On the Origin of Solar Radio Noise

ANDREW V. HAEFF
Naval Research Laboratory, Washington, D. C.
(Received November 22, 1948)

The observed anomalous radio-frequency radiations from the sun are associated with sun spot activity and are believed to be generated within intermingling streams of charged particles issuing from active areas of the sun. Such streams have the property of greatly amplifying initial space-charge fluctuations over a range of frequencies determined by the density and velocity distribution of particles in the stream. The theory of generation of radio energy resulting from space-charge interaction between streams of charged particles is reviewed and applied to the solution of the solar radio noise problem. From estimates of average density and velocity distribution of solar particles the frequency of the most intense radiation (30 to 60 megacycles) and the absolute value of radiation intensity at the surface of the earth (7 to 2)$\times 10^{-22}$(watt/cm^2/cycle/sec.) are computed and found to agree well with measurements. The most probable spectral distribution of the anomalous solar radiation is derived in the form

$$E/E_m = (\lambda/\lambda_m)^2 \exp\{2[1-(\lambda/\lambda_m)]\},$$

where E_m is the maximum intensity corresponding to the wave-length λ_m.

I. INTRODUCTION

RECENT studies of solar radiation by means of radio telescopes[1] indicate that in addition to a steady component corresponding to radiation from a black body at a temperature of about 6000°K, there occur bursts of intense radiation within the radio spectrum which are associated with solar flares. While the steady radiation originates over the entire surface of the sun, the source of these anomalous bursts appears to lie in the vicinity of sun spots. The intensity of anomalous radiations, as measured at the surface of the earth, exceeds the normal radiation by a factor of 10^6 to 10^7 in the frequency region below 50 megacycles.[1]

It has been suggested by several investigators[2] that the abnormally intense radiations are caused by oscillations of the electron gas or plasma existing in the outer layers of solar atmosphere and corona. It has also been suggested that the electron plasma is excited by the passage of fact electrons or other corpulscles originating in the sun spots. It has always been assumed, however, that the frequency of such radiation must be near the natural frequency of oscillation of the electron plasma.[2] Furthermore, no detailed picture of the mechanism of excitation of the plasma has ever been given.

It is the purpose of this paper to present a description of a new mechanism of generation of radio energy which is believed to be responsible for the observed anomalous solar noise, and to interpret the observed data on the intensity of solar radiation and its spectral distribution in terms of the new theory.

[1] E. V. Appleton and J. S. Hey, Phil. Mag. **37**, 73–84 (1946).
[2] I. S. Shklovsky, Astronom. J. (U.S.S.R.) **23**, 333–347 (1946).

II. MECHANISM OF GENERATION OF RADIO ENERGY BY STREAMS OF CHARGED PARTICLES

Consider a stream of electrons of velocity v_1 injected into an electron cloud moving with velocity v_2. Because of Coulomb forces between electrons the two streams will interact. In addition to a familiar scattering due to individual electron encounters there occurs additional scattering due to interaction of electrons with electric fields associated with space-charge waves. As was shown by the author in a recent paper[3] this space-charge-wave and electron interaction is a first-order effect considerably larger than that due to "collisions" and is believed responsible for the observed abnormally high temperature of electron clouds.[4]

As is shown in reference 3, the space-charge-wave interaction in streams of charged particles results, under certain conditions, in imparting to the space occupied by the streams the characteristics of a medium having negative attenuation. This means, that under such conditions an initial perturbation which may exist in the stream (such as caused by statistical fluctuations, for example) will be amplified in an exponential manner as the disturbance propagates along the stream. The amplification process continues until the available energy is exhausted. This energy is derived from the kinetic energy of the particles in the stream so that the energy spectrum of the composite electron cloud will be substantially modified after a prolonged coexistence of streams of different energy. The kinetic energy is thus partially transformed into the energy of the electromagnetic fields associated with space-charge waves and can be observed as

[3] A. V. Haeff, Naval Research Laboratory Report No. R-3306, June 24, 1948.
[4] A. V. Haeff, Phys. Rev. **74**, 1532 (1948).

radiation emanating from the streams of charged particles.

