Quantum-Theory Restrictions on the General Theory of Relativity

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The limitations set by the uncertainty principle on the measurement of the curvature of space are determined by passing a small test particle around a geodesic triangle. In measuring the curvature of space around a Schwarzschild-solution particle of finite size and physically realized density, it is found that the mass must be at least of the order of 10^4 kilograms, or the curvatures cannot be measured with an accuracy equal to the order of magnitude of the terms in the defining Eqs. $G_{ik}=0$. For mass points, the lower limit is 10^{-5} g. For smaller masses, the curvatures can only be measured with less accuracy and only over large regions of space. Similar limitations apply to alternative laws of gravitation involving higher derivatives of the metric. It is concluded that in any theory which attempts to unite quantum theory with the general theory of relativity, the relation of the metric to the energy momentum tensor, $G_{ik}-g_{ik}G/2=-KT_{ik}$, must appear only in the large and in a statistical sense, i.e., for large regions of space and large numbers of elementary particles.

In this paper we endeavor to answer the question: What restrictions does the uncertainty principle place on the measurement of the curvature of space, and thus on the conceptions of the general theory of relativity? We shall do this by considering the definition of curvature as the ratio of the sum of the angles of a geodesic triangle minus π to its area in the limit of small area, and then determine the errors in measurement of angles and area when a small test particle or light quantum is passed around the triangle.

The basic measurements (to obtain the metric) of general relativity are coordinate and time. The basic measurements of quantum theory are coordinate and momentum. It is just the limitation of quantum theory on general relativity that coordinate and time cannot be determined alone, but must be tied up with a measurement of momentum or energy, which in turn are restricted by the uncertainty principle. As will be seen, this consideration becomes of especially simple and classical interpretation when one measures curvature in a time-like or xt plane.

We first recall the expressions for the curvatures of a space. The Riemannian curvature of a space, or product of the principal curvatures (Gauss curvature) of the surface formed by geodesics through a linear combination of two orthogonal directions at a point, is given by

$$M_{mn}=R_{ijkl}e_m{}^i e_n{}^j e_m{}^k e_n{}^l, \tag{1}$$

where R_{ijkl} is the Riemann-Christoffel tensor and $e_m{}^i$, $e_n{}^i$ are the contravariant components of the two orthogonal unit vectors $\mathbf{e}_m$, $\mathbf{e}_n$ which determine the surface.[1] The Riemannian curvature at a point is also definable as the ratio, in the limit of a small area, of the difference from π of the sum of the angles of a geodesic triangle to its area.

[1] C. E. Weatherburn, *Riemannian Geometry and Tensor Calculus* (Cambridge University Press, London, 1938), p. 119.

The sum of the Riemannian curvatures for all pairs of orthogonal directions formed with one given direction, is the mean curvature $\bar{M}_n$, given by

$$\bar{M}_n=\textstyle\sum_m M_{mn}=-G_{ik}e_n{}^i e_m{}^k, \tag{2}$$

where G_{ik} is the Ricci-Einstein tensor.

The scalar or total curvature is the sum of all the mean curvatures

$$(M)_{\text{sc.}}=\textstyle\sum_n \bar{M}_n=-G_{ik}g^{ik}. \tag{3}$$

We now ask what is the order of magnitude of these curvatures in a case of physical interest. To answer this question we observe that if one writes the Ricci-Einstein tensor in mixed form in (2), that is,

$$\bar{M}_n=-G_k{}^i e_n{}^k e_{ni}, \tag{4}$$

the dimensionality of the unit vectors cancels out. The same is true if (1) is written in mixed form. Furthermore, if the coordinate system and field is such that $G_i{}^k$ has only diagonal elements, these elements are the mean curvatures (ignoring sign) since the sum in (4) reduces to one term. Also, if $G_i{}^k$ is diagonal and expressed as a sum of terms, the individual terms are of the order of the Riemannian curvatures, or the elements of the Riemann-Christoffel tensor.

