Production of π-Mesons in Nucleon-Nucleon Collisions

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The cross section for the production of π-mesons in nucleon-nucleon collisions is calculated at energies just above the threshold. The process is treated in complete analogy with photonic bremsstrahlung: the π-meson field is coupled to the nucleon and an empirical potential between the two nucleons ensures momentum and energy conservation. The nuclear potential is taken from the Berkeley experiments on neutron-proton scattering at 90 Mev. In this treatment the symmetric scalar theory yields zero cross section if the recoil of the nucleons is neglected. The symmetric pseudoscalar theory (with pseudovector coupling) leads to the results given in Tables I–III. The cross section obtained is several orders of magnitude smaller than the cross section obtained by Morette and Peng on the basis of a thoroughgoing field-theoretic approach.

1. INTRODUCTION

IN connection with the construction of the large synchro-cyclotrons, a theoretical prediction of the cross section for the production of π-mesons in nucleon-nucleon collisions is desirable. A comparison with experiment of the predicted angular distribution and energy spectrum of the π-mesons will throw light on certain properties of the π-meson and the nature of its coupling with nucleons.

Although it is now apparent that the fundamental idea of the two-meson hypothesis is correct[1] (in accordance with which only the π-meson is produced in a nucleon-nucleon collision whereas the μ-meson has a negligible interaction with nucleons and arises solely as a decay product of the π-meson), a reliable meson theory of nuclear forces still does not exist. The possibility of using one kind of meson to explain the tensor character of nuclear forces while at the same time avoiding the inevitable $1/r^3$ singularity has not yet been demonstrated. Moreover, it is possible that several kinds of mesons are strongly coupled to nucleons and the resulting field theory of nuclear forces may become quite complicated.

In view of the uncertain status of meson field theories, we have adopted a different approach to the problem of meson production. We have not treated meson production as a third-order process in which one real meson is created and one virtual meson is created and destroyed, as Morette and Peng[2] have done. Instead, we have regarded meson production as a second-order process in which one step consists of the creation of a meson by one of the nucleons, and the other step consists of the scattering of the resulting nucleon by the second nucleon via the nuclear potential between them. The advantage of our approach is twofold: (1) the nuclear potential may be chosen so as to give the best fit to those scattering experiments which involve momenta transfers coming into play in the meson problem (e.g. the Berkeley neutron-proton experiments at 90 Mev,[3] and (2) the cross section only depends on the square of the coupling constant rather than the sixth power so that the correct value of the coupling constant is not as crucial.

It might be objected that our method of calculation neglects the "exchange" terms taken into account in the consistent third-order field-theoretic calculation. These "exchange" terms arise from the creation of the real meson in the second step (in the scheme of perturbation theory) while the virtual meson is created in the first step and destroyed in the third. The "ordinary" terms in which the virtual meson is created and destroyed in the first two steps or the last two steps comprise the terms which are essentially taken into account by our method. Fortunately, it can be shown[4] that the "exchange" terms are of order (μ/M) (μ is the mass of the π-meson, M is the nucleonic mass) compared to the "ordinary" terms in the energy region with which we are concerned—directly above the threshold—so that the error incurred is small. Of course, if the non-relativistic approximation for the nucleons, in which the recoil momenta of the nucleons are neglected, leads to a vanishing cross section for a particular type of coupling, the "ex-

[1] R. E. Marshak, Phys. Rev. **75**, 700 (1949).
[2] C. Morette and H. W. Peng, Proc. Roy. Irish Acad. **51A**, 217 (1948).
[3] Hadley, Kelly, Leith, Segrè, Wiegand, and York, Phys. Rev. **73**, 1114 (1948); Brueckner, Hartsough, Hayward, and Powell, Phys. Rev. **75**, 555 (1949).
[4] This can be shown by writing out the different types of perturbation-theoretic terms or by means of a more direct method developed by R. P. Feynman; we are indebted to Professor Feynman for performing the calculation at our request.

change" terms will be as important, possibly more important, then the ordinary terms, in determining the final result.[5]