As is shown in Appendix I which reproduces for the convenience of the reader the pertinent details of the author's previous analysis[3] of space-charge-wave and particle interaction effects, the amplitude of perturbation V at a distance z from its origin can be expressed as

$$V = V_0 \cos(\omega t - \omega z/v) \cdot (\cosh \gamma_r z + \cos \gamma_i z) \cdots \quad (1)$$

where V_0 is the initial amplitude of the disturbance of frequency ω; v is the average velocity of the inhomogeneous electron stream, and γ_r and γ_i are the real and the imaginary components of the propagation constant. When $\gamma_r z \gg 1$ the increase in energy of the original disturbance, or the energy gain, G, can be approximated as

$$G = (V/V_0)^2 = e^{2\gamma_r z} \cdots. \quad (2)$$

In the case of two-velocity streams the amplifying properties can be indicated by the value of the "inhomogeneity factor," (see Appendix I) which is defined as the ratio of the fractional velocity difference δ/v of the two components of the stream, to the fractional plasma frequency ω_1/ω, or

$$x = (\delta/v) \cdot (\omega/\omega_1) \cdots. \quad (3)$$

The real component of the propagation constant has finite value when the inhomogeneity factor lies within the range from 0 to $\sqrt{2}$.* The inhomogeneous electron gas or plasma having the inhomogeneity factor within this range will behave as a medium of negative attenuation where original perturbation can be amplified to an extent limited only by the length of the stream and the amount of energy available for the process.

Since streams of charged particles which are present in sun spots consists of electrons, alpha-particles, ionized helium atoms or other charged particles, and resemble the inhomogeneous streams discussed in the above theory it is reasonable to assume that space-charge-wave and particle interaction takes place in these streams and can be responsible for abnormal radiations characterized by energy distributions corresponding to very high temperatures of the plasma when observed over a limited range of frequencies. It is important to note that the frequency of the disturbance which can be amplified by the space-charge interaction within inhomogeneous clouds does not have to be near the plasma frequency as has been always assumed by previous investigators.[2] In fact, as long as the inhomogeneity factor does not exceed a critical value ($\sqrt{2}$ for the case $\omega_1 = \omega_2$), disturbances of all frequencies are amplified to some degree up to the limiting frequency.

Since the length, z, of the streams of solar particles along which the space-charge wave amplification can take place, is quite large, it is clear that a limit to amplification or to the maximum amount of energy that can be generated and eventually radiated will be set not by the magnitude of the attenuation constant but by the amount of available energy which can be converted from the kinetic energy of the particles in the stream into the electromagnetic energy of the field of the space-charge waves. The available energy can be estimated in the following manner. Because of collisions and space-charge wave scattering the energy spectrum of the composite stream gradually changes. After a prolonged travel of the stream a final energy distribution will be attained which can be characterized by the condition that the real component of the propagation constant becomes equal to 0, so that no more kinetic energy of the particles can be transformed into the electromagnetic energy of the space-charge wave field. As shown in Appendix II, a stream of particles having a continuous velocity distribution such that $d\omega_n^2/dv = K$ (const.) has zero value of the real component of the propagation constant. Other distributions, for example $dv/d\omega_n = 0$ also show no amplifying properties. Experimental results obtained with the Electron Wave Tubes[3] indicate that the "true" final distribution is rather complex and resembles the first-mentioned but with superposition of the Maxwellian distribution.

If it is assumed that the stream is initially composed of two components, one having a current I_1 and velocity corresponding to potential V_1, and the other a current I_2 and potential V_2, and that the final composition of the stream is characterized by the condition $d\omega_n^2/dv = \text{const.}$, then the maximum rate at which the kinetic energy of the particles in the stream can be converted into the energy of electromagnetic fields, is (see Appendix III):

$$P = \tfrac{1}{2}(V_1 - V_2) \cdot (I_1 - I_2) \cdots. \quad (4)$$

In general, it is reasonable to assume that the available energy is of the order of the initial difference of energy of the components of the stream so that

$$P \approx \Delta V \cdot I. \quad (4')$$

This energy is the total available energy so that the energy per unit frequency range will depend upon the frequency band width over which amplification takes place. As was shown by the author in reference 3, the band width (defined as the frequency range over which the amplification is within 3 decibels of the amplification at the optimum frequency) is proportional to the optimum

* A more general solution (when $\omega_1 \neq \omega_2$) gives a still wider range of the inhomogeneity factor over which amplification of energy can take place although the frequency band width is reduced. See "General Solution of the Two-Beam Electron Wave Tube Equation" by A. V. Haeff, H. Arnett, and W. Stein, submitted to the Proc. I. R. E.

