For optical light $\lambda\sim 10^{-5}$ we find

$$q \gg 10^{17}\ \text{cm} \sim 0.1\ \text{light year}. \qquad (44)$$

The important conclusion from these examples is not that a given curvature can always be measured somehow, but that (a) the curvature is defined (in the sense of the limitations of quantum-theory measurement) only in the *large*, and (b) the domain of largeness is fundamentally determined by the momentum of the test particle with which the curvature is measured. Here we are led again to the idea that the conception of curvatures and, when these are equated in the form $G_{ik}+\frac{1}{2}g_{ik}G$ to the stress tensor KT_{ik}, the conceptions of energy, mass, and momentum, are only defined for quite large masses and large volumes of space.

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Theoretical Analysis of Ionization Chamber Data on Large Air Showers*

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The decoherence curve of large air showers is analyzed under the simplifying assumption of a constant lateral structure function for all air showers. The following results emerge: (1) Contrary to statements found in the literature, ordinary shower theory leads one to expect a rise in the decoherence curve at distances much less than the characteristic lateral distance r_1. (2) The theoretical decoherence curve calculated under the assumption of a constant structure function rises *more* sharply near the origin than the experimental points. This indicates that the effective structure function is less peaked than the one we used (due to Molière). (3) The dependence of the structure function upon the age of the shower tends to lessen this discrepancy. However, quantitative estimates make it appear doubtful that one can get agreement between theory and experiment without assuming a rather high multiplicity of the event which starts the shower.

(1) QUALITATIVE CONSIDERATIONS CONCERNING THE LATERAL STRUCTURE OF AIR SHOWERS

THE experiments which are to be interpreted here are connected with the so-called *decoherence curve* of large air showers. Two ionization chambers are placed a distance $2a$ apart. The chambers are biased so that they only respond if more than a certain amount of ionization is produced in each chamber, the bias being the same for both chambers. If the dimensions of the chambers can be neglected compared with their separation, the bias can be interpreted as meaning that each chamber responds only if the *density* of shower electrons passing through it is greater than a certain minimum amount, which we call ρ.

One then measures the coincidence counting rate W as a function of this minimum density ρ and of the half-distance a between the chambers. *A curve of $W(\rho, a)$ vs. a (keeping the bias constant) is called a decoherence curve.* We shall call a curve of $W(\rho, a)$ *vs.* ρ (keeping the separation a constant) a *density response curve.*

Historically, decoherence curves were measured first with Geiger-Müller counters.[1] It can be shown that a set of Geiger-Müller counter measurements giving the coincidence rate as a function of the distance between the counter trays and of the area of the counter trays (keeping the number of counters in each tray constant) is mathematically equivalent to a set of ionization chamber data, giving their coincidence rate as a function of the distance between the chambers and of the chamber bias. However, ionization chamber data are much preferable for the following reasons:

(1) A single set of measurements, in which pulses of different sizes are recorded, is required instead of a large number of measurements with counter trays of different areas.

(2) In Geiger-Müller counter measurements, there often exists the possibility of getting spurious counts from a single particle traveling horizontally, since one particle can easily produce a pulse. In ionization chamber measurements, ten to twenty particles must pass through the chamber before a pulse is recorded; hence, there are no spurious counts due to that source.

(3) For the same reason, statistical fluctuations are much less important in ionization chamber data.

(4) The mathematical analysis is incomparably simpler.

In discussing large air showers, we shall consider only the electron-photon component, since this component accounts for most of the ionizing radiation observed without heavy shielding on top of the detecting equipment. There seems to be reason to believe[2] that there are on the average about fifty electrons for every heavy ionizing particle in big air showers. (The word "electron" is used for both negatrons and positrons.)

We assume that the shower is started by an "initiating electron" near the top of the atmosphere.

* Assisted by the joint program of the ONR and the AEC.

[1] Auger, Maze, Ehrenfest, and Freon, J. de phys. et rad. 1, 39 (1939).

[2] G. Cocconi and K. Greisen, Phys. Rev. 74, 62 (1948); J. E. Treat and K. Greisen, Phys. Rev. 74, 414 (1948).

The initiating electron is very probably *not* a primary particle.[3] Since observations are mostly carried on below a considerable atmospheric depth (twenty radiation units in our case), the observations do not distinguish whether the initiating particle is an electron or a photon, nor can we tell just where the shower starts, as long as it is within a few radiation units from the top of the atmosphere. Recently, it has been suggested[4] that the event which initiates the shower is of a highly multiple character. The consequences of this hypothesis will be discussed in Section 6.

The usual shower theory treats the average longitudinal development of a shower under the assumption that the particles are all concentrated around the shower axis (the direction of the initiating particle). The effects (neglected in the usual shower theory) which lead to a lateral displacement of the shower particles from the axis of the shower are:

(1) scattering of electrons by air nuclei (Coulomb scattering);
(2) scattering of photons by electrons in air molecules (Compton scattering);
(3) the angular deviation of the particle in pair production from the direction of the parent photon;
(4) the angular deviation of the Bremsstrahlung photon from the direction of the parent electron.

One can estimate the angles involved in these processes. It turns out that the scattering of electrons by air nuclei gives most of the effect so that the other processes (2), (3), and (4), may be neglected. The scattering is predominantly multiple; i.e., deflections through a finite angle are mostly due to many successive scattering events, each event involving a very small angle, rather than to a single scattering event. The mean square angle of deviation $\langle\theta^2\rangle_{\mathrm{Av}(dt)}$ due to multiple Coulomb scattering of a particle of energy E passing through a layer of matter of thickness dt (measured in radiation lengths) is given by[5]

$$\langle\theta^2\rangle_{\mathrm{Av}(dt)}=(E_s/E)^2dt. \tag{1.1}$$

The "scattering energy" E_s is 21 Mev.

Qualitatively, the development of a shower may be described as follows: The initiating electron defines the axis of the shower. Before the maximum is reached, the number of particles rapidly increases while their average energy decreases. Beyond the maximum, the average energy of the particles stays roughly constant (at about the critical energy $\epsilon=86$ Mev in air), while the total number of particles decreases until the shower dies.

The lateral distribution of electrons is different for electrons of different energies. The root-mean-square Coulomb scattering angle is inversely proportional to the energy of the electrons. Hence, high energy electrons are found close to the shower axis, while low energy electrons are spread out over much larger lateral dimensions. The high energy electrons form a dense *shower-core*. The density of particles in the core of the shower is determined by the number of high energy electrons in the shower, *not* by the total number of electrons in the shower.

We can estimate the mean square distance $\langle r^2(E)\rangle$ of an electron of energy E from the shower axis in the following way. First, we observe that the parent of this electron was a particle of much higher energy (about three to ten times as much). Hence, on a first approximation, we can neglect the spreading of the ancestors of the electron in question. (This is true for E greater than the critical energy ϵ; we will see later that it is not true for $E\lesssim\epsilon$.) The mean square scattering angle per radiation unit of matter traversed is given by (1.1). The mean square lateral displacement can be estimated by taking the mean square angle over the range of the particle, and multiplying by the square of the range; i.e.,

$$\langle r^2\rangle\sim\langle\theta^2\rangle_{\mathrm{Av(range)}}\cdot[\mathrm{range}]^2. \tag{1.2}$$

The range of high energy electrons in air showers is of the order of magnitude of one radiation length, the distance in which the energy is on the average reduced to $1/e$ of its original value by radiation processes. We therefore get the rough estimate

$$\langle r^2(E)\rangle\sim(E_sX_0/E)^2 \text{ for } E\gtrsim\epsilon. \tag{1.3}$$

For electrons of the critical energy ϵ, (1.3) gives

$$\langle r^2(\epsilon)\rangle\sim(E_sX_0/\epsilon)^2\equiv r_1^2, \tag{1.4}$$

The *characteristic lateral unit of length* r_1 defined by (1.4) gives an idea of the lateral extent of a shower over most of its length. For air, r_1 is about one-fifth of a radiation unit. At Echo Lake, Colorado, r_1 is about one hundred meters.

For electrons below the critical energy, it is not permissible to neglect the ionization loss. Indeed, the range of an electron of energy $E<\epsilon$ is of order $(E/\epsilon)X_0$, not of order X_0. If we put this into (1.2), we get

$$\langle x^2(E)\rangle\sim(E_s/E)^2(E/\epsilon)(EX_0/\epsilon)^2=(E/\epsilon)r_1^2. \tag{1.5}$$

This result must be interpreted with care. $\langle x^2(E)\rangle$ is *not* the mean square distance from the shower axis, but rather the mean square lateral distance from the position of the parent particle. Since $\langle x^2(E)\rangle$ *decreases* with decreasing energy, *one is not justified in neglecting the distribution of the ancestry of electrons of below critical energy*. This is borne out by more detailed calculations of $\langle r^2(E)\rangle$.[6]

[3] R. I. Hulsizer and B. Rossi, Phys. Rev. **73**, 1402 (1948).
[4] H. W. Lewis, J. R. Oppenheimer, and S. A. Wouthuysen, Phys. Rev. **73**, 127 (1948).
[5] B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 240 (1941).
[6] L. Landau, J. Phys. U.S.S.R. **3**, 237 (1940); S. Z. Belenky, J. Phys. U.S.S.R. **8**, 9 (1944).