2. METHOD OF CALCULATION

We consider the production of π-mesons by the collision of two nucleons when the nucleons possess energies just above the threshold for production (i.e. $2\mu c^2$ or 290 Mev[6]). Although, even at these energies, the velocities of the nucleons are not small compared to the velocity of light, we employ the non-relativistic approximation for the nucleons. This is consistent with the neglect of the "exchange" terms referred to in the introduction. For the interaction of the two nucleons with each other we use a symmetrical potential of the most general type which does not contain tensor forces. It is possible to include convergent tensor forces in the theory but we have not done so because the best fit with the neutron-proton scattering cross section at 90 Mev, from which we shall take our potential, is obtained by neglecting tensor forces.[7] The most general charge-independent central-force interaction can be written in the form:

$$V(r)=V^{(1)}(r)+\sigma_1\cdot\sigma_2V^{(2)}(r)+\tau_1\cdot\tau_2V^{(3)}(r)+(\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2)V^{(4)}(r),\quad(1)$$

where the $V^{(i)}$ may all be distinct and contain the radial dependence of the forces. For the coupling of the π-meson with the nucleon, we first treat the case of a symmetric pseudoscalar field with pseudovector coupling. We then consider the case of a symmetric scalar field with scalar coupling. Other types of coupling can be investigated in a similar fashion, but the symmetric scalar and pseudoscalar fields will serve as illustrations.

Since the nucleon may be a proton or neutron and may have either direction of spin, we use a four-component wave function for the nucleon, namely, $\psi_\rho=(\psi_1,\psi_2,\psi_3,\psi_4)$ and four-component representations of the ordinary spin operators and the isotopic spin operators, namely:

$$\sigma_x=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad\sigma_y=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix},$$
$$\sigma_z=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\quad(2)$$

and

$$\tau_x=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad\tau_y=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix},$$
$$\tau_z=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.\quad(3)$$

These operators satisfy the relations:

$$\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij};\quad\tau_i\tau_j+\tau_j\tau_i=2\delta_{ij};\quad\sigma_i\tau_j-\tau_j\sigma_i=0.\quad(4)$$

Any operator in the ordinary spin-isotopic spin space may be expressed as a linear combination of the sixteen linearly independent matrices: $1,\sigma_i,\tau_i,\sigma_i\tau_j$, including the operators in (1). The total Hamiltonian for symmetric pseudoscalar mesons may then be written in second quantized form as follows (we set $\hbar=c=1$ and use the summation convention throughout):

$$H=\int dx\bigg\{\frac{1}{2M}(\nabla\psi_\rho^*)\cdot(\nabla\psi_\rho)+\tfrac{1}{2}[\pi_i^2+(\nabla\phi_i)^2+\mu^2\phi_i^2]+(4\pi)^{\frac{1}{2}}\frac{g}{\mu}\psi_\rho^*(\tau_{\rho\mu})_i(\sigma_{\mu\nu}\cdot\nabla\phi_i)\psi_\nu+\int\psi_\rho^*(x)\psi_\sigma^*(x')J_{\rho\mu}{}^{\alpha}J_{\sigma\nu}{}^{\alpha}\times V^{(\alpha)}(x-x')\psi_\mu(x)\psi_\nu(x')dx'\bigg\}.\quad(5)$$

In (5), ϕ is the meson wave function, π the canonically conjugate momentum, g the meson-nucleon coupling constant and J^α represents the operators for $\alpha=1,2,3,4$, respectively (i.e. $V=J_1{}^\alpha\cdot J_2{}^\alpha V^{(\alpha)}$). The commutation rules for the ψ and ϕ are

$$[\psi_\rho^*(x),\psi_\sigma(x')]_+=-i\delta_{\rho\sigma}\delta(x-x'),$$
$$[\psi_\rho(x),\psi_\sigma(x')]_+=[\psi_\rho^*(x),\psi_\sigma^*(x')]_+=0,\quad(6)$$
$$[\pi_i(x),\phi_j(x')]=-i\delta_{ij}\delta(x-x'),$$

all other quantities commuting.

We expand the wave functions in Fourier series in a box of volume Ω. For the nucleon, we have

$$\psi_\rho(x)=\Omega^{-\frac{1}{2}}A_p{}^m u_\rho{}^m e^{ip\cdot x},$$
$$\psi_\rho^*(x)=\Omega^{-\frac{1}{2}}A_p{}^{m*}u_\rho{}^{m*}e^{-ip\cdot x},\quad(7)$$

where

$$u_\rho{}^{m*}u_\rho{}^{m'}=\delta_{mm'},\quad u_\rho{}^{m*}u_\sigma{}^m=\delta_{\rho\sigma}.$$

The u's may conveniently be taken to be

$$u^1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad u^2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad u^3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad u^4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\quad(8)$$

[5] W. Horning and M. Weinstein, Phys. Rev. 72, 251 (1947).
[6] Private communication from R. Serber.
[7] R. Serber and R. Christian, private communication; see, however, the note added in proof at the end of this paper.

where the superscripts 1, 2, 3, 4 represent proton spin up, proton spin down, neutron spin up, neutron spin down, respectively.