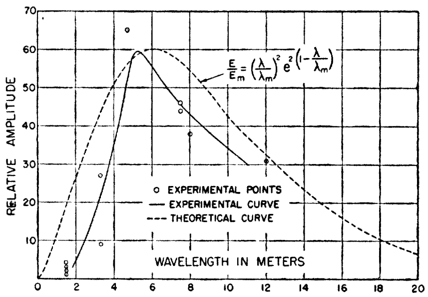

FIG. 1. Comparison of the theoretically derived spectral distribution (Formula (13)) with observations of Appleton and Hey (reference 1).

frequency and inversely proportional to the square root of the product of propagation constant and the effective length z of the stream, i.e.

$$\Delta f = f/(\gamma_r z)^{\frac{1}{2}}. \quad (5)$$

In the case of streams of solar particles, the effective length of the streams is limited by saturation effects and it can be assumed that the product $(\gamma_r z)$ remains approximately constant. Therefore, we may assume that the band width is proportional to the optimum frequency, and the power per unit band width is

$$P/\Delta f \propto P/f. \quad (6)$$

It is reasonable to assume that clouds of varying compositions are generated by the sun spot activity. Therefore, there exists a certain space and time distribution of clouds having different particle density, velocity and inhomogeneity. Each individual cloud having a particular value of the inhomogeneity factor will amplify mostly those disturbances where frequency lies in the vicinity of

$$\omega_{opt} = \tfrac{1}{2}\sqrt{3}\,\omega_1 v/\delta. \quad (7)$$

that is, for which the inhomogeneity factor is near its optimum value. The frequency at which maximum intensity of radiation from a particular cloud will be expected should correspond to the difference in velocity (2δ), to the average value of space charge density (and thus of the corresponding plasma frequency ω_1), and to the average velocity of particles (v). The over-all spectral distribution can then be determined by averaging over all clouds both in time and space.

The problem of finding the frequency at which maximum intensity of radiation is to be expected then reduces to the problem of determining the average velocity of particles, their charge to mass ratio, their density and the most probable difference in energy of the component streams.

The study of magnetic storms indicates that sun spot activity sometimes causes streams of particles originating from the surface of the sun to arrive at the earth. The magnetic field due to current generated by these streams of charged particles is responsible for variations of the earth magnetic field. From measurements of the magnitude of these magnetic disturbances it has been estimated[5] that the cross section of the streams when they arrive at the earth is limited to an area of $(100\text{–}200)^2$ kilometers and the total current is of the order of $5\cdot 10^5$ amperes. Thus the average current density is of the order of $(5-1.25)\cdot 10^{-10}$ amp./cm^2. Neglecting focusing effects of the earth magnetic field and assuming that the particles propagate from the sun in radial directions so that the particle density decreases as the inverse square of the distance from the sun, we arrive at the estimate of the average current density at the surface of the sun:

$$I_s = (92.9\cdot 10^6/0.43\cdot 10^6)^2\cdot(5-1.25)10^{-10} = (23-5.75)10^{-6}\ \text{amp./cm}^2. \quad (8)$$

It has been noted that the beginning of the magnetic storms takes place usually about one day after intense flares are observed at the sun. This indicates that the particles emanating from the sun move with an average velocity of the order of

$$v = [92.9\cdot 10^6(\text{miles})/24(\text{hours})] = 2.5\times 10^8\ \text{cm/sec}. \quad (9)$$

It has been shown that solar particles which arrive at the earth are either ionized helium or hydrogen atoms.[5] Whatever the nature of these particles it is reasonable to assume that at the origin of a corpuscular stream there existed a corresponding stream of electrons moving with the same average velocity. Assuming that the initial temperature of the particles corresponds to the observed surface temperature of the solar disk (6000°K), the most probable value of the velocity difference of the electrons would be approximately

$$2\delta \approx (3k_0T/m)^{\frac{1}{2}} = 5.4\times 10^7\ \text{cm/sec}. \quad (10)$$

where k_0 is Boltzmann's constant.