The above considerations lead to the following qualitative picture of a shower. The shower first spreads out as it goes down into the atmosphere. The more energetic particles stay near the axis, their less energetic offspring spread out somewhat, *their* offspring spread even farther, etc. This makes the shower appear like a fir tree. This behavior does not persist, however. After a few radiation units, electrons of about the critical energy make up most of the population of the shower (this happens long before the average energy is reduced to ϵ). These low energy electrons have a limited range and do not spread any more. Hence, after a few radiation units, the shower does not spread out any more, its lateral dimensions staying constant, of order of magnitude r_1, (1.4). From then on, the shower looks like a cylinder, provided we measure all distances in radiation units. If we measure distances in meters, the cylinder becomes compressed as it goes down into the atmosphere since the radiation unit is proportional to the density of the air. In other words, the cylinder becomes a funnel.

The core of the shower is made up of energetic particles ($E\sim 10^9-10^{11}$ ev). The relative number of these energetic particles decreases as the shower passes through its maximum and gets older. The number of electrons of energy greater than E ($E>\epsilon$) passes through a maximum *before* the shower as a whole passes through its maximum. For a shower initiated by an electron of energy 10^{16} ev, the maximum of the shower as a whole occurs about 17.5 radiation units down. The maximum of electrons of energies larger than 10^9 ev occurs about two radiation units before that. We see that the relative importance of the core of the shower decreases with age.

In what follows, we shall restrict ourselves to the *average* development of air showers. At first sight, one might think that correlations of density fluctuations in the shower introduce large errors into the comparison of theory and experiment. This is true for the longitudinal development of the shower, but is *not* true for measurements of the lateral structure at any one level in the atmosphere, provided that many particles are required to set off the ion chambers. The reason is the difference in the "memory" and "amplification" properties of the shower for longitudinal and lateral fluctuations. In the longitudinal development, we have a cumulative process in which a given fluctuation is not only "remembered" for many radiation lengths but exerts an "amplified" effect because of the multiplication along the shower. In the lateral development, the electrons observed at a given distance from the shower axis go there almost entirely in the last few radiation lengths. Furthermore, once they get out from the core by an appreciable distance, they are already close to critical energy and do not multiply by a very large factor. Hence, the lateral development of the shower features both a short "memory" and a small "amplification" for fluctuations.

It is therefore reasonable to expect that the fluctuations in the lateral structure of the shower have a distribution not too far from the distribution which follows from the assumption that all the particles are statistically independent. If this is true, the root-mean-square deviation from the mean number of particles N expected at a given place will be of the order of $N^{\frac{1}{2}}$. We require at least twenty particles through the chamber before it responds. Hence, the fluctuations in density have an effect of the order of twenty-five percent on an individual count. Since we are dealing with a statistical distribution (the decoherence curve) rather than with each shower separately, this fluctuation is likely to be averaged out to a large extent. It is therefore permissible to work with average values and to neglect fluctuations from the average *provided that we restrict ourselves to events in one plane of observation perpendicular to the shower axis.* Fluctuations may interfere badly as soon as we try to correlate the experimental information at the plane of observation with hypotheses about the distribution-in-energy and in place-of-origin of the initiating particles of the shower.

We shall also assume that all showers come in vertically. This assumption can be checked experimentally by triggering a cloud chamber in coincidence with the ionization chambers. This experiment was done.[7] The result is that the angles involved are of order 20° or less for most showers.

The angles are expected to be small because a shower hitting at a large angle has to go through a thick layer of atmosphere compared to a vertical shower. There are fewer showers which can do this.

The large angle showers *do* contribute importantly to the decoherence curve at really large separations (500 to 1000 meters). The reason is that the effective chamber separation is less for showers coming in at an angle than for vertical showers. The two effects compensate in such a way that the maximum contribution comes not from the vertical direction but from some intermediate angle. This is especially so because a shower which can set off two counters separated by many times r_1 is necessarily a giant shower which is not affected much by the path length through the atmosphere (indeed, the shower may be so large that it does not even reach its maximum going straight down, in which case a bigger path length is even helpful). These considerations are due to Cocconi.[8] We shall not treat this subject here since we are interested in the decoherence curve for small separations between the chambers.

(2) THE LATERAL STRUCTURE FUNCTION OF AN AIR SHOWER

Let $\rho(E_0,t,r)$ be the average density of all shower electrons, irrespective of energy, per unit area per-

[7] R. W. Williams, Phys. Rev. **74**, 1689 (1948).
[8] G. Cocconi, Phys. Rev. **72**, 350 (1947).

pendicular to the axis of the shower at a distance r from the shower axis. We assume the shower was started by an initiating electron of energy E_0, a distance t radiation units above the plane of observation. Then the total number of electrons at this depth in the atmosphere is, in the notation of Rossi and Greisen,[5]

$$\Pi(E_0,0,t)=\int_0^{\infty}\rho(E_0,t,r)2\pi r dr. \quad (2.1)$$

In order to correlate theory with experiment, we should not consider electrons down to zero energy, but rather we should make the lower limit on E equal to the minimum energy necessary to penetrate the wall of the ionization chamber. This does not make a significant difference for the experiments in question.

We now define the *lateral structure function* $f(E_0,t,r)$ of the shower by

$$f(E_0,t,r)\equiv\rho(E_0,t,r)/\Pi(E_0,0,t). \quad (2.2)$$

It follows from this definition and (2.1) that the integral of the structure function over the whole plane of observation is unity:

$$\int_0^{\infty}f(E_0,t,r)2\pi r dr=1. \quad (2.3)$$

Our qualitative considerations have shown that the structure function depends upon the age of the shower. A young shower, before it reaches its maximum, has a large proportion of high energy particles, hence a dense core. This makes the structure function peaked near the origin. As the shower becomes older, the core becomes less dense; hence the peak of the structure function becomes relatively less important.

For values of $r\ll r_1$ and t not much beyond $t_{\max}$, electrons of energy $E\gg\epsilon$ give most of the contribution to the structure function $f(E_0,t,r)$. In that case, one can give a rough estimate of the behavior of the structure function:[9]

$$f(E_0,t,r)\sim r^{s-2}\text{ provided }r\ll r_1, \quad (2.4)$$

where s is a quantity defined by Eq. (2.104), reference 5. s is related to the "age" of the shower, being smaller than one before the maximum and larger than one after the maximum. Thus, for young showers, the peak is very strong; for showers at their maximum, the structure function is inversely proportional to the radial distance (for $r\ll r_1$); for very old showers, s approaches 2 and the above estimate breaks down (electrons of critical energy then determine the behavior of the structure function for all values of r, not just for $r\gtrsim r_1$).

[9] I. Pomeranchuk, J. Phys. U.S.S.R., **8**, 17 (1944). A. Migdal, J. Phys. U.S.S.R., **9**, 183 (1945).

The only detailed calculation of a structure function available at this time is due to Molière.[10] Molière restricts himself to the maximum of the shower. He starts by calculating the structure function for particles of a single energy E under the assumptions described as "approximation A" in the review paper by Rossi and Greisen;[5] i.e., Molière calculates the partial structure function $f(E_0,E,t_{\max}r)$ for electrons of energy $E\gg\epsilon$. It is somewhat difficult to estimate the accuracy of the Molière function, since long numerical computations are involved and the calculation has not been reported in detail so far. The fact that the distribution-in-angle (an intermediate step in the calculation) turns out to be negative for angles near $\theta=3(E_s/E)$,[11] is not encouraging. On the other hand, Molière's value for the mean square distance $\langle r^2\rangle(E_0,E,t_{\max})$ of particles of energy E from the shower axis at the shower maximum turns out to be correct within about 15 percent.

Molière himself states the value of $\langle r^2(E_0,E,t_{\max})\rangle$ implied by his partial structure function. It is

$$\langle r^2(E_0,E,t_{\max})\rangle=0.835(E_sX_0/E)^2. \quad (2.5)$$

This can also be seen by differentiating his Fourier transform $\varphi(\rho)$ twice, and setting $\rho=0$. While this value of $\langle r^2(E_0,E,t_{\max})\rangle$ is supposedly appropriate to the maximum of the shower, actually the only property of the shower-distribution used was that $\pi(E_0,E,t_{\max})\sim E^{-2}$ (the notationis as in reference 5). This, however, is just the behavior of the track-length $z_\pi(E_0,E)$. Hence Molière's $\langle r^2(E_0,E,t_{\max})\rangle$ can be compared directly with $\langle r^2(E_0E)\rangle$ computed on a track-length basis; i.e.,

$$\langle r^2(E_0E)\rangle\equiv\int_0^\infty\langle r^2(E_0,E,t)\rangle\pi(E_0,E,t)dt/\int_0^\infty\pi(E_0,E,t)dt, \quad (2.6)$$

This latter quantity can be calculated accurately under approximation A by a method due to Landau[6] or alternatively by an equivalent method due to Nordheim.[5,12] The result in either case is

$$\langle r^2(E_0E)\rangle=0.723(E_sX_0/E)^2\text{ provided }E\ll E_0. \quad (2.7)$$

The near agreement between (2.5) and (2.7) confirms Molière's basic calcalution. (2.7) differs somewhat from the result of Landau and Belenky in spite of the fact that we used their own method of calculation. The reason is that they employed approximate expressions for the track-lengths which are valid for $E\ll E_0$ but not for $E\sim E_0$. It turns out that an accurate expression for $z_\pi(E_0,E)$ for values of $E\sim E_0$ is essential to obtain the correct answer, even though the answer itself refers to an energy $E\ll E_0$. The result (2.7) was derived using the Rossi-Greisen Mellin transform directly.