Exactly the conditions stated above are met in the fundamental example of greatest interest, the Schwarzschild solution of the equations of general relativity. Thus, if the metric is

$$ds^2=-\exp(\lambda)dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+\exp(\nu)dt^2,$$

we easily find[2] for the $G_i{}^k$

$$\begin{aligned} G_1{}^1&=-\exp(-\lambda)(\nu''/2-\lambda'\nu'/4+\nu'^2/4-\lambda'/r),\\ G_2{}^2&=G_3{}^3=-\exp(-\lambda)(1+r(\nu'-\lambda')/2)/r^2+1/r^2,\\ G_4{}^4&=\exp(-\lambda)(-\nu''/2+\lambda'\nu'/4-\nu'^2/4-\nu'/r), \end{aligned} \tag{5}$$

[2] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, London, 1923), p. 85. Also R. C. Tolman, *Relativity Thermodynamics and Cosmology* (Oxford University Press, London, 1934), p. 203.

and the Riemannian curvatures can be calculated to be

$$\begin{aligned} M_{12}&=M_{13}=-\exp(-\lambda)\lambda'/2r,\\ M_{14}&=\exp(-\lambda)(\nu''/2+\nu'^2/4-\nu'\lambda'/4),\\ M_{23}&=-(1-\exp(-\lambda))/r^2,\\ M_{24}&=M_{34}=\exp(-\lambda)\nu'/2r. \end{aligned} \tag{6}$$

To relate (5) and (6), using (2), one must use for directions along the three space-like axis, unit vectors whose co- and contravariant components are imaginary and of opposite sign. Thus $e_i=\delta_i{}^j g_{jj}{}^{\frac{1}{2}}$, $e^i=g^{ii}e_j$. This requirement is dictated by the use of an indefinite metric, so that lowering or raising subscripts changes signs along the first three axes, but the magnitude of unit vectors must be positive. (See reference 1, Eqs. (24)–(27), p. 46.)

It is known that the solution of Eqs. $G_i{}^k=0$ for λ, and ν is given by $\exp(\nu)=\exp(-\lambda)=1-2M/r$, and taking advantage of the fact that λ and ν are small, we have that

$$\nu''=2\nu/r^2,\quad \nu'=-\nu/r,\quad -\lambda=\nu=-2M/r.$$

This shows at once that there are two different orders of magnitude of terms which appear in (5) and (6),

$$\nu/r^2,\quad \nu^2/r^2, \tag{7}$$

or in c.g.s. units[3]

$$(GM/c^2r)(1/r^2),\quad (GM/c^2r)^2(1/r^2). \tag{8}$$

Terms of order $1/r^2$ also appear but always in combinations of order $(\exp(\nu)-1)/r^2\simeq\nu/r^2$. That there can be no terms $1/r^2$ directly in the Riemann-Christoffel tensor follows from the fact that for flat space and any coordinate system $R_{ijkl}\equiv 0$. Equations (7) and (8) give the orders of magnitude by which space, according to the general theory of relativity, differs from flatness. All of the curvatures in (6) are of the first order of magnitude. The second order of magnitude also appears directly in M_{14}.

Evidently for a complete determination of the curvature of space by triangulation, the curvature of the space is completely determined only if the error of observation is smaller than the second or smaller order of magnitude. We understand by "completely determined," a determination to the same accuracy as the defining differential equation $G_i{}^k=0$. Otherwise, the field will be incompletely determined. The curvature of a space is an observable in the sense of quantum mechanics if the errors, imposed by the uncertainty principle involved in determining it by triangulation with a test particle, are less than the smaller of the quantities (8). Otherwise, the curvature is unobservable or perhaps incompletely an observable.

We consider first the case that the quantities (8) are in decreasing order of magnitude, as will be the case so long as

$$(GM/c^2r)<1, \tag{9}$$

and we shall find that instructive information is obtained by considering the restrictions imposed by having the errors of observation separately less than these two different orders of magnitude.