For the meson, we have

$$\begin{aligned}\phi_i(x)&=\Omega^{-\frac{1}{2}}(2\omega_k)^{-\frac{1}{2}}(a_k^s v_i^s+a_{-k}^{s*}v_i^{s*})e^{ik\cdot x},\\ \pi_i(x)&=i\Omega^{-\frac{1}{2}}(\tfrac{1}{2}\omega_k)^{\frac{1}{2}}(a_k^{s*}v_i^{s*}-a_{-k}^{s}v_i^{s})e^{-ik\cdot x},\end{aligned}\qquad(9)$$

with

$$\omega_k=(\mu^2+k^2)^{\frac{1}{2}}$$

and where

$$v_i^{s*}v_j^s=\delta_{ij},\quad v_i^{s*}v_i^{s'}=\delta_{ss'}.$$

It is convenient to choose

$$v^1=\begin{pmatrix}1/\sqrt{2}\\ +i/\sqrt{2}\\ 0\end{pmatrix},\quad v^2=\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\\ 0\end{pmatrix},\quad v^3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix},\qquad(10)$$

where the superscripts 1, 2, 3 represent positive, negative and neutral mesons, respectively.

The operators satisfy the commutation rules:

$$\begin{aligned}&[A_p^{m*},A_{p'}^{m'}]_+=\delta_{mm'}\delta_{pp'},\\ &[A_p^{m*},A_{p'}^{m'*}]_+=[A_p^m,A_{p'}^{m'}]_+=0,\\ &[a_k^s,a_{k'}^{s'*}]=\delta_{ss'}\delta_{kk'},\end{aligned}\qquad(11)$$

all other quantities commuting.

The Hamiltonian now takes the form

$$\begin{aligned}H=&A_p^{m*}A_p^mE_p+(a_k^sa_k^{s*}+a_k^{s*}a_k^s)\omega_k/2\\ &+\frac{i(4\pi)^{\frac{1}{2}}g}{\mu\Omega^{\frac{1}{2}}(2\omega_k)^{\frac{1}{2}}}(u^{m*}|\tau_i(\sigma\cdot k)|u^{m'})A_p^{m*}A_{p'}^{m'}\\ &\times(a_k^sv_i^s+a_{-k}^{s*}v_i^{s*})\delta(p-p'-k)\\ &+A_p^{m*}A_{p'}^{m'*}A_{p''}^{m''}A_{p'''}^{m'''}V_{p''-p}^{(\alpha)}\\ &(u^{m*}|J^\alpha|u^{m''})\cdot(u^{m'*}|J^\alpha|u^{m'''})\\ &\times\delta(p+p'-p''-p'''),\end{aligned}\qquad(12)$$

where

$$V_P^{(\alpha)}=\frac{1}{\Omega}\int V^{(\alpha)}(x)e^{iP\cdot x}dx.$$

We shall calculate to first order in V and g the transition probability from a state in which we have two nucleons with momenta p_0 and $-p_0$ and in spin states m_1 and m_2, respectively, to a state in which we have two nucleons with momenta $p-\frac{1}{2}k$ and $-p-\frac{1}{2}k$ and in spin states m_1' and m_2', respectively, and a meson with momentum k and isotopic spin state s. The transition scheme is

$$\begin{array}{ccccc} & \nearrow & (p-\frac{1}{2}k,m_1')(-p+\frac{1}{2}k,m'') & \searrow & \\ & \nearrow & (-p-\frac{1}{2}k,m_2')(p+\frac{1}{2}k,m'') & \searrow & \\ (p_0,m_1)(-p_0,m_2) & & & & (p-\frac{1}{2}k,m_1')(-p-\frac{1}{2}k,m_2')(k,s).\\ & \searrow & (p_0,m_1)(-p_0-k,m'')(k,s) & \nearrow & \\ & \searrow & (-p_0,m_2)(p_0-k,m'')(k,s) & \nearrow & \end{array}\qquad(13)$$