The expression (2.7) still needs a correction because we express $\langle r^2\rangle$ in units of the radiation length X_0, and X_0 decreases as we go down into the atmosphere. We define the fractional decrease in X_0 per radiation length, q, by

$$q\equiv -X_0^{-1}(dX_0/dt). \quad (2.8)$$

$q=0.041$ at Echo Lake, and it is roughly proportional to t^{-1}. If we neglect this altitude variation of q, the coefficient in

[10] G. Molière, *Cosmic Radiation*, ed. by W. Heisenberg (Dover Publications, New York, 1946), Chapter 3.

[11] The author thanks Drs. H. A. Bethe and L. Eyges for calling his attention to this point.

[12] L. W. Nordheim, Phys. Rev. **59**, 929 (1941). J. Roberg, Phys. Rev. **62**, 304 (1942). J. Lanossy, *Cosmic Rays* (Oxford University Press, London, 1948), Chapter VIII.

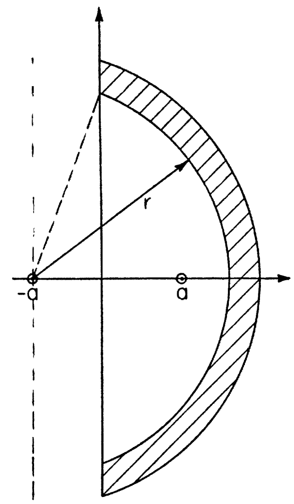

FIG. 1. Two ionization chambers, 1 and 2, are placed at $x=\pm a$, $y=0$. A shower whose core hits in the right half-plane will be counted provided it has sufficient density at the position of chamber 2.

Eq. (2.7) (called Q from here on) is given by

$$Q=0.723\frac{1-0.87q+0.43q^2}{(1-0.99q+0.19q^2)(1-1.99q+0.74q^2)}. \quad (2.9)$$

We can understand the dependence of Q upon q as follows: Since the particles which we measure at a given lateral distance from the shower axis got their lateral displacement somewhat earlier, we should replace X_0 in (2.7) by X_0 measured some distance Δt above the place of observation. This means $X_0(t)$ is replaced by $X_0(t-\Delta t)\cong X_0(t)[1+q\Delta t]$. Since X_0 enters squared, Q is given approximately by

$$Q\cong Q_0[1+2q\Delta t],$$

where $Q_0=0.723$ is the value computed by neglecting this effect. Expansion of (2.9) shows that $\Delta t\cong 1.05$ radiation units; i.e., *the high energy particles which we observe got their lateral displacement on the average one radiation unit above the place of observation.* This agrees with Janossy's result. This effect is not negligible. It amounts to a ten percent increase in Q at Echo Lake. Indeed, at Echo Lake we find that $\langle r^2(E_0E)\rangle$ is given by

$$\langle r^2(E_0E)\rangle=0.786(E_sX_0/E)^2. \quad (2.10)$$

Even if Molière's partial structure function for particles of a single energy were rigorously correct for high energy ($E\gg\epsilon$) particles, his over-all structure function, summing the contributions of electrons of *all* energies, might still be in error. Molière states that he used Arley's approximation[13] to find the total number of low energy electrons. This approximation is quite poor,[14,15] and it would falsify his over-all structure function even if he had at his disposal the exact partial structure function for low energy electrons.

An even more serious error in Molière's calculation comes from the fact that he did not use the correct partial structure function for electrons near the critical energy. Rather he extrapolated the high energy partial structure function right down to the critical energy. It can be shown[15a] that this procedure overestimates the lateral spread of these low energy electrons considerably ($\langle r^2(E)\rangle$ is too large by about a factor of two at $E=\epsilon$). Since electrons of about critical energy make up most of the population of the shower, this error in the partial structure function is preserved in the total structure function.

Underestimation (a la Arley) of the number of low energy electrons is likely to make Molière's structure function more peaked than the true one. On the other hand, the incorrect partial structure function for low energies will make Molière's structure function less peaked (extending farther out) than the true one. A rough estimate[15a] shows that the over-all effect makes the Molière function *less* peaked than the true one. Just how much more is not known at this time.

It must be emphasized that Molière's calculation, even though we have no estimate of error for it, is the best one available at this time.** For this reason we shall choose it as the basis for comparison between theory and experiment. We shall assume that the Molière structure function represents the lateral structure of all showers. This assumption implies that *we disregard the variation of the structure function with age.* An estimate of the error thus introduced will be given in Section 6. It may be appropriate at this point to emphasize that the main merit of the theory to be developed here lies in the qualitative understanding of the main features of the decoherence curve, not in the quantitative results.

The over-all structure function of Molière can be approximated for small values of r by an analytic expression due to H. A. Bethe.[16]

$$\left.\begin{aligned}f(E_0,t_{\max},r)&=r_1^{-2}\phi(r/r_1),\\ \phi(x)&\cong Cx^{-1}(1+4x)\exp[-4x^{\frac{3}{4}}] \text{ for } x\lesssim 0.5.\end{aligned}\right\}(2.11)$$

r_1 is the characteristic lateral unit of length (1.4). C is a constant, $C\cong 0.450$. Since we are concerned with distances of the order of one to ten meters while r_1 is 106 meters at Echo Lake, we can use (2.11) without introducing a large error. (Roughly ninety percent of the contribution to the counting rate of two chambers even twelve meters apart comes from showers whose cores hit no farther than fifty meters ($r/r_1\cong 0.5$) away.)

We see that the Molière function has the pre-

[13] N. Arley, Proc. Roy. Soc. **A168**, 519 (1938).
[14] S. Z. Belenky, Comptes Rendus, U.S.S.R. **30**, 608 (1941).
[15] M. Schonberg, Ann. Acad. Brasil. Sci. **12**, 281 (1940).
[15a] J. Roberg and L. Nordheim, Phys. Rev. **75**, 444 (1949). The author wishes to thank Professor Nordheim for calling his attention to this point and for giving him the numerical results of the above paper prior to its publication.
** Pomeranchuk and Migdal, reference 9, put the proper emphasis on the age-dependence of the structure function, but their work is more qualitative in character.
[16] H. A. Bethe, private communication.

dicted r^{-1} dependence near the origin. It should be emphasized, though, that the exponential factor is very important. It gives an appreciable deviation from the r^{-1} law already at distances of order $(r/r_1)\sim 0.01$ (i.e., one meter at Echo Lake).

(3) THE THEORETICAL FORMULA FOR THE COUNTING RATE NEGLECTING THE DIMENSIONS OF THE CHAMBER. THE INVERSION THEOREM

Let $S(N)$ be the number of showers crossing the plane of observation per unit time, with their axes falling within a unit area, such that the total number of particles crossing the plane is greater than N. We want to determine $W(\rho, a)$, the coincidence counting rate of two ionization chambers of dimensions much smaller than a, and biased so that a particle density greater than ρ is necessary before either chamber will register.

One can see from Fig. 1 that the recording of showers whose cores pass through the half-plane $x>0$ will be limited by the ionization chamber at $x=-a$. That is, if it registers, the other one will certainly register. (We are neglecting fluctuations.) Similarly, showers whose core passes through the half-plane $x<0$ will be limited by the chamber at $x=+a$.

Consider those showers whose cores pass through the shaded circular strip in Fig. 1. They will be recorded provided

$$Nf(r)\geqslant \rho. \tag{3.1}$$

The area of the circular strip is

$$dA=2\ \mathrm{arc\,cos}(a/r)rdr. \tag{3.2}$$

There are $S[\rho/f(r)]$ such showers per unit time. We get the contribution of the right half-plane to the counting rate $W(\rho, a)$ by summing over all these circular strips; i.e., by integrating from $r=a$ to $r=\infty$. By symmetry, the showers whose cores pass through the left half-plane give the same result. Hence

$$W(\rho, a)=\int_a^\infty 4r\ \mathrm{arc\,cos}(a/r)S[\rho/f(r)]dr. \tag{3.3}$$

(3.3) is an integral equation connecting the experimental function $W(\rho, a)$ of ρ and a with the unknown function $S[\rho/f(r)]$ of ρ and r. It turns out that (3.3) has an explicit solution. This is due to the fact that the kernel involves only the ratio of r to a. The solution is

$$S[\rho/f(r)]=(2\pi)^{-1}\int_r^\infty (a^2-r^2)^{-\frac{1}{2}}(\partial^2W/\partial a^2)da. \tag{3.4}$$

The kernel $(a^2-r^2)^{-\frac{1}{2}}$ has an infinite (but integrable) peak at the lower limit $a=r$. *Therefore, $S[\rho/f(r)]$ is essentially proportional to the second derivative of $W(\rho, a)$ with respect to a, at the point $r=a$.* This is a rather unfortunate fact. It means that *it is practically impossible to get the detailed behavior of $f(r)$ from the experimental data.* The statistical accuracy necessary before the second derivative of W has any meaning would require years of taking data. Conversely, this result explains the agreement between theory and experiment, found by investigators who *assumed* a structure function $f(r)$, and a frequency function $S(N)$ and determined the corresponding $W(\rho, a)$ from (3.3). In the light of this discussion, such agreement is to be expected for any reasonable choice of $f(r)$, but it proves very little about the correctness of this choice of $f(r)$.