Consider the errors imposed by the uncertainty principle on the measurement of curvature by passing a particle around a geodesic triangle, and measuring its angles and area. If the length of the side of the triangle is q, the position of the test particle will be uncertain within a band of width δq around the edges of the triangle. Hence, the uncertainty of the angle measurements will be of order $\delta q/q$. The curvature is

$$\kappa=\left(\sum_{i=1}^{3}\varphi_i-\pi\right)\Big/\text{area of triangle.} \tag{10}$$

If the area S is $\sim q^2$, the uncertainty δS in the area $\sim q\delta q$. Therefore, the uncertainty in the curvature is

$$\begin{aligned}\delta\kappa&=[\sum(\delta\varphi_i/S)^2+((\sum\varphi_i-\pi)\delta S/S^2)^2]^{\frac{1}{2}}\\ &\simeq\delta q/q^3.\end{aligned} \tag{11}$$

This expression could also have been written down almost from inspection, on observing that the principal uncertainty in the curvature is due to the angle measurements, if the area of the band of uncertainty around the triangle is small compared to the total area of the triangle.

In the case that the triangle lies in the xt plane, the triangle is spanned by sending the test particle out along the axis and return (path ADA in Fig. 1), and measuring the time interval consumed. The angles are determined by velocity component (cf. momentum) measurements at the beginning and end of each side and these converted to angle measurements, using a cartesian-polar coordinate transformation. Imaginary units for t are used in the case of triangulation in the xt plane.

For the xt plane one has to distinguish between two cases, whether the velocity of the test particle is very much less than, or of order of, the velocity of light. For $v\ll c$ the area of the triangle ABC, Fig. 2, is of order qct, the uncertainty in the angles is $\sim\delta q/ct$. This follows from the fact that all the sides are of length $\sim ct$. Then the uncertainty in the curvature is $\sim\delta q/c^2t^2q$. For $v\sim c$ the area of the triangle ADC is $\sim c^2t^2$ and the uncertainty in angle is $\sim\delta q/ct$, so that the uncertainty in curvature $\sim\delta q/c^3t^3$.

We now have the following restrictions on the measurements of the test particle:

(A) The restriction of the uncertainty principle,

$$\delta p\delta q\gtrsim h. \tag{12}$$

[3] See reference 2, p. 202.

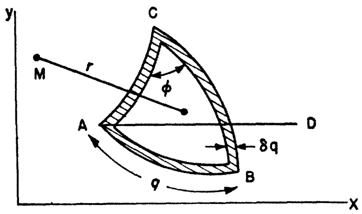
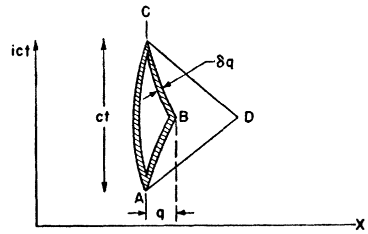


FIG. 1. Geodesic triangle to determine curvature of space at a distance r from a Schwarzschild particle of mass M.

FIG. 2. Triangulation in the xt plane.



(B) A restriction in virtue of the fact that the basic measurements of relativity x and t (rods and clocks) and those of quantum theory (coordinate q and momentum p) are not independent. For the case that x and q are the same, we have

$$q = pt/m. \tag{13}$$

In the case that coordinates are deduced from direct measurements of momentum and time (as would be the case if one combined the momentum measurements made at the vertices of the triangles with time interval measurements, to determine angles and lengths of sides), Eq. (13) gives

$$\delta q/q \geq \delta p/p. \tag{14}$$

When the inequality sign is used, (14) can be roughly interpreted as a closure condition. This means that when the test particle is projected on the third side of the triangle, it will not return to its starting point or even remain in the plane of the first two sides within an error δq, unless the uncertainty in angle $\delta q/q$ is greater than the relative uncertainty in the momentum $\delta p/p$. We say "roughly" interpreted since p and δp, q and δq are not parallel but rather refer to average measurements at the three vertices of the triangle.

(C) The restriction

$$q \gg h/p. \tag{15}$$

This requires the triangle to be much more than one de Broglie wave-length on a side. Otherwise, the region of observation is not even defined to be a triangle as opposed to any other geometrical figure.

(D) The condition $q \ll r$. This requires that the dimensions of the triangle must be small compared to the distance to the particle whose field we are measuring. Otherwise one cannot properly speak of the measurement of the field $G_k{}^i$ at a point (actually average over a region of dimension q).

