On the Transmission Coefficient of a Circular Aperture

C. J. Bouwkamp
Laboratorium voor Wetenschappelijk Onderzoek, N. V. Philips' Gloeilampenfabrieken, Eindhoven, Nederland
February 28, 1949

Recently Levine and Schwinger[1] developed a theory for approximately solving the problem of diffraction of a scalar plane wave by an aperture in an infinite plane screen. In particular they obtained for the transmission coefficient of a circular aperture, in the case of normal incidence, the approximate expression

$$t^{(2)}=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.047823(ka)^4+\cdots], \quad (1)$$

which was then compared with my previous result,[2]

$$t=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.027427(ka)^4-0.004393(ka)^6+\cdots], \quad (2)$$

obtained by the method of spheroidal wave functions.

For a long time, however, I doubted the correctness of the last two numerical coefficients in (2). In fact both are wrong, and in view of the interesting results of Levine and Schwinger, it may be worth while to mention the correct expression. This reads

$$t=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.0507755(ka)^4+0.0002613(ka)^6+\cdots]. \quad (3)$$

It is to be observed that the approximation of (1) to (3) is far better than to (2). On the other hand, however, it is remarkable that the first-order approximation of Levine and Schwinger, *viz.*,

$$t^{(1)}=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.049061(ka)^4+\cdots], \quad (4)$$

is closer to the correct expression (3) than is the second-order approximation (1).

I derived Eq. (3) in two completely different ways. First, by the method of spheroidal wave functions as before. As I have shown,[2] one has, up to and including terms of the order of $(ka)^{10}$,

$$t=(8/27\pi^2)(ka)^4[b_1{}^4X_1{}^2(1)][\{X_1'(0)\}^4+4(ka)^6b_1{}^4/81\pi^2]^{-1}, \quad (5)$$

in which[3]

$$b_1=1-(3/2\cdot5^4\cdot7)(ka)^4+(2/3\cdot5^6\cdot7)(ka)^6+\cdots,$$
$$X_1(1)=1+(1/5^2)(ka)^2-(3/2\cdot5^3\cdot7^2)(ka)^4-(641/2\cdot3^4\cdot5^5\cdot7^2)(ka)^6+\cdots,$$
$$X_1'(0)=1-(3/2\cdot5^2)(ka)^2+(51/2^3\cdot5^3\cdot7^2)(ka)^4+(6641/2^4\cdot3^4\cdot5^5\cdot7^2)(ka)^6+\cdots.$$

Inserting these expressions in (5), and expanding in powers of ka, we obtain (3). The exact values of the coefficients are $311/5^3\cdot7^2=0.0507755$, $2612/3^4\cdot5^3\cdot7^2-4/81\pi^2=0.0002613$.

In the second method, we do not use properties of spheroidal wave functions; it proceeds rather along the lines of Levine and Schwinger's paper cited. I shall briefly indicate the new approach, details of which will be published elsewhere.

Let $\exp(ikz)/ik$ represent a plane wave impinging normally upon the screen. Then the resultant wave field in the aperture may be determined from an integro-differential equation.[1] This aperture field can be developed in ascending powers of ika, *viz.*,

$$\varphi=\sum_{n=0}^{\infty}\varphi_n(ika)^n,$$

in which the coefficients depend on ρ, the distance to the center of the aperture. Setting

$$\varphi_n=\sum_{m=0}^{\infty}B_{n,m}P_{2m+1}[1-(\rho/a)^2]^{\frac12},$$

in which P denotes the Legendre polynomial, while

$$B_{n,m}=0\begin{cases}m>n/2, & n \text{ even},\\ m>(n-3)/2, & n \text{ odd},\end{cases}$$

I obtained the recurrence relation

$$B_{n,m}=(-1)^{m+1}(m+\tfrac34)\frac{\Gamma(m+1)}{\Gamma(m+\frac32)}\sum_{\sigma=0}^{n-2}\sum_{\tau=0}^{\infty}(-1)^{\tau}\frac{\Gamma(\tau+\frac32)}{\Gamma(\tau+1)}$$
$$\times\frac{\Gamma[-(\sigma/2)+(n-1/2)]\Gamma[-(\sigma/2)+(n+1)/2]}{\Gamma[-(\sigma/2)+\tau+(n+2)/2-m]\times\Gamma[-(\sigma/2)-\tau+(n+2)/2+m]\times\Gamma[-(\sigma/2)+\tau+(n+5)/2+m]\times\Gamma[-(\sigma/2)-\tau+(n-1)/2-m]}B_{\sigma,\tau},$$

from which the coefficients $B_{n,m}$ can be obtained successively, in virtue of

$$B_{0,0}=-2/\pi;\quad B_{0,m}=0(m>0);\quad B_{1,m}=0(m\geq0).$$

The transmission coefficient then becomes

$$t=Re\left[2ika\int_0^a\varphi(\rho)\rho d\rho\right]=\frac23\sum_{n=0}^{\infty}(-1)^nB_{2n+3,0}(ka)^{2n+4},$$

which leads to (3) because

$$B_{3,0}=4/9\pi^2;\quad B_{5,0}=-32/225\pi^2;\quad B_{7,0}=1244/55125\pi^2;$$
$$B_{9,0}=16/729\pi^4-10448/4465125\pi^2.$$

[1] H. Levine and J. Schwinger, Phys. Rev. 74, 958 (1948).
[2] C. J. Bouwkamp, Thesis, Groningen (1941).
[3] J. Math. Phys. 26, 79 (1947).