Using the above estimated values of the current density, the average velocity and the most probable value of the velocity difference we can compute from Eq. (7) the frequency at which maximum energy will be generated in such an electron cloud:**

[5] M. N. Gnevyshev, Astronom. J. (U.S.S.R.), 25, 109–122 (1948).

** It is interesting to note that the same frequency will be obtained if it is assumed that the streams consist of ionized hydrogen, for example, rather than electrons. Indeed:

$$[\omega]_{\text{ions}} = \tfrac{1}{2}\sqrt{3}(\omega)_{\text{ions}}(v/\delta)_{\text{ions}} = \tfrac{1}{2}\sqrt{3}(\omega)_{\text{electrons}}(m/M)^{\frac{1}{2}}\frac{v}{\delta_{\text{el}}(m/M)^{\frac{1}{2}}} = [\omega]_{\text{electrons}},$$

where $[\omega]_{\text{ions}}$ and $[\omega]_{\text{electrons}}$ represent optimum frequency in the case of ion or electron streams, respectively, and $(\omega)_{\text{ion}}$ and $(\omega)_{\text{el}}$ represent the corresponding plasma frequencies.

$$\omega = \tfrac{1}{2}\sqrt{3}\omega_1 v/\delta = 1.22\times 10^{14}(I_s/v)^{\frac{1}{2}}\cdot(v/\delta)$$
$$= 1.22\times 10^{14}[(23-5.75)10^6/2.5\cdot 10^8]^{\frac{1}{2}}$$
$$\times(2.5\times 10^8/2.7\times 10^7)$$
$$= (32-64)\text{ megacycles.} \quad (11)$$

Considering the uncertainties in estimates of cloud density and velocity this value checks surprisingly well with the frequency of 50 megacycles corresponding to maximum of solar radio noise as observed by Appleton and Hey.[1]

In order to estimate the time and space averaged spectral distribution of the abnormal radio noise we may proceed as follows. We assume that the average density and velocity are the same for all clouds but that different clouds have different velocity difference. The probability that a cloud has a velocity difference 2δ is $e^{-k\delta}$, and the available energy in such a cloud is proportional to δ according to the relation (4′). Since the band width is proportional to the optimum frequency f (Eq. 5), and this frequency is inversely proportional to δ, (from 7) the energy per unit frequency range will be proportional to δ^2. Since $\delta \approx 1/f \approx \lambda$ we finally arrive at the result that the energy density of abnormal noise when averaged over space and time can be represented as

$$E_\lambda = A\lambda^2 e^{-k\lambda}\cdots \quad (12)$$

where A and k are constants. By differentiation of Eq. (12), k can be expressed in terms of the wavelength λ_m at which maximum of energy density E_m is observed. Expressing the energy density at a wave-length λ in units of energy density E_m at the optimum wave-length λ_m, we obtain the following relation:

$$E/E_m = (\lambda/\lambda_m)^{\frac{1}{2}}\exp[-2(\lambda/\lambda_m - 1)]. \quad (13)$$

This relation is plotted in Fig. 1 which also shows experimental points obtained by Appleton and Hey.[1] The maximum of the theoretical curve has been made to coincide approximately with the observed value. Considering the fact that the theoretical assumption of the constancy of the average density and velocity of particles in the clouds is only a convenient approximation, and also considering the fact that the experimental data are rather incomplete, the agreement between the theory and observations is believed to be as good as can be expected at this stage of the study.

On the basis of the theory given in this paper it is also possible to estimate the absolute value of the radiation density. The power generated per square centimeter of the cross section of the solar stream can be computed from Eq. (4′):

$$P_s \approx \Delta V\cdot I = 0.8(\text{volt})(23-5.75)10^{-6}(\text{amp./cm}^2)$$
$$= (18-4.6)10^{-6}\text{ watt/cm}^2. \quad (14)$$

With the peak of intensity occurring at 50 megacycles the effective band width (taken from Fig. 1) is about 106 megacycles so that the power density (per cycle per second) E_s at the active area of sun is

$$E_s = P_s/\Delta f = (17-4.3)10^{-14}\text{ watt/cm}^2\text{/cycle/sec.} \quad (15)$$