(E) The condition

$$mc^2 \ll Mc^2, \tag{16}$$

where m is the *total* mass of the test particle. This condition states that the gravitational effect of the test particle including its kinetic energy must be small compared to that of the Schwarzschild particle of rest mass M being measured. This condition, whether the test particle has a finite rest mass or is a light quantum, implies

$$p \ll Mc, \tag{17}$$

since for

$$v \ll c, \quad m \ll M, \quad p = mv \ll Mc,$$

or, if

$$v \sim c, \quad pc \sim mc^2 \ll Mc^2.$$

The form (16) will be useful for slow particles in measuring in the xt plane, otherwise the form (17) will be used.

Summarizing, we have that the error in measurement of the Riemannian curvature is $\delta\kappa \sim \delta q/q^3$ for curvatures in a space-like plane, and $\delta q/c^2t^2$, or $\delta q/c^3t^3$ for curvatures in a time-like plane as measured by slow ($v \ll c$) or fast ($v \sim c$) particles, respectively.

We have the following restrictions on the measurements

$$\begin{aligned} &\delta p \delta q \gtrsim h, && \text{(a)} \\ &\delta q/q \geq \delta p/p, && \text{(b)} \\ &q \gg h/p, && \text{(c)} \\ &q \ll r, && \text{(d)} \\ &m \ll M, \quad \text{or} \quad p \ll Mc. && \text{(e)} \end{aligned} \tag{18}$$

The first form of (e) implies the second, but not conversely.

We now ask what is implied by the condition that $\delta\kappa$ be much less than the larger of the two orders of magnitude (8) of the curvature itself. This requires, for a space-like section,

$$\delta q/q^3 \ll (GM/c^2r)(1/r^2). \tag{19}$$

Using (18) to eliminate δq, δp, q, p, r we find

$$(ch/GM^2) \ll 1. \tag{20}$$

Putting in numerical values for the physical constants, we find $M \gg 10^{-5}$ g. Thus 10^{-5} g is a lower limit for mass points, the space-like part of whose metric is observably different from flatness. It is also to be noted that once the critical mass is exceeded, the curvatures of this order of magnitude are *everywhere* defined. This is a consequence of the fact that q and r appear to the same power on both sides of (19) so that 18 (d) can eliminate them both at once. Physically, it means that the side of the measurement triangle q, and δq may be increased in just such a way, as the curvature decreases with increasing r, to make accurate observations always possible.

Let us now examine the restriction imposed by the second order of magnitude, for curvatures in space-like section. In this case we have

$$\delta q/q^3 \ll (GM/c^2r)^2(1/r^2), \tag{21}$$

which, when combined with (18), eventually gives

$$(c^2r/GM)(ch/GM^2)\ll 1. \tag{22}$$

When solved for r, (22) states that only within a sphere around the particle of the order of this radius is the curvature of space, or its departure from flatness, observable to the accuracy of the defining Eqs. $G_i^k=0$. If we combine this restriction with accepted possible values for density and ask for the limiting mass just outside of which the departure of the metric from flatness is completely observable, we find a lower limit of mass much greater than that obtained from the first order of magnitude. Let ρ be the density and r_0 a characteristic dimension of the particle. Then $M\simeq\rho r_0^3$, and for the curvature to be just completely observable at the surface $(r=r_0)$ of the particle, we must have

$$(M/\rho)^{\frac{1}{3}}\ll G^2M^3/c^3h, \quad M\gg 10^7/\rho^{1/8}\text{ g}. \tag{23}$$

This is a quite severe restriction on the mass. As the density varies from 10^{-26} (density of interstellar space) to 10^{15} (density of nuclear matter) the limiting mass which can be thought to bend space definitively just outside itself varies from 10^{10} to 10^5 g.