(4) THE BEHAVIOR OF THE DECOHERENCE CURVE AT SMALL DISTANCES

The experiments[7] show that the coincidence counting rate $W(\rho, a)$ has approximately a power law dependence upon ρ; i.e.,

$$W(\rho, a)\cong \rho^{-\gamma}V(a), \tag{4.1}$$

where the exponent γ is a slowly varying function of a. In order to discuss the asymptotic behavior of the decoherence curve near $a=0$, we shall assume that γ is constant. Some evidence in favor of this assumption comes from the experiments of Cocconi[17] who investigated the distribution-in-density of showers somewhat smaller than the ones considered here. The results of Cocconi can be interpreted to mean that $S(N)\sim N^{-1.5}$ for N of order 10^3-10^5. Since showers of this order of magnitude contribute most of the counting rate in Williams' experiments for separations less than a meter, we can infer that the assumption

$$S(N)=\sigma N^{-\gamma}=\sigma N^{-1.5} \tag{4.2}$$

will give a good approximation to the behavior of the decoherence curve near zero separation.

We now show that $V(a)$ behaves in the following way for small values of a:

$$V(a)\cong V(0)-ka^{2-\gamma}. \tag{4.3}$$

Referring back to Fig. 1, we see that the region between the dotted line and the y axis would contribute to the counting rate of the left-hand chamber if the other chamber were moved over to $x=-a$. Hence, we must integrate $S[\rho/f(r)]$ over the area of this strip and multiply by 2 to get the differene between $W(\rho, 0)$ and $W(\rho, a)$. We introduce polar coordinates to get

$$W(\rho, 0)-W(\rho, a)=2\int_{-\pi/2}^{\pi/2}d\phi\int_0^{a\sec\phi} rdrS[\rho/f(r)].$$

We use Eq. (4.2) and the limiting form of the

[17] G. Cocconi, A. Loverdo, and V. Tongiorgi, Phys. Rev. **70**, 846 (1946).

structure function for small values of r:

$$f(r) \cong C/r_1 r. \quad (4.4)$$

Equation (4.2) then gives

$$W(\rho, 0) - W(\rho, a) = \rho^{-\gamma}[V(0) - V(a)],$$

where

$$V(0) - V(a) = 2\sigma \int_{-\pi/2}^{\pi/2} d\phi \int_0^{a \sec\phi} r dr f^\gamma(r). \quad (4.5)$$

If we now substitute (4.4), we find that

$$V(0) - V(a)$$

$$\cong \left[4\sigma (C/r_1)^\gamma (2-\gamma)^{-1} \int_0^{\pi/2} (\sec\phi)^{2-\gamma} d\phi \right] a^{2-\gamma}.$$

The coefficient in brackets is the k of Eq. (4.3). The importance of this result does not lie in the value of the coefficient. It lies in the fact that *the theory predicts an extremely sharp rise in the decoherence curve for small distances.* Indeed, formula (4.3) implies an infinite slope of $V(a)$ at $a=0$.

We have already seen that predictions like this cannot be trusted for very small distances. First of all, the chambers are *not* infinitely small—this will be discussed in Appendix A. Second, the structure function is *not* the Molière function for these small distances. This will be discussed in Section 6.

However, the qualitative result stands. *From ordinary shower theory, one should expect a rise in the decoherence curve for distances considerably less than the characteristic lateral unit* r_1. The usual argument against this is that most of the counts are due to showers whose cores hit far away. This is not true for very small separations. Essentially there are two effects here: (1) the cores of showers of a given number of particles N have to pass through an area which becomes smaller as N becomes smaller, and (2) the total number of showers per second per unit area is a rapidly decreasing function of N. The two effects work against each other. The net result depends upon the exponent γ in the power law of the distribution-in-number $S(N)$. For γ between one and two, we get a sharp peak at the center.

A few remarks may be in order here concerning Molière's calculation of the decoherence curve for Geiger-Müller counters. Molière finds that the theoretical decoherence curve should flatten off at distances of the order of three meters, in contradiction with the experimental data. On the other hand, a lengthy but elementary argument of the general nature of the one just given enables us to state that the result (4.3) also holds for Geiger-Müller counter decoherence curves; the only difference lies in the coefficient k.

A possible explanation for Molière's result is that he did not use a power law distribution for $S(N)$, but rather a power law for the distribution-in-energy of assumed primary electrons. The resulting $S(N)$ is not a pure power law. In particular, it rises less rapidly for small values of N. Hence, our assumption of a pure power law for $S(N)$ gives a larger rise of the decoherence curve near zero than Molière's assumption. However, it is doubtful that even under this assumption the decoherence curve should flatten off quite that rapidly. One is tempted to believe in an error in Molière's calculation, stemming either from the use of approximate functions for $\Pi(E_0,0,t)$ or from the very complicated numerical integrations involved in this method.

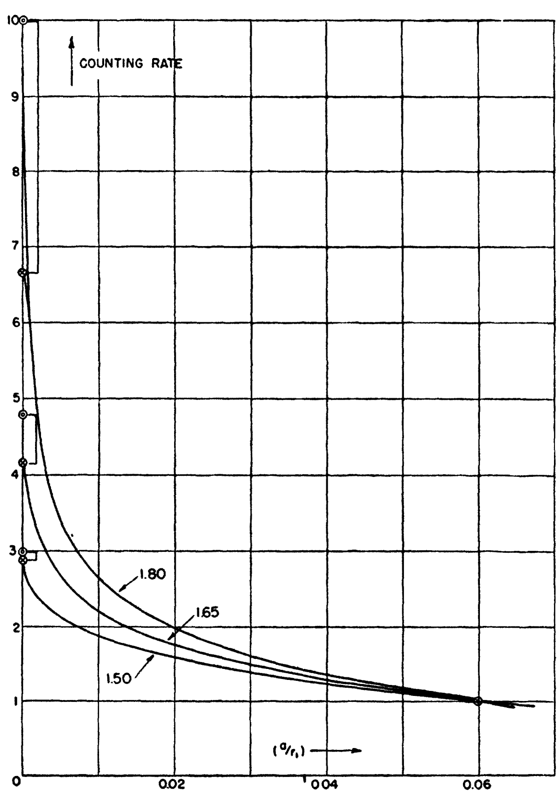

FIG. 2. Ordinate: Counting rate. Abscissa: Half-separation a between the chambers in units of the characteristic lateral distance r_1. The values of the exponents γ in the distribution-in-number $S(N)$ are indicated next to the curves. The correction for the point at $a=0$ is discussed in Appendix A.

The asymptotic form (4.4) of the structure function is not very accurate. $V(a)$, as defined by (4.1), was therefore calculated numerically from the Molière function (2.11) for various values of γ.

The results are given in Fig. 2. They are normalized to unity at $a=0.06$ (twelve meters separation at Climax). For each value of γ there are two points at zero separation. The upper one is uncorrected for the finite size of the chamber. The lower one is corrected for this effect. (See Appendix A.) The numbers next to the curves show what values of γ were assumed.

The theoretical decoherence curves show a drop of 2 or 3 (depending upon γ) between $a=0.01$ and $a=0.06$ (i.e., 2 and 12 meters chamber separation at Echo Lake). A factor of this order is observed experimentally.[7] On the other hand, the theoretical curves definitely disagree with experiment for very short distances. There the experimental curves flatten off whereas the theoretical ones keep rising, even after one takes the finite chamber size into account. This will be discussed in the next section.

We would only like to point out here that the existence of this disagreement is not in contradiction with the remarks of Section 3. $S[\rho/f(r)]$ is proportional to the second derivative of W only where $f(r)$ is regular. In the present case, the theory predicts not only an infinite second derivative at $a=0$, but also an infinite first derivative. The experiments, even with rather poor statistics, can distinguish between a finite and an infinite slope, and they give a finite slope.

(5) THE DISTRIBUTION-IN-NUMBER $S(N)$

We start by recalling the experimental data[7] concerning the exponent γ in Eq. (4.1). These data are summarized in Table I. We see that the effective exponent γ_{eff} of the density response curves increases with chamber separation; $\gamma_{\text{eff}}=1.5$ for chambers close together, $\gamma_{\text{eff}}=1.9$ for chambers separated by twelve meters ($a\cong 0.06r_1$).

This behavior of γ_{eff} takes place because the size of the showers which contribute most to the counting rate depends upon the chamber separation. As the chamber separation increases, showers containing increasingly larger numbers of particles make the major contribution to the counting rate $W(\rho, a)$.

There are two reasons for γ_{eff} to vary with the number of particles in the showers which contribute most to the counting rate: (1) the distribution-in-number $S(N)$ of the showers may not follow a pure power law; (2) the structure function may be (and we know it actually is) age-dependent.

The first effect needs no explanation. In order to understand the second effect, let us assume for the moment that the distribution-in-number $S(N)$ follows a pure power law, $S(N)=\sigma N^{-\gamma}$. The age-dependence of the structure function implies that we overestimate the sharpness of the peak in old showers by using the Molière function. Since the structure function is normalized by Eq. (2.3), a smaller peak near zero implies that $f(r)$ has larger values for larger values of r. This situation is shown schematically in Fig. 3.

If the event which originates the shower is known both as to kind and height above the equipment, then the number of particles N in the shower also defines the age of the shower. We may therefore consider the structure function as a function of N and r:

$$f=f(N, r). \tag{5.1}$$

We define the minimum effective number of particles $\bar{N}=\bar{N}(\rho, r)$ by

$$\bar{N}f(\bar{N}, r)=\rho. \tag{5.2}$$

TABLE I. Effective exponent γ_{eff} as a function of chamber separation, for a bias $\rho=460$ electrons/m^2.