As reported by Appleton and Hey[1] the area of the active sun spot (S) during the period of their observations was estimated to be about 2×10^{-3} of the area of the sun's hemisphere:

$$S = 2.10^{-3}\cdot 2\pi(0.432\times 10^6)^2$$
$$= 2.35\times 10^9\text{ sq. miles.} \quad (16)$$

Assuming that the whole area of the sun spot was effective in generating abnormal radiations the power density E_e at the earth's surface (a distance $R = 92.9\times 10^6$ miles away) can be easily computed to be

$$E_e = E_s S/4\pi R^2 = (17.0-4.3)10^{-14}$$
$$\times[2.35\times 10^9/4\pi(92.9\times 10^6)^2]$$
$$= (7.7-2)10^{-22}\text{ watt/cm}^2\text{/cycle/sec.} \quad (17)$$

which agrees well with the observed value of 10^{-22} watt/cm^2/cycle/sec. as measured by Appleton and Hey.[1]

Figure 2 shows the theoretical curve of the spectral distribution of the abnormal radiations as computed in this paper, (Eq. (13)) together with experimental observations in the radio spectrum and the theoretical "Black Body" solar spectrum at 6000°K.

Some recent observations by J. P. Hagen and his coworkers at the Naval Research Laboratory indicate that appreciable excess noise is generated during the existence of active sun-spots at wavelengths even below 1 cm. This indicates that particle streams of higher density than estimated in the present paper can also be produced.

Because of a rather good agreement between the theoretical results and the observed data, it is believed that a more detailed analysis of abnormal solar radiations on the basis of the present theory

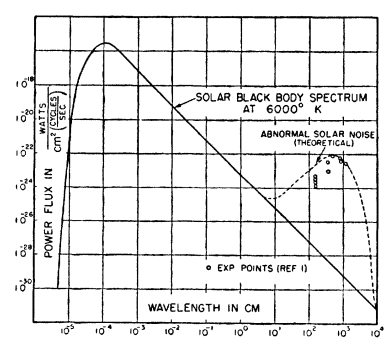

FIG. 2. Solar radiation spectrum showing a second maximum in the region of radio frequencies. These abnormal radiations are caused by streams of charged particles issuing from sun spots.

will be profitable. It is also hoped that experimental data giving instantaneous spectral distribution over wide frequency bands will be available in the future to permit a more direct analysis.

APPENDIX I.

Theory of Space-Charge Wave and Particle Interaction

Consider that n-streams of charged particles of charge densities ρ_n moving in the z-direction with velocities v_n are injected into a space common to all streams. Taking into account only first order perturbations, we define the space charge densities, currents and velocities as follows

$$\rho_n' = \rho_n + \bar{\rho}_n, \tag{1a}$$

$$v_n' = v_n + \bar{v}_n, \tag{2a}$$

$$i_n' = i_n + \bar{i}_n = i_n + \bar{\rho}_n v_n + \bar{v}_n \rho_n. \tag{3a}$$

The first symbols represent average or d.c. quantities, and the marked symbols represent first order perturbation quantities which are assumed to vary with time and distance in an exponential manner similar to the variation of the field $\bar{E}$:

$$\bar{E} = \bar{E}_0 e^{-\Gamma z + j\omega t} \tag{4a}$$

where Γ is the propagation constant, and ω is the frequency of the disturbance having an initial amplitude $\bar{E}_0$ at the origin ($z=0$). The Poisson's equation

$$\left(\nabla \cdot \epsilon \bar{E} = \sum_1^n \rho_n'\right),$$

the equation expressing conservation of charge $((\partial/\partial t)\rho_n' + (\partial/\partial z)i_n' = 0)$ and the force equation

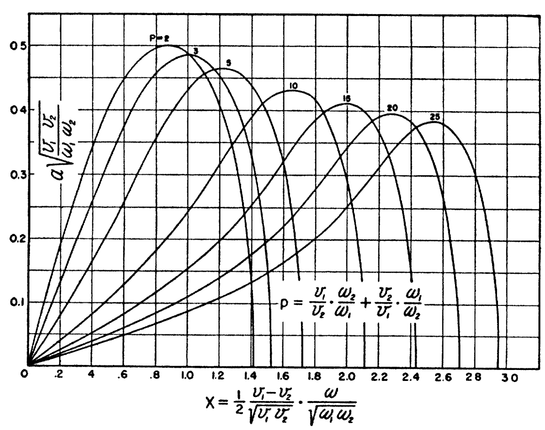

FIG. 3. Variation of the real component of propagation constant with the inhomogeneity factor (x), for different values of the distribution index (p).