One obvious implication of the restrictions above is that any relation between the metric of space and the tensor of matter, such as $G_{ik}-g_{ik}G/2=-KT_{ik}$ must hold only in the statistical sense, and fail for elementary particles. Thus one should expect that in any unified field theory which attempts to unite quantum theory with the general theory of relativity, the curvature of space should arise as a statistical concept valid only for very large numbers of particles, since the above discussion shows that the curvature is simply not defined in the sense of measurements on elementary particles. A related viewpoint, which was perhaps first suggested by Mach and has also been discussed by Einstein,[4] is that the conception of mass requires the presence of other very large masses. One can have equivalence between inertial and gravitational mass, but the latter conception is simply not defined except in the presence of large masses.

Let us now consider the case where we carry out the triangulation in an xt plane. We must then have, in order to measure the curvature to first order for $v\ll c$,

$$\delta q/c^2t^2q\ll(GM/c^2r)(1/r^2). \tag{24}$$

Eliminating t with (13), and using (18), one finds

$$h^2/GM^3r\ll 1, \tag{25}$$

or $r\gg h^2/GM^3$. In a time-like section, curvatures are defined to the first order only *outside* a sphere of this radius. If we express the inequality (25) in terms of the density ρ, as in (23), we can find a lower limit to the mass *everywhere* outside of which the curvature of a time-like section is defined to the first order. This is

$$M\gg(h^2\rho/G)^{3/10}, \tag{26}$$

or

$$M\gg 10^{-14}\rho^{3/10}. \tag{27}$$

Thus the lower limiting mass for this condition *increases* with increasing density. However, it should be noted that the restriction only applies for test particles of $v\ll c$, and so is not a completely binding one. In all cases the lower limit for the mass is exceedingly small, but is again, either for atomic or nuclear densities, much greater than the mass of one elementary particle. This conclusion, at first rather surprising, is still in keeping with the statistical point of view mentioned above, since, if the probability distribution (cf. ρ) of a particle of any mass is distributed thinly enough (i.e. over a large enough volume), everywhere *outside* this distribution it has (as shown below) an observable Newtonian gravitational field.

The inequality (25), which is independent of c, can also be deduced from the problem of measuring the classical gravitational attraction of a particle of mass M with a test particle of mass m by measuring the momentum change Δp_g when m passes M at a distance r. If one assumes the force $f\simeq GMm/r^2$ acts for a time $t\sim rm/p$ and imposes the condition (1), the change in momentum Δp_g, due to gravitational attraction, must be larger than the uncertainty in momentum Δp_u arising from the uncertainty principle, (2) $\delta q\ll r$, (3) $m<M$, (4) $\delta p\delta q\gtrsim h$, (5) $r\gg h/p$, exactly the inequality (25) is obtained. Thus the measurement of curvature to first order in the xt plane with slow particles is equivalent to the measurement of a classical gravitational potential.

While the above restriction is less severe than that obtained for the curvatures in a space-like plane, it does suggest that for elementary particles $(M\simeq 10^{-24})$ even the classical gravitational field is, in practice, not defined, since the distance and time required for the measurement is greater than the volume of the universe or its history can provide.

If we measure curvature in the xt plane with a particle of $v\sim c$, we must have for first order accuracy, $\delta q/c^3t^3\ll(GM/c^2r)(1/r^2)$. Evidently this is exactly the same condition (19) obtained previously and leads to nothing new, since $ct\sim q$ for the triangle ADC in Fig. 2.

[4] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1945), p. 99 ff.

For second order accuracy for slow particles in an xt plane, we have

$$\delta q/c^2t^2q \ll (GM/c^2r)^2(1/r^2), \tag{28}$$

and using (13) and (18), we find $ch/GM^2 \ll 1$. This is the same as (20) for first order accuracy in an xy plane, so that this situation leads to nothing new. Also, nothing new is obtained by considering second order accuracy for particles with $v \sim c$ as we again obtain (21).