Chamber separation in meters at Echo Lake	a/r_1	γ_{eff}
0.015	0.0007	1.50
0.36	0.0017	1.67
1.0	0.0047	1.56
7.0	0.0332	1.85
12.2	0.0575	1.90

We shall assume that Eq. (5.2) has a unique solution for every r and ρ of interest. Then Eq. (3.3) becomes

$$W(\rho, a)=\int_a^\infty 4r \text{ arc } \cos(a/r)S[\bar{N}(\rho, r)]dr. \tag{5.3}$$

Next, we determine the dependence of $\bar{N}$ upon ρ. For a constant structure function, $\bar{N}$ is of course proportional to ρ (see Eq. (3.1)). Figure 3 shows that for *small* values of r, $f(N, r)$ increases with N, while for large values of r, $f(N, r)$ decreases with N. Over a small interval in ρ, N may be assumed to behave like ρ^α where α is less than unity for small r, greater than unity for large r.

We now substitute a pure power law for $S(N)$. Since most of the contribution to W comes from distances r of order of magnitude a, we conclude that the dependence of $W(\rho, a)$ upon ρ is roughly

$$W(\rho, a)\sim\rho^{-\alpha\gamma}, \tag{5.4}$$

with α evaluated at $r=a$. This shows that even a pure power law for $S(N)$ will lead to *effective exponents γ_{eff} which increase with increasing separation due to the age-dependence of the structure function.*

Since we have no quantitative information about the age-dependence of the structure function, we shall neglect effect (2) altogether in the quantitative discussion of this section, restricting ourselves to

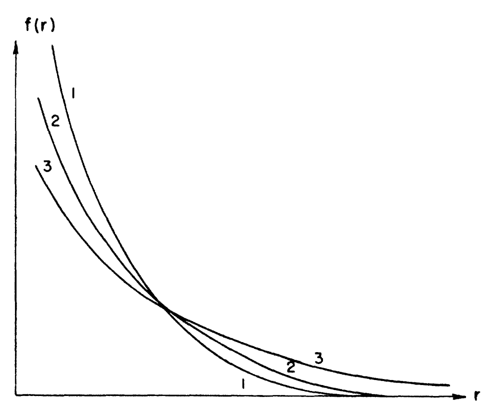

FIG. 3. Schematic picture of the age-dependence of the structure function. Curves 1, 2, and 3 are in order of increasing age of the shower. The structure function decreases with age for small values of r, increases with age for large values of r.

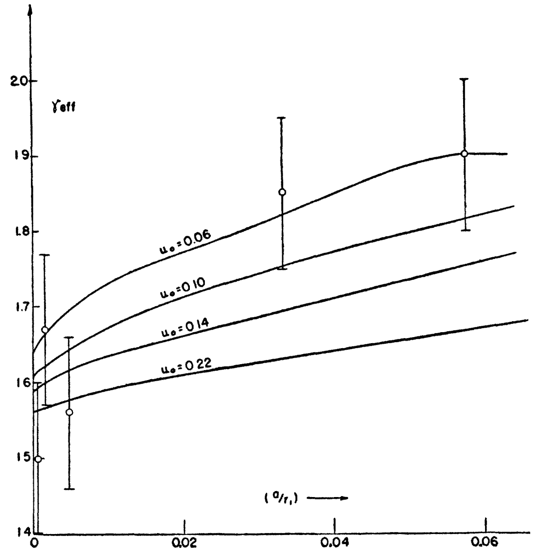

FIG. 4. Ordinate: Effective exponent γ_{eff} of $W(\rho, a)$ as a function of ρ. Abscissa: Half-distance a between the chambers in units of the characteristic lateral distance r_1. The curves are computed from formula (5.6). The experimental data are taken from Table I.

pointing out the qualitative changes introduced by this effect. Some very rough quantitative estimates will be given in Section 6.

We therefore assume that the structure function is still the same for all showers, but $S(N)$ is not a pure power law any more. This latter assumption is rather appealing, for the following reason: if we assume a power law for the distribution-in-*energy* of the initiating electrons, the distribution-in-number $S(N)$ at the place of observation will *not* be a power law throughout the range of N. It will be approximately a power law over small ranges of N, with exponents γ which depend upon N. γ will equal the exponent of the initiating distribution-in-energy for those values of N and the initiating energy for which the shower just reaches its maximum at Echo Lake. For bigger N (and therefore bigger E_0 and younger showers) γ will be larger. For smaller N (older showers past their maximum at Echo Lake) γ will be less.

We recall that Cocconi[17] found a power law for $S(N)$ with an exponent $\gamma\cong 1.5$. The showers which he observed were on the average smaller than the ones observed by Williams.[7] It is therefore reasonable to assume that $S(N)$ goes like $N^{-1.5}$ for low N (of order 10^3-10^5) and then drops off more steeply. The value of γ_{eff} for the largest separations used by Williams is of order 1.9. We therefore make the following assumption for the distribution-in-number $S(N)$:

$$\left.\begin{aligned} S(N)&=\sigma_0 N^{-1.5} \text{ for } N<N_0,\\ S(N)&=\sigma_1 N^{-1.9} \text{ for } N>N_0.\end{aligned}\right\} \tag{5.5}$$

From the definition of $S(N)$, it follows that it must be a continuous function. This determines the ratio of σ_0 to σ_1 for any given choice of the "breakpoint" N_0. It is an easy matter to substitute the assumption (5.5) into the basic formula (3.3) for the counting rate. We define the "breakpoint distance" u_0 by

$$f(u_0)\equiv\rho/N_0, \tag{5.6}$$

and the function $T(a, u, \gamma)$ by

$$\left.\begin{aligned} T(a,u,\gamma)&\equiv\int_u^\infty 4r \arc\cos(a/r) f^\gamma(r)dr \quad \text{for} \quad u>a,\\ T(a,u,\gamma)&\equiv T(a,a,\gamma) \quad \text{for} \quad u\leqslant a.\end{aligned}\right\} \tag{5.7}$$

We observe that $T(a, a, \gamma)$ is proportional to the counting rate for a pure power-law $S(N)$. Furthermore, $T(a, u, \gamma)$ can be obtained by a very small amount of numerical work once the pure power-law functions have been computed.

With these definitions, the coincidence counting rate becomes

$$W(\rho,a)=\sigma_0\rho^{-1.5}[T(a,a,1.5)-T(a,u_0,1.5)]+\sigma_1\rho^{-1.9}T(a,u_0,1.9). \tag{5.8}$$

The effective exponent γ_{eff} can be defined by

$$\gamma_{eff}=-(\rho/W)(\partial W/\partial\rho). \tag{5.9}$$

This definition means that γ_{eff} is the (negative) slope of a plot of $\log W$ against $\log\rho$. It follows that γ_{eff} is given by

$$\gamma_{eff}=\frac{1.5f(u_0)^{-1.5}[T(a,a,1.5)-T(a,u_0,1.5)]+1.9f(u_0)^{-1.9}T(a,u_0,1.9)}{f(u_0)^{-1.5}[T(a,a,1.5)-T(a,u_0,1.5)]+f(u_0)^{-1.9}T(a,u_0,1.9)}. \tag{5.10}$$

In Fig. 4, we have plotted γ_{eff} *vs.* a for various values of u_0. (a and u_0 are measured in units of the characteristic lateral shower-distance r_1.) Also shown in the figure are the experimental values. We have assigned an error of ± 0.10 to the values of γ_{eff} in Table I. This is somewhat less than the limit of error from a differential plot (see reference 7), but it is felt that ± 0.10 is a reasonable estimate of the probable error of the coefficient obtained from an integral plot.

We see that the points fall on a smooth curve within the experimental error. Unfortunately, none of the theoretical curves is a very good fit. A value of $u_0\cong 0.06$ is a good fit for the points at 7 and 12 meters separation, while $u_0\cong 0.14$ is a better fit for the smaller separations.

We point out that a function $S(N)$ which goes from $N^{-1.5}$ to $N^{-1.9}$ more gradually than our simple assumption (5.5) would make the theoretical curves in Fig. 4 even flatter.

The age-dependence of the structure function makes γ_{eff} smaller for small separations, larger for large separations (see (5.4)); hence the theoretical curves in Fig. 4 will become steeper if this effect is taken into account, in better agreement with the experimental data. However, it is uncertain whether the effect will be pronounced enough to lead to quantitative agreement between theory and experiment.

Showers near their maximum (for which the Molière $f(r)$ is presumably a reasonable approximation) contribute proportionately more to the counting rate at 7 meters and 12 meters than at close distances. We therefore choose the "best" value of the breakpoint distance u_0 to fit the points farther away; i.e., $u_0=0.06$. One can then assume that the discrepancy between the theoretical and observed values of γ_{eff} at small separations is due to the age-dependence of the structure function (see the next section, however, for another important effect).

A value of $u_0=0.06$ with a minimum density $\rho=460$ particles per m^2 corresponds, according to (5.6) to a breakpoint N_0 of 1.03×10^6, or within the accuracy involved here,

$$N_0=10^6 \text{ particles.} \tag{5.11}$$

We now compare the theoretical and experimental *decoherence* curves under the assumptions (5.5) and (5.11) (γ_{eff} was related to the density response curve). We shall determine the coefficient $\sigma_0 N_0^{-1.5}=\sigma_1 N_0^{-1.9}$ in front of the distribution by fitting the points at 7 and 12 meters. The result is shown in Fig. 5.

We see that the trend of the curve is correct except at very small separations (less than one meter). The theoretical curve drops by a factor of two, approximately, between $a/r_1=0.01$ and $a/r_1=0.05$ (i.e., between 2 meters and 10 meters chamber separation). The experimental points are in rough agreement with this.

