angular momentum, the situation may be entirely different.

From (57), noting that for symmetric theory

$$\left[\left(f-\frac{\mu g}{2M}\right)^2+2f_v^2-g_v^2\right]=0.239$$

for a mixture of vector and pseudoscalar mesons, we see that unless g_v is very large, U_0 is still about -15 kev. Thus, if they converge, the other couplings do not give a substantially different neutron-electron interaction.

To summarize, the results of the above investigation seem sufficiently positive to warrant examination of other couplings and higher corrections. However, use of the newer methods seems advisable for more secure treatment of divergences.

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APPENDIX

If $u_{\mathbf{p}}$ and $u_{\mathbf{p}0}$ are Dirac 4-component columns corresponding to positive energies E_p and E_{p0}, then for any Dirac operator $\mathfrak{D}$ an operator $\mathfrak{D}'$ of the form const$+$(**const**$\cdot\boldsymbol{\sigma}$) can be found so $(u_{\mathbf{p}}^*\mathfrak{D}u_{\mathbf{p}0})=(u_{\mathbf{p}}^*\mathfrak{D}'u_{\mathbf{p}0})$. The operators and their replacements (indicated by $\rightarrow$) are as follows: (Of course, $1\rightarrow1$ and $\boldsymbol{\sigma}\rightarrow\boldsymbol{\sigma}$.)

$$\beta\rightarrow\frac{E_pE_{p0}+M^2-\mathbf{p}\cdot\mathbf{p}_0-i\boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{p}_0}{M(E_p+E_{p0})}$$

$$\boldsymbol{\alpha}\cdot\mathbf{c}\rightarrow\frac{\mathbf{c}\cdot\mathbf{s}+i\boldsymbol{\sigma}\cdot\mathbf{c}\times\boldsymbol{\Delta}}{E_{p0}+E_p}$$

$$\beta\boldsymbol{\alpha}\cdot\mathbf{c}\rightarrow\frac{\mathbf{c}\cdot\boldsymbol{\Delta}+i\boldsymbol{\sigma}\cdot\mathbf{c}\times\mathbf{s}}{2M}-\left(\frac{E_{p0}-E_p}{E_{p0}+E_p}\right)\frac{\mathbf{c}\cdot\mathbf{s}+i\boldsymbol{\sigma}\cdot\mathbf{c}\times\boldsymbol{\Delta}}{2M}$$

$$\beta\boldsymbol{\sigma}\cdot\mathbf{c}\rightarrow-\frac{E_pE_{p0}+M^2}{M(E_p+E_{p0})}-\frac{2i\mathbf{c}\cdot\mathbf{s}\times\boldsymbol{\Delta}+(\mathbf{s}^2-\boldsymbol{\Delta}^2)\boldsymbol{\sigma}\cdot\mathbf{c}+2(\mathbf{c}\cdot\boldsymbol{\Delta})(\boldsymbol{\sigma}\cdot\boldsymbol{\Delta})-2(\mathbf{c}\cdot\mathbf{s})(\boldsymbol{\sigma}\cdot\mathbf{s})}{4M(E_p+E_{p0})}$$

$$\alpha_1\alpha_2\alpha_3\rightarrow\frac{i\boldsymbol{\sigma}\cdot\mathbf{s}}{E_p+E_{p0}}$$

$$\alpha_1\alpha_2\alpha_3\beta\rightarrow-\frac{i\boldsymbol{\sigma}\cdot\boldsymbol{\Delta}}{2M}+\left(\frac{E_{p0}-E_p}{E_{p0}+E_p}\right)\frac{i\boldsymbol{\sigma}\cdot\mathbf{s}}{2M}$$

where $\mathbf{s}=\mathbf{p}_0+\mathbf{p}$, $\boldsymbol{\Delta}=\mathbf{p}_0-\mathbf{p}$, and $\mathbf{c}$ is an arbitrary ordinary vector.

Note on Meson Contribution to the Lamb Shift

An estimate of the shift of the 2s hydrogen level caused by the meson charge cloud around the proton is of some interest. As may be seen from the non-relativistic approximation or from (39) directly, the nucleon contributions are not very important, while the meson contribution is by (38) just the negative of that for the neutron case. Thus, in analogy to (30), we have for the proton

$$K_{\mathbf{pp}0}\approx+\frac{\pi eg^2}{2L^3}\int d\tau V(\mathbf{r})\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})\frac{\boldsymbol{\Delta}^2}{M^2}C_\Delta{}^m=-\frac{\pi eg^2}{2L^3}\int d\tau V(\mathbf{r})\nabla^2\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})\frac{C_\Delta{}^m}{M^2}.$$

By Green's theorem, this is, apart from a surface integral,

$$(-\pi eg^2/2L^3)\int d\tau\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})\nabla^2V(\mathbf{r})(C_\Delta{}^m/M^2).$$

Now for a proton localized at the origin, the product of the initial and final wave functions is $\delta(\mathbf{r})$ rather than $L^{-3}\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})$. Also, $\nabla^2V=-4\pi\rho(\mathbf{r})=+4\pi e\psi^*\psi$, where $\psi(\mathbf{r})$ is the electron wave function, so that the change in energy due to smearing of the proton charge is approximately

$$-(\pi eg^2C_\Delta{}^m/2M^2)\int d\tau\delta(\mathbf{r})\cdot4\pi e\psi^*\psi=-(2\pi^2C_\Delta{}^me^2g^2/M^2)|\psi(0)|^2.$$

For an s electron, $|\psi(0)|^2=(\pi n^3a_0{}^3)^{-1}$ ($a_0=$Bohr radius). Thus with $2\pi^2C_\Delta{}^m\approx-0.5$ and $g^2\approx40$ we obtain $+0.08$ Mc for the shift of the 2s level, considerably less than the present experimental error.