We can summarize all of the above by showing that they follow from the two basic inequalities (9) and (22):

$$(GM/c^2r) < 1, \tag{29}$$

$$(c^2r/GM)(ch/GM^2) \ll 1. \tag{30}$$

Multiplying (30) by successive powers of (29) we find a series of inequalities, (30) to (34), each of which implies, with (29), or (30) all those that follow it.

$$ch/GM^2 \ll 1, \tag{31}$$

$$h/Mcr \ll 1, \tag{32}$$

$$Gh/c^3r^2 \ll 1. \tag{33}$$

Equation (31) is the same as (20), as the condition for measuring curvature to the first order. (32) states that the field of a mass point is defined only for distances greater than its Compton wave-length, in practice not a very severe restriction. (33) gives a lower limit to the distance of definition independent of the mass, which is also exceedingly small.

Equations (29), (31) to (33) are respectively independent of h, r, G, and M. A fifth inequality independent of c can be obtained by multiplying (33) and (30).

$$h^2/GrM^3 \ll 1. \tag{34}$$

This is the same as (25), for measuring Newtonian potentials, or curvatures in the xt plane with slow particles.

We can discuss here a question raised by Eddington.[5] How shall we choose for the law of gravitation between different tensors whose divergence vanishes identically? $G_k{}^i$ involves only the second derivatives of the metric, hence, to measure it completely, only second order accuracy is required. Other possible tensors involve higher derivatives. For nth order accuracy, in triangulating to obtain the curvature, it follows from (7) and (8) that one must have

$$\delta q/q^3 < (GM/c^2r)^n(1/r^2), \tag{35}$$

which leads to

$$(c^2r/GM)^{n-1}(ch/GM^2) < 1, \tag{36}$$

$n=1$, 2 giving just (31) and (30).

[5] See reference 2, pp. 141–143.

For $n=3$, as required by Eddington's alternative, and introducing the density from $r_0^3 \sim M/\rho$, we find, for third order accuracy, in the curvature to be measurable just outside a particle of mass M

$$(c^2M^{\frac{1}{3}}/GM\rho^{\frac{1}{3}})^2(ch/GM^2) \ll 1,$$
$$M \gg 10^{14}/\rho^{1/5}\text{ g}. \tag{37}$$

From this, one can conclude that the distinction between the different proposals of Eddington for the gravitational law could not be measured, save for extremely large masses.

All of the above discussion was predicated on the assumption that $GM/c^2r < 1$. For $GM/c^2r > 1$ the orders of magnitude are reversed. This only occurs for extremely large masses and densities. Using $M \sim \rho r_0^3$, we find that for the orders of magnitude to be reversed at the surface of a particle of mass M,

$$M > c^3/G^{\frac{3}{2}}\rho^{\frac{1}{2}}, \quad M > 2\cdot 10^{42}/\rho^{\frac{1}{2}}. \tag{38}$$

Such a situation is realized, if ever, only in the initial stages of an expanding universe.

In closing this discussion we may consider the case where curvature is *given*, say that for some very large mass, and we ask for the relation between the size of the triangle and the properties of the test particle in order that this given curvature may be measured. If $1/R^2$ is the given Gauss curvature, we must have

$$\delta q/q^3 \ll 1/R^2, \tag{39}$$

and using the first three of the inequalities (18) (the remaining ones are not applicable), we find

$$h/pq^3 \ll 1/R^2. \tag{40}$$

Evidently this inequality can always be satisfied by making p and q large, but it is instructive to consider a few examples. Suppose we are triangulating in the neighborhood of the sun with optical light $h/p = \lambda \sim 10^{-5}$ cm. For the first order, $M = 2\cdot 10^{33}$ g, $r = 7\cdot 10^{10}$ cm,

$$1/R^2 \simeq GM/c^2r^3 \simeq 10^{-27}\text{ cm}^{-2}. \tag{41}$$

It is essentially a curvature of this order of magnitude which is measured by an Einstein light deflection experiment. Equation (40) gives, with $h/p = 10^{-5}$ cm,

$$q \gg 10^7\text{ cm} \tag{42}$$

as the side of the measurement triangle.

For a particle of mass one gram moving with a velocity of one centimeter per second, one finds the side of the triangle must be $q \gg 1$ cm.