On the other hand, there is decided disagreement for $(a/r_1)<0.01$. The theoretical curve rises sharply up to a value of 1.73 at zero separation. (This is already corrected for the finite chamber size; see Appendix A.) The experimental points indicate a flattening off. (The indication that the curve actually drops for small distances must not be taken seriously.) This discrepancy can again be attributed to the age-dependence of the structure function and is again explained qualitatively by the smaller peak if the structure function for older showers. It will be discussed quantitatively in Section 6.

It may be of interest to put down some numbers here for $S(N)$. We obtain

$$\left.\begin{aligned} S(N)&=1.8\times10^{-3}(N/10^6)^{-1.5}, \quad N<10^6,\\ S(N)&=1.8\times10^{-3}(N/10^6)^{-1.9}, \quad N>10^6,\end{aligned}\right\} \tag{5.12}$$

where $S(N)$ *is in* $(meter)^{-2}$ $(hour)^{-1}$. In particular, (5.12) tells us that there are 0.0018 showers per hour whose cores hit within one square meter and which have more than 10^6 particles at the place of observation. It hardly needs to be emphasized that *(5.12) is only an order-of-magnitude estimate.* In view of the uncertainty in the theory, we prefer not to give any estimate of error at all. The reader is referred to Appendix B for an alternative way of determining $S(N)$ which gives a result different by more than a factor of two from (5.12).

The qualitative agreement between theory and experiment in Fig. 5 for larger distances cannot be used as an argument in favor of the Molière function, in view of our remarks in Section 3.

(6) SOME ESTIMATES CONCERNING THE INFLUENCE OF THE AGE-DEPENDENCE OF THE STRUCTURE FUNCTION AND SOME SPECULATIONS ABOUT MULTIPLE PRODUCTION

We shall now discuss in more detail whether the discrepancies between the theoretical and experi-

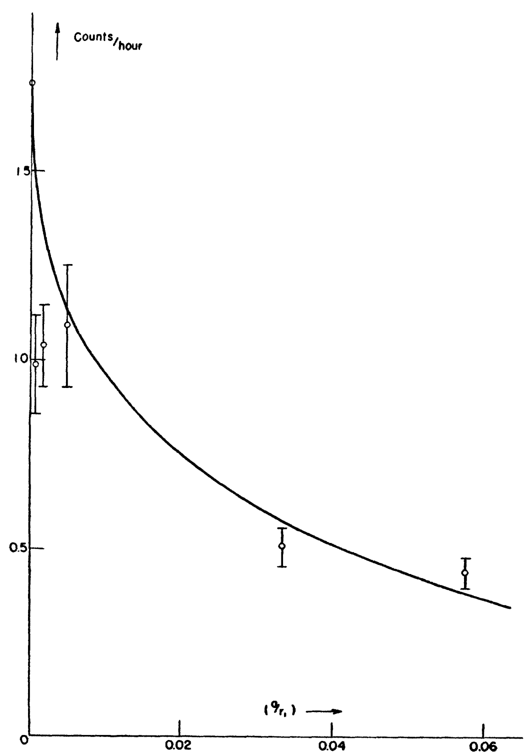

FIG. 5. The decoherence curve. Ordinate: Counts per hour with a bias of 460 particles/m². Abscissa: Half-separation between chambers in units of r_1. The experimental data are due to R. W. Williams. (See reference 7.) The theoretical curve is fitted to the points at $a=0.0332$ and $a=0.0575$.

mental values of $W(\rho, a)$ at small distances can be accounted for by the age-dependence of the structure function, under various assumptions about the way large air showers are initiated.

We shall use the estimate of Pomeranchuk and Migdal[9] for the behavior of the structure function at small distances

$$f(r) \sim r^{s-2}, \tag{6.1}$$

where s defines the age of the shower. $s=1$ corresponds to the shower maximum. In Section 4 we have derived the asymptotic form of the decoherence curve, (4.3). The "γ" in (4.3) came from the $[f(r)]^\gamma$ in the integral (4.5), under the assumption that $f(r)$ behaves like r^{-1}. If $f(r)$ behaves like r^{s-2} instead, the asymptotic behavior of the decoherence curve near $a=0$ becomes

$$V(a) \cong V(0) - ka^{2-(2-s)\gamma}. \tag{6.2}$$

Of course, the constant k is also changed.

For $s=1$ (showers at their maximum), this reduces to (4.3), as it must. But now consider older showers for which

$$2-(2-s)\gamma > 1, \quad s > 2-\gamma^{-1}. \tag{6.3}$$

Then $V(a)$ will come in horizontally at $a=0$, rather than with a vertical tangent. Since we are talking about older showers, it is reasonable to use Cocconi's[17] result for γ; i.e., $\gamma \cong 1.5$. *We then classify a shower as "old" if* $s > 4/3$.

It may be worth while to point out that this classification involves not merely the properties of any one shower, but also their distribution-in-number. Unless we know something about the exponent γ, we do not know which showers we are to classify as "old" for our purposes.

It is useful to put a few numbers in here, just to get an idea of the orders of magnitude involved. We shall make two different assumptions:

(1) The shower is initiated by an electron at the top of the atmosphere.

(2) The shower is initiated by a (secondary) electron four radiation units (144 g/cm²) below the top of the atmosphere. We choose this number because the absorption length of the N radiation[18] in the atmosphere is of this order of magnitude.

We can determine which shower has $s=1.33$ under these two assumptions, by using formula (2.104) of reference 5. We then estimate what regions of the decoherence curve will be affected seriously by these showers. A shower containing only a few particles cannot affect the counting rate of two ionization chambers if they are placed far enough apart. Indeed, the largest possible half-separation $a_{\max}$ is given by

$$Nf(a_{\max}) = \rho \tag{6.4}$$

where ρ is the chamber bias. This corresponds to the case where the shower hits just midway between the chambers.

Unfortunately, we cannot use (6.1) here for $f(r)$ since the constant in front is important. We therefore have to fall back upon the Molière distribution. The results are given in Table II. The first column is the distance over which the shower develops, the second is the energy of the initiating electron for $s=1.33$, the third the number of particles at the place of observation. The fourth and fifth columns give the maximum chamber separations over which the shower can be effective.

These numbers are disconcertingly small. On Fig. 5, the experimental results start deviating from the theoretical curve at distances of the order of one meter, probably even two meters if we want to get a smooth curve.

Of course, the Molière function, being too large at the origin, will falsify the numbers in Table II, but the schematic Fig. 6 illustrates that we are likely to overestimate $a_{\max}$ in this way. For large enough values of (ρ/N), the true intersection point P gives a lower value of $a_{\max}$ than the intersection point Q with the Molière function. Since the values of r involved here are extremely small, we are very probably in that region of ρ/N.

It is not possible to say at this time whether a correct calculation will improve the situation or whether the qualitative discrepancy in Fig. 5 will remain. The estimates given here make us inclined to believe that under either one of the two assumptions about the generation of the shower presented so far, a mathematically correct application of shower theory will not yield agreement between theory and experiment. (We emphasize that the disagreement we are talking about here is in the opposite direction from the one sometimes referred to in the literature.[19,9] The theoretical curve rises too steeply near the origin.)

This leads us to believe that we may have been underestimating the age of the showers which are responsible for our counting rates. Suppose that air showers are not produced in an event involving only one initiating electron (or photon), but rather in an event of high multiplicity ν, involving elec-

TABLE II. (6.1) Maximum chamber separations over which an "old" shower can produce a count.

$t-t_0$ (radiation units)	E_0 (ev)	N	$(a_{\max}/r_1)$	Chamber separation at Echo Lake (meters)
16	3.23×10^{12}	1.42×10^{3}	1.5×10^{-4}	0.03
20	3.70×10^{13}	1.18×10^{4}	10^{-3}	0.20

[18] B. Rossi, Rev. Mod. Phys. 20, 537 (1948).

[19] L. Janossy, *Cosmic Rays* (Oxford University Press, London, 1948), Chapter VIII.

trons (or photons) originating in the same place and in roughly the same direction. In that case, the value of N in Table II have to be multiplied by ν. If ν is of order 10^2 to 10^3, we are right in the region of N which contributes most to the counting rate for moderate separations. In that case, the flattening off of the decoherence curve becomes eminently understandable.

There is another effect of multiple production tending to make the decoherence curve come in with zero slope at $r=0$. This is the granular structure of such a many-core shower. After all, the initiating particles do not all have exactly the same directions. Whereas the angles are small, the distance to the place of observation is large.

It is not certain just how one should estimate the angles involved in such a multiple-production process. We shall assume that a primary (presumably a proton) creates a large number ν of short-lived mesons, each of which then generates a shower. In analogy with electron pair creation,[5] we shall assume that the mesons get sidewise momenta of order μc, where μ is the meson mass. (We shall assume $\mu c^2=160$ Mev; i.e., the mass of a π-meson.) The angle between the meson and the primary is then of order $\mu c^2/E$ where E is the energy of the meson. If the total energy of each meson goes into the shower, we can relate E to the total number of particles N at the place of observation in the following way. The number of particles in each component shower of this many-core air shower is N/ν. This determines E through

$$N/\nu=\Pi(E, 0, t), \tag{6.5}$$

where $\Pi(E, 0, t)$ is given by formula (2.104), reference 5 We shall assume $t=16$ radiation units; i.e., the showers are created 4 radiation units (144 g/cm^2) below the top of the atmosphere.

The result of this estimate is shown in Fig. 7, in which we have drawn lines of constant (estimated) core-separation in the N-ν-plane.