The transition scheme (13) would yield eight distinct matrix elements if we did not take advantage of the assumption that $p_0\ll M$. Making this assumption, we may neglect $\frac{1}{2}k$ compared with $p\pm p_0$ so that $p^2=p_0^2-M\omega_k$ and we may replace $V_{p\pm p_0\pm\frac{1}{2}k}^{(\alpha)}$ by $V_{p\pm p_0}^{(\alpha)}$. With this approximation, neglecting the nucleonic recoil energies and summing over m''. the transition matrix element reduces to

$$\begin{aligned}H'=&\frac{i(4\pi)^{\frac{1}{2}}g}{\mu\Omega^{\frac{1}{2}}\omega_k(2\omega_k)^{\frac{1}{2}}}\{V_{p-p_0}^{(\alpha)}[(m_1|J^\alpha|m_1')\\ &\cdot(m_2|K^\alpha|m_2')+(m_1|K^\alpha|m_1')\cdot(m_2|J^\alpha|m_2')]\\ &+V_{p+p_0}^{(\alpha)}[(m_1|J^\alpha|m_2')\cdot(m_2|K^\alpha|m_1')\\ &+(m_1|K^\alpha|m_2')\cdot(m_2|J^\alpha|m_1')]\},\end{aligned}\qquad(14)$$

where

$$K^{\alpha*}=J^\alpha(\tau_iv_i^{s*})(\sigma\cdot k)-(\tau_iv_i^{s*})(\sigma\cdot k)J^\alpha.\qquad(14a)$$

Before proceeding, we must make our notation a little more specific. The quantity J^1 is a scalar, J^2 and J^3 are vectors, while J^4 is a tensor. We may adopt a uniform notation if we use two tensor indices as follows:

$$\begin{aligned}J^1:&\quad J_{ik}^1=\delta_{ik}/\sqrt{3},\\ J^2:&\quad J_{ik}^2=\sigma_i\delta_{ik},\\ J^3:&\quad J_{ik}^3=\tau_i\delta_{ik},\\ J^4:&\quad J_{ik}^4=\sigma_i\tau_k.\end{aligned}\qquad(15)$$

Then,

$$\begin{aligned}H'=&-\frac{ig(4\pi)^{\frac{1}{2}}}{\mu\Omega^{\frac{1}{2}}\omega_k(2\omega_k)^{\frac{1}{2}}}\{V_{p-p_0}^{(\alpha)}[(m_1|J_{ik}^\alpha|m_1')\\ &\times(m_2|K_{ik}^\alpha|m_2')\\ &+(m_1|K_{ik}^\alpha|m_1')(m_2|J_{ik}^\alpha|m_2')]\\ &+V_{p+p_0}^{(\alpha)}[(m_1|J_{ik}^\alpha|m_2')(m_2|K_{ik}^\alpha|m_1')\\ &+(m_1|K_{ik}^\alpha|m_2')(m_2|J_{ik}^\alpha|m_1')]\}.\end{aligned}\qquad(16)$$

Taking the absolute square of H' and summing over the final spin states of the nucleons, we get (since

$J_{ik}{}^{\alpha}=J_{ik}{}^{\alpha})$

$$
\begin{aligned}
|H'|^2=&-\frac{2\pi g^2}{\mu\Omega\omega_k{}^3}\\
&\times\{[V_{p-p_0}{}^{(\alpha)}V_{p-p_0}{}^{(\beta)}+V_{p+p_0}{}^{(\alpha)}V_{p+p_0}{}^{(\beta)}]\\
&\times[(m_1|J_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_1)(m_2|K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_2)\\
&+(m_1|J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_1)(m_2|K_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_2)\\
&+(m_1|K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_1)(m_2|J_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_2)\\
&+(m_1|K_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_1)(m_2|J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_2)]\\
&+2V_{p-p_0}{}^{(\alpha)}V_{p+p_0}{}^{(\beta)}[(m_1|J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_2)\\
&\times(m_2|K_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_1)+(m_1|J_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_2)\\
&\times(m_2|K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_1)+(m_1|K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_2)\\
&\times(m_2|J_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_1)+(m_1|K_{ik}{}^{\alpha}J_{jl}{}^{\beta}|m_2)\\
&\times(m_2|J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}|m_1)]\}. \qquad (17)
\end{aligned}
$$