On the Transmission Coefficient of a Circular Aperture

Harold Levine and Julian Schwinger
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
March 25, 1949

In connection with the accompanying letter of Bouwkamp,[1] a few remarks appear desirable as a sequel to our paper.[2] A small numerical error in one term of an expansion for the second variational approximation to the transmission coefficient requires correction; the first two variational approximations, with an additional term in each beyond those given previously, are

$$t^{(1)}=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.0490612(ka)^4-0.0008054(ka)^6+\cdots],$$
$$t^{(2)}=(8/27\pi^2)(ka)^4[1+0.32(ka)^2+0.0507755(ka)^4+0.0002613(ka)^6+\cdots].$$

The latter result is identical with the exact expression (3) of Bouwkamp, for all powers of ka retained.

A statement of the accuracy incorporated in any transmission coefficient $t^{(N)}$ of the foregoing sequence is readily obtained. By virtue of the stationary property of the coefficient, its deviations are proportional to the square of the error in the aperture wave function. Thus, with a wave function of the form $(1-(\rho^2/a^2))^{\frac12}$ (or the first-order Legendre polynomial with argument $(1-(\rho^2/a^2))^{\frac12}$), the expansion for $t^{(1)}$ is in error by a term of relative order $(ka)^4$, since the error in the wave function is of relative order $(ka)^2$ (see Bouwkamp). Similarly, with a wave function of the form

$$A_1(1-(\rho^2/a^2))^{\frac12}+A_2(1-(\rho^2/a^2))^{\frac32}$$

(or a linear combination of the first- and third-order Legendre polynomials with argument as before), the expansion for $t^{(2)}$ is in error by a term of relative order $(ka)^8$, since the error in the wave function is of relative order $(ka)^4$. In general, a wave function constructed from a linear combination of the first N odd Legendre polynomials with argument $(1-(\rho^2/a^2))^{\frac12}$ yields a variational transmission coefficient $t^{(N)}$ whose expansion is exact through terms of relative order $(ka)^{4N-2}$. Furthermore, the stationary property of the transmission coefficient leads one to expect that the inexact numerical factor of the term $(ka)^{4N}$ will be in close accord with the correct value.

[1] C. J. Bouwkamp, this issue.
[2] H. Levine and J. Schwinger, Phys. Rev. **74**, 958 (1948).

Two-Component Wave Equations

HERBERT JEHLE
*Brace Laboratory of Physics, University of Nebraska, Lincoln, Nebraska**
April 4, 1949

RECENTLY some interest has come up in two-component wave equations. As is well known, those wave equations, in contrast to the four-component Dirac equation, can only be made covariant with respect to the Lorentz group, not including covariance with respect to reflections.

The following antilinear two-component equation presents some interest,

$$\gamma^\kappa(\partial/\partial x^\kappa - i\varphi_\kappa)\psi = \mu\psi^*, \tag{1}$$

(ψ^* = conjugate complex of ψ), where

$$\tfrac{1}{2}(\gamma^{\kappa*}\gamma^\lambda + \gamma^{\lambda*}\gamma^\kappa) = -g^{\kappa\lambda}1, \tag{2}$$

with $g^{00} = +1$, $g^{11} = g^{22} = g^{33} = -1$. In the case $\varphi_\kappa = 0$, we obtain by iteration the ordinary linear second-order wave equation. Equation (2) can be satisfied with

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

With the Bargmann operator

$$\alpha = -\tilde{\alpha} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

we can form a real current vector

$$s^\kappa = \psi^\dagger \alpha \gamma^\kappa \psi \tag{5}$$

($\psi^\dagger$ = Hermitian conjugate).