The figure shows that showers of 10^5 particles have cores separated by distances of order 10 cm if $\nu=10$, of order 0.9 meter if $\nu=100$, of order 5 meters if $\nu=1000$. Williams observed a few events with his "arrangement A" in which there were about that number of particles in the shower, with an apparent unique coer over distances of one meter. A multiplicity of order 20–40 would not contradict this result, yet it would already give an appreciable structure to the shower core. True, the equipment used in the experiments could not resolve such a granular structure explicitly. But the structure would appear through the substitution of a much flatter "effective" structure function instead of our $f(r)$. This helps to explain the flattening off of the decoherence curve.

We want to emphasize that these estimates are very rough and are based upon an assumption (sidewise momentum of order μc) which may well be completely wrong. In addition, our estimate of the multiplicity ν depends upon the assumed effective height of shower-production, becoming smaller as this height increases. We have also indicated in Fig. 7 the curve of multiplicity *vs.* number of particles predicted by the Oppenheimer-Lewis theory.[4] These multiplicities are somewhat high, if we use our estimate of the angles involved, since showers of 10^5 particles contribute appreciably to the counting rate near zero separation, and these showers would have core-separations of order 1 meter.

We remark that the Oppenheimer-Lewis theory itself would give much larger angles than we have estimated, but this has been disproven experimentally.[7] It appears to the author that it is not justifiable to dismiss the possibility of high multiplicity, but small angles, on the grounds of theoretical arguments which involve rather doubtful meson field theories.

Since we do not accept the Oppenheimer-Lewis estimate of the angles involved in the production process, there is no obvious reason to accept their prediction for the multiplicity of production. Indeed, it is likely that the Oppenheimer-Lewis multiplicity provides an upper limit for what we can expect. A somewhat lower multiplicity, with ordinary relativistic angles of production, seems consistent with the data analyzed here.

In conclusion, the following points emerge from our analysis:

(1) *Contrary to statements found in the literature, ordinary shower theory leads one to expect a rise in the decoherence curve at distances much less than the characteristic lateral distance r_1.*

(2) *The theoretical decoherence curve calculated under the assumption of a constant structure function rises more sharply near the origin that the experi-*

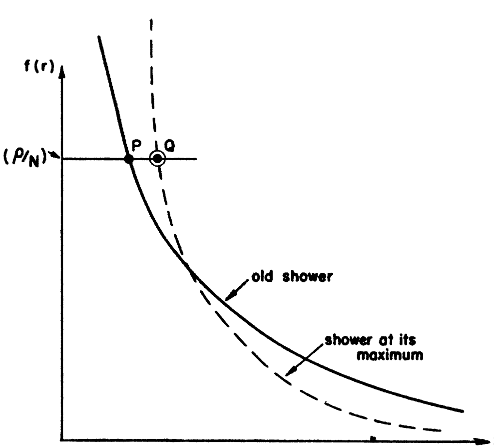

FIG. 6. The Molière structure function is likely to overestimate the area of influence of an old shower.

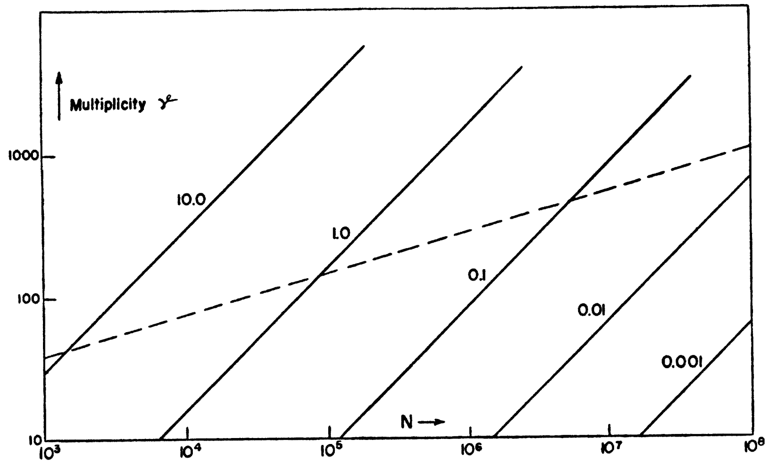

FIG. 7. Estimate of the granular structure of many-cored showers. Ordinate: Multiplicity of production ν. Abscissa: Total number of particles in the shower at the place of observation. The straight lines are lines of constant separation between cores, given in meters. The dashed curve is the multiplicity-number relation predicted by the Oppenheimer-Lewis theory. (See reference 4.)

mental points. This indicates that the effective structure function is less peaked than the Molière function.

(3) *While the age-dependence of the structure function tends to lessen this discrepancy, the quantitative estimates of Section 6 make it appear doubtful that one can get agreement between theory and experiment without assuming a rather high multiplicity of the event which starts the shower.*

ACKNOWLEDGMENTS

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APPENDIX A

The Correction for the Finite Dimensions of the Chambers

We recall that the chambers are biased so that they record only if the ionization exceeds a certain minimum amount. The chambers used were of cylindrical shape, about half a meter long and ten centimeters in diameter. We shall therefore neglect the width of the chambers, but not their length. A minimum value of the ionization can then be interpreted as a minimum number of particles which have to pass through the chamber before it records.

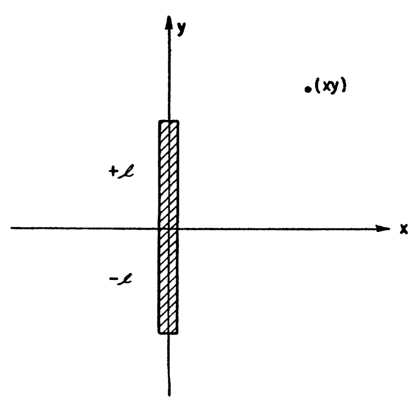

FIG. 8. A shower of N particles whose core hits at the point (xy) will send n particles through the shaded ionization chamber, n being given by formula (A.1).

We shall determine $W(\rho, a)$ for $a=0$, i.e., for a single chamber. Consider Fig. 8. If the core of a shower of N particles with a density function $\rho(r)=Nf(r)$ passes through the point (xy), the number of particles n passing through the chamber is

$$n=\int\int_{\text{Area of Chamber}} dx'dy'\rho[((x-x')^2+(y-y')^2)^{\frac{1}{2}}]. \qquad (A.1)$$

If we neglect the width of the chamber, the double integral becomes a single integral over y' between $-l$ and l with x' set equal to zero and $\int dx'$ replaced by T, the width of the chamber. Assuming a constant structure function,

$$\rho(r)=Nf(r),$$

(A.1) becomes

$$n=Ng(xy) \qquad (A.2)$$

where

$$g(xy)=T\int_{-l}^{l} dy'f[(x^2+(y-y')^2)^{\frac{1}{2}}]. \qquad (A.3)$$

Let r_0 ($\sim 0.01r_1$) be the largest value of r for which (4.4) is an adequate approximation for our purposes. If both $r\ll r_0$ and $l\ll r_0$, we can replace $f(r)$ in (A.3) by the expression (4.4). The integral can then be evaluated explicitly, with the result

$$g(x, y)\cong(2lT)(C/r_1R)\cdots r\ll r_0 \text{ and } l\ll r_0 \qquad (A.4)$$

where the variable R is defined by

$$R^{-1}\equiv(2l)^{-1}\log\left(\frac{[x^2+(y-l)^2]^{\frac{1}{2}}-(y-l)}{[x^2+(y+l)^2]^{\frac{1}{2}}-(y+l)}\right). \qquad (A.5)$$

R is zero at $x=y=0$; for large values of x and y, R approaches $r\equiv(x^2+y^2)^{\frac{1}{2}}$.

For values of x and y much larger than l, the dimensions of the chamber can be neglected. In that case, formula (A.3) gives

$$g(xy)\cong(2lT)f(r), \quad r\gg l. \qquad (A.6)$$

We now observe that both (A.4) and (A.6) can be written as

$$g(xy)\cong(2lT)f(R). \qquad (A.7)$$

In practice the condition $l\ll r_0$ is satisfied, since $l\cong 0.003r_1$ for the chambers used. Equation (A.7) then holds for $R\ll r_0$ and for $R\gg l$. Since these two regions of validity overlap, it is evidently a good approximation to assume that (A.7) holds for all values of R.

Since $S(N)\cdot dx\cdot dy$ showers of more than N particles have cores which pass through the elementary area $dx\cdot dy$, per second, and since (A.2) implies that N must exceed $n/g(xy)$ in order that a count be recorded, we conclude that the

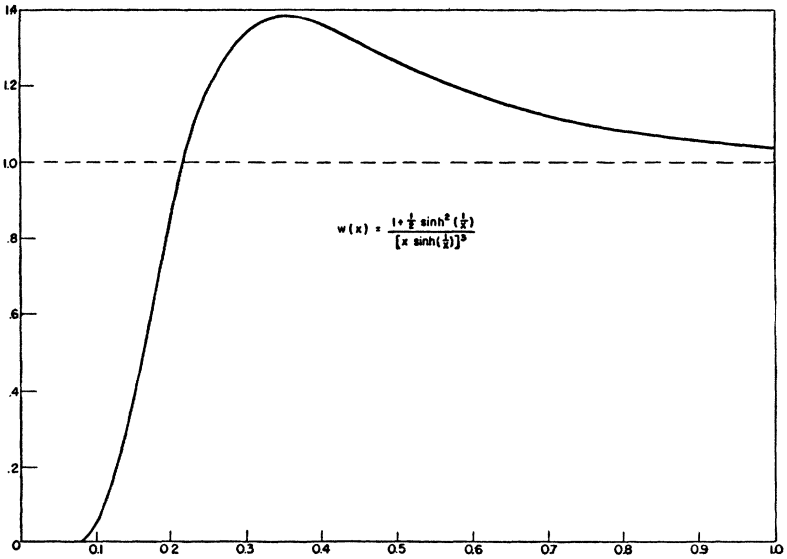

FIG. 9. Ordinate: Weight function $w(x)$ in formulas (A.11), (A.12). Abscissa: $x=R/l$.

counting rate $W(n,0)$ of a single chamber is given by

$$W(n,0)=\int\int S[n/g(xy)]dxdy, \quad \text{(A.8)}$$

where the integral extends over the whole xy plane.