If we now introduce the projection operators:

$$
\Lambda=\begin{cases}\frac{1}{2}(1+\tau_z) & \text{if the particle is a proton}\\ \frac{1}{2}(1-\tau_z) & \text{if the particle is a neutron}\end{cases} \qquad (18)
$$

and average over the initial spin states, we find

$$
\begin{aligned}
|H'|^2=&-\frac{\pi g^2}{2\mu^2\Omega\omega_k{}^3}\\
&\{[V_{p-p_0}{}^{(\alpha)}V_{p-p_0}{}^{(\beta)}+V_{p+p_0}{}^{(\alpha)}V_{p+p_0}{}^{(\beta)}]\\
&\times[Sp(J_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_1)Sp(K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_2)\\
&+Sp(J_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_2)Sp(K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_1)\\
&+Sp(J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_1)Sp(K_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_2)\\
&+Sp(J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_2)Sp(K_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_1)]\\
&+2V_{p-p_0}{}^{(\alpha)}V_{p+p_0}{}^{(\beta)}[Sp(J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_2K_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_1)\\
&+Sp(J_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_2K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_1)\\
&+Sp(K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_2J_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_1)\\
&+Sp(K_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_2J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_1)]\}. \qquad (19)
\end{aligned}
$$

The subscripts 1 and 2 on Λ refer to the nucleons with momenta p_0 and $-p_0$, respectively. The evaluation of the spurs in (19) is straightforward and yields the results (we have written $\Lambda=\frac{1}{2}(1+\delta\tau_z)$ where $\delta=+1$ for proton and $\delta=-1$ for neutron and $\epsilon^s=-i(v_1{}^{s*}v_2{}^s-v_2{}^{s*}v_1{}^s)$):

$$
\begin{aligned}
P_1{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)&\equiv Sp(J_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_1)Sp(K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_2)\\
&=-16k^2\begin{pmatrix}0&0&0&0\\0&2(1+\epsilon^s\delta_2)&0&2\delta_1\delta_2(v_3{}^s)^2\\0&0&[2(1-\delta_1\delta_2(v_3{}^s)^2)+\epsilon^s(\delta_1+\delta_2)]&0\\0&P_1{}^{24}&0&[2(2-\delta_1\delta_2(v_3{}^s)^2)+\epsilon^s(\delta_2+3\delta_1)]\end{pmatrix}, \qquad (20a)
\end{aligned}
$$

$$P_2{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)\equiv Sp(J_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_2)Sp(K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_1)=P_1{}^{\alpha\beta}(\Lambda_2,\Lambda_1,s), \qquad (20b)$$

$$P_3{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)\equiv Sp(J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_1)Sp(K_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_2)=-16k^2\begin{pmatrix}0&0&0&0\\0&-P_1{}^{24}&0&-P_1{}^{22}\\0&0&0&-P_1{}^{33}\\0&-P_1{}^{22}&-P_1{}^{33}&-P_1{}^{24}\end{pmatrix}, \qquad (20c)$$

$$P_4{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)\equiv Sp(J_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_2)Sp(K_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_1)=P_3{}^{\alpha\beta}(\Lambda_2,\Lambda_1,s), \qquad (20d)$$

$$
\begin{aligned}
Q_1{}^{\alpha\beta}(\Lambda_1.\Lambda_2,s)&\equiv Sp(J_{ik}{}^{\alpha}K_{jl}{}^{\alpha*}\Lambda_2K_{ik}{}^{\beta}J_{jl}{}^{\beta}\Lambda_1)\\
&=-16k^2\begin{pmatrix}0&0&0&0\\0&0&[1+\epsilon^s\delta_2-\delta_1\delta_2(v_3{}^s)^2]&-Q_1{}^{23}\\0&Q_1{}^{23}&-[\delta_1\delta_2(1-(v_3{}^s)^2)+(\epsilon^s/2)(\delta_1+\delta_2)]&-[1+\delta_1\delta_2(1-2(v_3{}^s)^2)+(\epsilon^s/2)(3\delta_1+\delta_2)]\\0&Q_1{}^{24}&Q_1{}^{34}&[2-\delta_1\delta_2(1+(v_3{}^s)^2)+(\epsilon^s/2)(3\delta_1-\delta_2)]\end{pmatrix}, \qquad (21a)
\end{aligned}
$$

$$Q_2{}^{\alpha\beta}\equiv Sp(J_{ik}{}^{\alpha}J_{jl}{}^{\beta}\Lambda_2K_{ik}{}^{\alpha}K_{jl}{}^{\beta*}\Lambda_1)=-16k^2\begin{pmatrix}0&0&0&0\\0&0&Q_1{}^{23}&Q_1{}^{24}\\0&Q_1{}^{23}&-Q_1{}^{23}&-Q_1{}^{23}+Q_1{}^{24}\\0&Q_1{}^{24}&-Q_1{}^{23}+Q_1{}^{24}&[2+\delta_1\delta_2(1-3(v_3{}^s)^2)+\frac{3}{2}\epsilon^s(\delta_1+\delta_2)]\end{pmatrix}, \qquad (21b)$$