Let us denote a Lorentz transformation by

$$\partial/\partial x^{\lambda'} = a_\lambda{}^\kappa \partial/\partial x^\kappa, \quad s^\lambda = a_\kappa{}^\lambda s^{\kappa'}. \tag{6}$$

Lorentz covariance of Eqs. (1) and (5) can be formulated in the following way. Let us take α and γ^κ as a *fixed* set of matrices (3) and (4) and transform the other quantities so that (1) and (5) go over into

$$\gamma^\kappa(\partial/\partial x^{\kappa'} - i\varphi_\kappa')\psi' = \mu\psi^{*\prime}, \tag{7}$$

$$s^{\kappa'} = \psi^{\dagger\prime}\alpha\gamma^\kappa\psi'. \tag{8}$$

If ψ transforms under the Lorentz transformation (6) as

$$\psi' = S^{-1}\psi, \tag{9}$$

the covariance of Eqs. (1) and (7) demands

$$\gamma^\kappa a_\kappa{}^\nu = S^{*-1}\gamma^\nu S, \tag{10}$$

and the covariance of (5) and (8) requires

$$\alpha = S^\dagger \alpha S^*, \tag{11}$$

Eq. (11) implies S to be an unimodular, i.e., $|S| = 1$. Therefore, the matrix S contains only 3 complex parameters which is sufficient to satisfy Eq. (10) with 6 real parameters $a_\kappa{}^\nu$ of the Lorentz group.

A transformation $S^{*-1}\gamma^\nu S$ with any matrix S, applied to the γ^ν will preserve the relations (2), i.e., any set of matrices differing from Eq. (3) by an arbitrary S transformation will lead to nothing new.

The set of matrices

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$

however, is another solution of (2), not differing from (2) by an S transformation. If Eqs. (1) and (3) represent the motion of a positive charge in the potential φ_κ, then Eqs. (1) and (12) represent the motion of a negative charge in the same potential φ_κ, which is readily seen by forming the conjugate complex of Eqs. (1) and (12) and comparing it with Eqs. (1) and (3).

The wave equation (1) is covariant with respect to gauge transformations $\varphi_\kappa'' = \varphi_\kappa + \partial\Lambda/\partial x^\kappa$,

$$\text{with} \quad \left.\begin{array}{l} \psi'' = S^{-1}\psi, \quad \gamma^{\kappa''} = S^{*-1}\gamma^\kappa S, \quad \alpha'' = S^\dagger\alpha S^* \\ S = 1\exp(-i\Lambda), \end{array}\right\} \tag{13}$$

i.e., a not unimodular S. It is readily seen that $s^{\kappa''} = s^\kappa$.

The charge conservation law

$$\partial s^\kappa/\partial x^\kappa = 0 \tag{14}$$

follows from Eq. (5) by differentiation, observing that the left-hand side is, as s^κ, a real quantity and the right-hand side, by virtue of Eq. (1), purely imaginary.

We are investigating solutions of the *set* of Eqs. (1) with any matrices γ^ν satisfying Eq. (2). If ψ_1 and ψ_2 are two solutions of Eqs. (1) and (2), we can superimpose them with arbitrary real coefficients provided we have first adjusted their relative phases by a transformation (13) to the same gauge, i.e., to the same matrices γ^ν. In this theory, therefore, the phase relations between superimposable ψ-functions are fixed.

* Research carried out at the Institute for Advanced Study and University of Pennsylvania, Philadelphia, Pennsylvania.

Neutron Diffraction by Gases

N. Z. ALCOCK AND D. G. HURST
Division of Atomic Energy Research, National Research Council of Canada, Chalk River, Ontario, Canada
April 5, 1949

BY use of a modified crystal spectrometer at the heavy water pile to select neutrons of 0.07-ev energy, the intensity of neutrons scattered from gases was measured in the angular range 5° to 90°. The gases studied, oxygen and carbon dioxide, were contained in a steel vessel at room temperature and approximately 60 atmospheres pressure. A schematic diagram of the experimental arrangement is shown in Fig. 1.

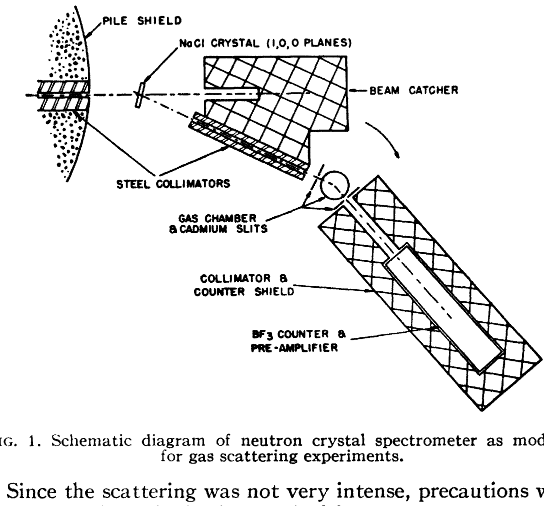

FIG. 1. Schematic diagram of neutron crystal spectrometer as modified for gas scattering experiments.

Since the scattering was not very intense, precautions were taken to reduce the background of fast neutrons penetrating the counter shield and slow neutrons scattered from the steel walls of the gas cell. The latter would be serious if the steel were not crystalline with a powder diffraction pattern. As there is no line of this pattern inside an angle of 30°, at small