We now make the approximation (A.7) and introduce the notation

$$\rho\equiv n/2lT \quad \text{(A.9)}$$

for the average particle density through the chamber. We then get

$$W(\rho,0)\cong\int_0^\infty S[\rho/f(R)]A(R)dR, \quad \text{(A.10)}$$

where $A(R)dR$ is the area in the xy plane enclosed between R and $R+dR$. A lengthy but elementary calculation gives the result

$$A(R)dR=w(R/l)2\pi RdR, \quad \text{(A.11)}$$

where the weight function w is given by

$$w(x)=[1+\tfrac{1}{2}\sinh^2(1/x)]\cdot[x\sinh(1/x)]^{-3}. \quad \text{(A.12)}$$

The weight function $w(x)$ is shown graphically in Fig. 9. We see that $w(x)\rightarrow 0$ as $x\rightarrow 0$. This cuts out the influence of the very small showers, as expected.

We proceed to calculate the correction ΔW to the counting rate for zero separation. We define

$$\Delta W=[W(\rho,0)]_l-[W(\rho,0)]_{l=0}. \quad \text{(A.13)}$$

We observe that we can obtain the limiting case of a point chamber from (A.10) and (A.11) by letting l go to zero (this is of course obvious from the start), in which case the weight function w becomes simply unity. We therefore have

$$\Delta W\cong\int_0^\infty S[\rho/f(r)][w(R/l)-1]2\pi RdR. \quad \text{(A.14)}$$

We remark that the integral extends from zero to infinity only formally. Practically, the factor $[w(R/l)-1]$ approaches zero for $R>l$ quite rapidly. Indeed, $w(R/l)=1+(7/120)(l/R)^4+\cdots$ for $R>l$. Hence, the main contribution to (A.14) comes from the region $R\lesssim\frac{3}{2}l$. For the chambers used $\frac{3}{2}l$ is about 0.40 meter, i.e., $4\times 10^{-3}r_1$ at Climax.

The asymptotic expression (4.4) for $f(r)$ is good out to r of order $10^{-2}r_1$. Hence, it is permissible to replace $f(r)$ by its asymptotic form (4.4) in the integral (A.14) for ΔW.

We again assume a power law for $S(N)$, i.e., $S(N)=\sigma N^{-\gamma}$. We then get

$$\Delta W=\rho^{-\gamma}\Delta V;$$

$$\Delta V\cong\sigma(C/r_1)^\gamma\int_0^\infty R^{-\gamma}[w(R/l)-1]2\pi RdR, \quad \text{(A.15)}$$

We can introduce $x=R/l$ as variable of integration, getting the *final result*

$$\Delta V\cong\sigma(C/r_1)^\gamma l^{2-\gamma}\int_0^\infty x^{-\gamma}[w(x)-1]2\pi xdx. \quad \text{(A.16)}$$

This formula for the correction has the advantage that it gives the dependence of ΔV upon r_1 and l explicitly. The dependence upon γ is not quite explicit, part of it being contained in the definite integral. The integral is a smooth function of γ, however. In Table III we give values of the integral for various values of γ. The integral is negative, indicating that the correction is in such a direction as to decrease the counting rate for a single chamber of finite size compared to a point chamber. This was, of course, to be expected.

APPENDIX B

The Counting Rate of the Triangle Arrangement

In order to check the Molière structure function experimentally, three of the four ionization chambers were placed

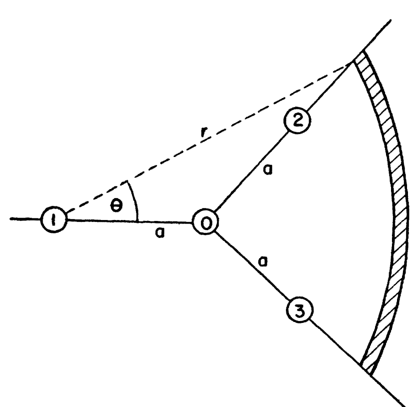

FIG. 10. A shower whose core passes through the shaded circular strip will be counted if it has sufficient density at the position of chamber 1.

TABLE III.

γ	$\int_0^\infty x^{-\gamma}[w(x)-1]2\pi x dx$
1.50	−2.08
1.60	−5.84
1.70	−10.42

at the corners of an equilateral triangle, with the fourth chamber in the center of the triangle.

We shall not discuss here the method used to test the validity of the Molière function. Suffice it to say that the Molière function agrees with experiment within the (rather large) experimental error, for the distances used (the side-length of the equilateral triangle was 12.2 meters).

We merely intend to derive the formula for the absolute counting rate of this arrangement. Consider Fig. 10. If all four chambers are biased to the same minimum density ρ, the showers whose core passes through the shaded circular strip will be counted if they have sufficient strength to set off chamber number one. This is true since chambers two, three, and four are nearer than chamber one for any point of the strip. The strip is defined by segments of circles centered at chamber number one, with radii between r and $r+dr$. The area of this strip is $2\theta rdr$, where θ is the angle indicated on the figure. The sine law of trigonometry gives the identity

$$r\sin((\pi/3)-\theta)=a\sin(2\pi/3),$$
$$\theta=(\pi/3)-\text{arc}\sin[(\sqrt{3}/2)(a/r)]. \qquad \text{(B.1)}$$

If the core of a shower hits within the shaded area, it will be recorded provided its total number of particles N is big enough to give a sufficient density ρ at chamber one, a distance r away:

$$N\geqslant\rho/f(r)$$

The number of such showers per unit area per second is $S(N)=S[\rho/f(r)]$. We cover one-third of the plane by letting r vary from a to ∞. We therefore have for the absolute counting rate $W(\rho, a)$ of the triangle arrangement

$$W(\rho,a)=\int_a^\infty S[\rho/f(r)]\cdot\left[2\pi-6\,\text{arc}\sin\left(\frac{\sqrt{3}}{2}\frac{a}{r}\right)\right]rdr. \qquad \text{(B.2)}$$

We again assume a power law for $S(N)$, $S(N)=\sigma N^{-\gamma}$; this gives

$$W(\rho,a)=\rho^{-\gamma}V(a),$$
$$V(a)=\sigma\int_a^\infty f^\gamma(r)\left[2\pi-6\,\text{arc}\sin\left(\frac{\sqrt{3}}{2}\frac{a}{r}\right)\right]rdr. \qquad \text{(B.3)}$$

The integral has been evaluated numerically, for $\gamma=1.9$.

The observed counting rate was 47 counts in 325 hours, at a minimum density $\rho=616$ particles/m². If we assume a single power law for $S(N)$ with exponent $\gamma=1.9$ (this agrees with the density response curve at the distances involved; see Table I), we obtain the following result for the absolute number of showers per square meter per hour, containing more than N particles, at Climax, Colorado:

$$S(N)=8.4\times10^{-4}(N/10^6)^{-1.9}(\text{meter})^{-2}(\text{hour})^{-1}. \qquad \text{(B.4)}$$

We have seen before (Section 5) that a single power law is not expected to be a good approximation to $S(N)$ for all values of N. The values of the exponents γ_{eff} at various distances (see Table I) make it appear reasonable to assume

$$S(N)=\begin{cases}\sigma_0 N^{-1.5} \text{ for } N<10^6,\\ \sigma_1 N^{-1.9} \text{ for } N>10^6,\end{cases} \qquad \text{(B.5)}$$

where σ_0, σ_1 are adjusted so as to make $S(N)$ a continuous function. The subsequent analysis is exactly analogous to that of Section 5. The result is:

$$\begin{aligned}S(N)&=8.4\times10^{-4}(N/10^6)^{-1.5}(\text{meter})^{-2}(\text{hour})^{-1},\ N<10^6\\ S(N)&=8.4\times10^{-4}(N/10^6)^{-1.9}(\text{meter})^{-2}(\text{hour})^{-1},\ N>10^6.\end{aligned} \qquad \text{(B.6)}$$

The coefficient is the same in (B.4) and (B.6), indicating that the showers with less than 10^6 particles contribute a very small part (actually about 5 percent) of the total counting rate.

(B.6) is directly comparable with (5.8). We see that there is a discrepancy of slightly more than a factor of two. Since the experimental inaccuracies cannot account for such a large error, this discrepancy must be ascribed to the rough nature of the theory used here.

The discrepancy is probably due to two causes: (1) The Molière structure function is too spread out; the true structure function, being more concentrated, will lead to a decoherence curve falling off more rapidly with distance. Since the measurements with the triangle arrangement count showers at a larger effective distance this will improve the comparison between (B.6) and (5.8). The fact that the triangle arrangement was used to count showers with greater minimum particle densities will also work in the same direction. (2) It is rather likely that $S(N)$ of (B.6), does not vary rapidly enough for very large N. The effective exponent for very large N is likely to be considerably greater than 1.9. A qualitative argument shows that this effect will also improve the agreement between (B.6) and (5.8).

Whether these two effects are sufficient to remove the discrepancy altogether cannot be determined at this time. Without a knowledge of the true structure function, one cannot even make a sensible estimate.