$m(dv_n'/dt) = m[(\partial v_n'/\partial t) + v_n(\partial v_n'/\partial z)] = e\bar{E}$ give:

$$-\epsilon\Gamma\bar{E} = \sum_1^n \bar{\rho}_n \tag{5a}$$

$$j\omega\bar{\rho}_n = \Gamma\bar{i}_n \tag{6a}$$

$$j\omega\bar{v}_n - v_n\Gamma\bar{v}_n = (e/m)\bar{E}. \tag{7a}$$

By using (3a) and (6a) we obtain

$$j\omega\bar{\rho}_n = \Gamma(\bar{\rho}_n v_n + \bar{v}_n \rho_n)\cdots \tag{8a}$$

so that

$$\bar{v}_n = \bar{\rho}_n / \Gamma\rho_n(j\omega - \Gamma v_n). \tag{9a}$$

When (9a) is substituted into (7a) we obtain

$$\bar{\rho}_n = -\frac{[(e/m)\rho_n]}{(\omega + j\Gamma v_n)^2}\Gamma\bar{E} \tag{10a}$$

so that (5a) can be written as

$$\Gamma\bar{E} = \sum_1^n \frac{[(e/m)(\rho_n/\epsilon)]}{(\omega + j\Gamma v_n)^2}\Gamma\bar{E}. \tag{11a}$$

Excluding trivial solutions ($\Gamma = 0$ and $\bar{E} = 0$) and replacing space charge densities by electron plasma frequencies defined as

$$\omega_n = [(e/m)(\rho_n/\epsilon)]^{\frac{1}{2}} \tag{12a}$$

we obtain the following equation from which Γ can be determined:

$$\sum_1^n \frac{\omega_n^2}{(\omega + j\Gamma v_n)^2} = 1. \tag{13a}$$

In the case of a continuous distribution of velocities (13a) can be replaced by

$$\int_{v_1}^{v_2} \frac{d\omega_n^2/dv}{(\omega + j\Gamma v)^2} dv = 1. \tag{14a}$$

In the case of only two streams of plasma frequencies ω_1 and ω_2, and velocities v_1 and v_2 the solution of (13a) can be written explicitly when $\omega_1 = \omega_2$ or when $\omega_1/v_1 = \omega_2/v_2$:

$$\Gamma = \alpha[x^2 + 1 \pm (4x^2 + 1)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{15a}$$

where for the case $\omega_1 = \omega_2$

$$\alpha = \frac{\omega_1}{\frac{1}{2}(v_1 + v_2)} \quad \text{and} \quad x = \frac{\frac{1}{2}(v_1 - v_2)}{\frac{1}{2}(v_1 + v_2)}\cdot\frac{\omega}{\omega_1} = \frac{\delta\omega}{v\omega_1}, \tag{16a}$$

and, for the case $\omega_1/v_1 = \omega_2/v_2$,

$$\alpha = \left(\frac{\omega_1\omega_2}{v_1 v_2}\right)^{\frac{1}{2}} \quad \text{and} \quad x = \frac{\frac{1}{2}(v_1 - v_2)}{(v_1 v_2)^{\frac{1}{2}}}\cdot\frac{\omega}{(\omega_1\omega_2)^{\frac{1}{2}}}. \tag{17a}$$

The factor x is called the inhomogeneity factor. It defines amplifying properties of the space traversed by electron streams. The general solution of the two-stream case when $\omega_1 \neq \omega_2$ and $\omega_1/v_1 \neq \omega_2/v_2$ is given by Haeff, Arnett, and Stein[6] from which Fig. 3 is reproduced. This figure shows the real component of the normalized propagation constant (γ_r/α) plotted against the inhomogeneity factor x for different values of the "distribution index" p defined as $p = v_1\omega_2/v_2\omega_1 + v_2\omega_1/v_1\omega_2$ and expressing the current distribution between the two streams. Maximum amplification of the space-charge-waves occurs when $\omega_1 = \omega_2$ or when $\omega_1/v_1 = \omega_2/v_2$ and the inhomogeneity factor is equal to $\frac{1}{2}\sqrt{3}$. Therefore, the frequency of optimum amplification is given by

$$\omega_{opt} = \tfrac{1}{2}\sqrt{3}v\omega_1/\delta \text{ when } \omega_1 = \omega_2. \tag{18a}$$