As a further example, consider the measurement of the radius of curvature of the universe. This is certainly not less than 10^{28} cm, so we have

$$h/pq^3 \ll 1/R^2 \simeq 10^{-56}. \tag{43}$$

For optical light $\lambda \sim 10^{-5}$ we find

$$q \gg 10^{17}\ \text{cm} \sim 0.1\ \text{light year}. \tag{44}$$

The important conclusion from these examples is not that a given curvature can always be measured somehow, but that (a) the curvature is defined (in the sense of the limitations of quantum-theory measurement) only in the *large*, and (b) the domain of largeness is fundamentally determined by the momentum of the test particle with which the curvature is measured. Here we are led again to the idea that the conception of curvatures and, when these are equated in the form $G_{ik}+\frac{1}{2}g_{ik}G$ to the stress tensor KT_{ik}, the conceptions of energy, mass, and momentum, are only defined for quite large masses and large volumes of space.

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Theoretical Analysis of Ionization Chamber Data on Large Air Showers*

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The decoherence curve of large air showers is analyzed under the simplifying assumption of a constant lateral structure function for all air showers. The following results emerge: (1) Contrary to statements found in the literature, ordinary shower theory leads one to expect a rise in the decoherence curve at distances much less than the characteristic lateral distance r_1. (2) The theoretical decoherence curve calculated under the assumption of a constant structure function rises *more* sharply near the origin than the experimental points. This indicates that the effective structure function is less peaked than the one we used (due to Molière). (3) The dependence of the structure function upon the age of the shower tends to lessen this discrepancy. However, quantitative estimates make it appear doubtful that one can get agreement between theory and experiment without assuming a rather high multiplicity of the event which starts the shower.

(1) QUALITATIVE CONSIDERATIONS CONCERNING THE LATERAL STRUCTURE OF AIR SHOWERS

THE experiments which are to be interpreted here are connected with the so-called *decoherence curve* of large air showers. Two ionization chambers are placed a distance $2a$ apart. The chambers are biased so that they only respond if more than a certain amount of ionization is produced in each chamber, the bias being the same for both chambers. If the dimensions of the chambers can be neglected compared with their separation, the bias can be interpreted as meaning that each chamber responds only if the *density* of shower electrons passing through it is greater than a certain minimum amount, which we call ρ.

One then measures the coincidence counting rate W as a function of this minimum density ρ and of the half-distance a between the chambers. *A curve of $W(\rho, a)$ vs. a (keeping the bias constant) is called a decoherence curve.* We shall call a curve of $W(\rho, a)$ *vs.* ρ (keeping the separation a constant) a *density response curve.*

Historically, decoherence curves were measured first with Geiger-Müller counters.[1] It can be shown that a set of Geiger-Müller counter measurements giving the coincidence rate as a function of the distance between the counter trays and of the area of the counter trays (keeping the number of counters in each tray constant) is mathematically equivalent to a set of ionization chamber data, giving their coincidence rate as a function of the distance between the chambers and of the chamber bias. However, ionization chamber data are much preferable for the following reasons:

(1) A single set of measurements, in which pulses of different sizes are recorded, is required instead of a large number of measurements with counter trays of different areas.

(2) In Geiger-Müller counter measurements, there often exists the possibility of getting spurious counts from a single particle traveling horizontally, since one particle can easily produce a pulse. In ionization chamber measurements, ten to twenty particles must pass through the chamber before a pulse is recorded; hence, there are no spurious counts due to that source.

(3) For the same reason, statistical fluctuations are much less important in ionization chamber data.

(4) The mathematical analysis is incomparably simpler.

In discussing large air showers, we shall consider only the electron-photon component, since this component accounts for most of the ionizing radiation observed without heavy shielding on top of the detecting equipment. There seems to be reason to believe[2] that there are on the average about fifty electrons for every heavy ionizing particle in big air showers. (The word "electron" is used for both negatrons and positrons.)

We assume that the shower is started by an "initiating electron" near the top of the atmosphere.

* Assisted by the joint program of the ONR and the AEC.

[1] Auger, Maze, Ehrenfest, and Freon, J. de phys. et rad. 1, 39 (1939).

[2] G. Cocconi and K. Greisen, Phys. Rev. 74, 62 (1948); J. E. Treat and K. Greisen, Phys. Rev. 74, 414 (1948).