APPENDIX II

Zero-Gain Velocity Distribution

If the velocity distribution in the inhomogeneous stream is such that $d\omega_n^2/dv = k$ then the solution of Eq. (14a) of Appendix I gives

$$1 = \int_{v_1}^{v_2} \frac{k\,dv}{(\omega + j\Gamma v)^2} = \frac{k}{j\Gamma}\left(\frac{1}{\omega + j\Gamma v_2} - \frac{1}{\omega + j\Gamma v_1}\right) = \frac{k(v_1 - v_2)}{(\omega + j\Gamma v_2)(\omega + j\Gamma v_1)} \tag{19a}$$

or, since

$$\omega_n = \int_{v_1}^{v_2} \frac{d\omega_n}{dv} dv = \int_{v_1}^{v_2} k\,dv = k(v_2 - v_1). \tag{20a}$$

Equation (19a) can be written as

$$\Gamma^2 - j\Gamma \frac{v_1 + v_2}{v_1 v_2}\omega - \frac{\omega^2 - \omega_n^2}{v_1 v_2} = 0. \tag{21a}$$

The solution of (21a) gives

$$\Gamma = j\frac{\omega}{v_1}\left\{\frac{1}{2}\left(1 + \frac{v_1}{v_2}\right) \pm \left[\frac{1}{4}\left(1 - \frac{v_1}{v_2}\right)^2 + \frac{v_1}{v_2}\left(\frac{\omega_n}{\omega}\right)^2\right]^{\frac{1}{2}}\right\}. \tag{22a}$$

Since the sum under the radical is always a positive quantity, the propagation constant Γ is always imaginary, regardless of velocity range or the magnitude of current. Therefore if the velocity distribution in the stream can be expressed as $d\omega_n^2/dv = k$, then no space charge amplification can take place.

[6] A. V. Haeff, H. Arnett, and W. Stein, submitted to the Proc. I. R. E. in Nov. 1948.

APPENDIX III

Estimate of Available Power

If it is assumed that the original velocity distribution is such that current I_1 is produced by charges moving with velocity v_1 corresponding to a potential V_1, and the current I_2 by charges of velocity v_2 corresponding to a potential V_2, then the total power carried by the stream at the origin is

$$P_1 = I_1 V_1 + I_2 V_2. \tag{23a}$$

Assuming the final distribution to be as described in the Appendix II, the total power carried by the stream at the end will be

$$P_2 = \int_{V_1}^{V_2} V(dI/dV)dV. \tag{24a}$$

But from the expression defining the distribution of current

$$d\omega_n^2/dv = k = e/m\epsilon(dI/vdv), \tag{25a}$$

we find that

$$I = I_1 + I_2 = \int_{v_1}^{v_2} (m/e)\epsilon k v\,dv = (\epsilon k/2)(m/e)(v_2^2 - v_1^2) = \epsilon k(V_2 - V_1), \tag{26a}$$

and

$$\frac{dI}{dV} = \frac{dI}{dv}\cdot\frac{dv}{dV} = \frac{m}{e}\epsilon k\frac{vdv}{dV} = \epsilon k = \frac{I_1 + I_2}{V_2 - V_1}. \tag{27a}$$

Therefore,

$$P_2 = \int_{V_1}^{V_2} \frac{I_1 + I_2}{V_2 - V_1} V\,dV = \tfrac{1}{2}(V_1 + V_2)(I_1 + I_2). \tag{28a}$$

Finally, the available power, or the power that can be converted from kinetic energy of the charged particles in the stream into the field energy of the space-charge waves and eventually radiated, is

$$P = P_1 - P_2 = (I_1 V_1 + I_2 V_2) - \tfrac{1}{2}(V_1 + V_2)(I_1 + I_2) = \tfrac{1}{2}(V_1 - V_2)(I_1 - I_2). \tag{29a}$$