$$Q_3^{\alpha\beta} \equiv Sp(K_{ik}{}^{\alpha} K_{jl}{}^{\beta *} \Lambda_2 J_{ik}{}^{\alpha} J_{jl}{}^{\beta} \Lambda_1) = -16k^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & [1+\epsilon^s\delta_1 - \delta_1\delta_2(v_3{}^s)^2] & -Q_3{}^{23} \\ 0 & Q_3{}^{23} & -Q_1{}^{33} & -[1+(\epsilon^s/2)(\delta_1-\delta_2)-\delta_1\delta_2] \\ 0 & Q_3{}^{24} & Q_3{}^{34} & Q_2{}^{44} \end{pmatrix}, \tag{21c}$$

$$Q_4^{\alpha\beta} \equiv Sp(K_{ik}{}^{\alpha} J_{jl}{}^{\beta} \Lambda_2 J_{ik}{}^{\alpha} K_{jl}{}^{\beta *} \Lambda_1) = -16k^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Q_3{}^{23} & Q_3{}^{24} \\ 0 & Q_3{}^{23} & Q_1{}^{23} & -Q_1{}^{23}+Q_1{}^{33} \\ 0 & Q_3{}^{24} & -Q_1{}^{23}+Q_1{}^{33} & Q_1{}^{33}+2Q_1{}^{23} \end{pmatrix}. \tag{21d}$$

If we now sum the P's and Q's, and use the notation

$$S_1{}^{\alpha\beta} = \sum_{n=1}^{4} P_n{}^{\alpha\beta}, \quad S_2{}^{\alpha\beta} = \sum_{n=1}^{4} Q_n{}^{\alpha\beta},$$

we get

$$S_1{}^{\alpha\beta} = -32k^2 A^s(\Lambda_1, \Lambda_2) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix};$$

$$S_2{}^{\alpha\beta} = -32k^2 A^s(\Lambda_1, \Lambda_2) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & -2 \end{pmatrix}, \tag{22}$$

where

$$A^s(\Lambda_1, \Lambda_2) = 2 + \epsilon^s(\delta_1+\delta_2) - 2\delta_1\delta_2(v_3{}^s)^2. \tag{22a}$$

The quantity $A^s(\Lambda_1, \Lambda_2)$ characterizes the relative probabilities for the production of the various types of charged mesons in the different nucleon-nucleon collisions and Table I lists its values.

Our theory (which is a weak coupling theory) correctly gives zero probability for the production of negative and positive mesons in proton-proton and neutron-neutron collisions, respectively. The equality of the number of neutral mesons and the number of charged mesons produced in a neutron-proton collision follows naturally, whereas the unexpected zero probability for the production of neutral mesons in like-particle collisions is a consequence of a subtle cancellation of matrix elements.

Since the density of final states is

$$\frac{\Omega M p d\Omega_p}{16\pi^3} \cdot \frac{\Omega k \omega_k d\Omega_k d\omega_k}{8\pi^3}$$

the differential cross section for the production of mesons of type s becomes

$$d\sigma^s(\Lambda_1, \Lambda_2) = \left(\frac{\sqrt{2}gMk}{4\pi^2\mu\omega_k}\right)^2 \left(\frac{kp}{p_0}\right) A^s(\Lambda_1, \Lambda_2)$$

$$\int d\Omega_p \{ [(V_{p-p_0}{}^{(2)})^2 + (V_{p+p_0}{}^{(2)})^2] + 2(V_{p-p_0}{}^{(2)} V_{p+p_0}{}^{(3)} + V_{p+p_0}{}^{(2)} V_{p-p_0}{}^{(3)}) - (V_{p-p_0}{}^{(2)} + V_{p+p_0}{}^{(4)})^2 - (V_{p-p_0}{}^{(4)} + V_{p+p_0}{}^{(2)})^2 + [(V_{p-p_0}{}^{(3)})^2 + (V_{p+p_0}{}^{(3)})^2] - (V_{p-p_0}{}^{(3)} + V_{p+p_0}{}^{(4)})^2 - (V_{p-p_0}{}^{(4)} + V_{p+p_0}{}^{(3)})^2 + 2(V_{p-p_0}{}^{(4)} + V_{p+p_0}{}^{(4)})^2 \} d\Omega_k d\omega_k. \tag{23}$$

Equation (23) is the general expression for the differential cross section for the production of pseudoscalar mesons of type s in a collision of two nucleons of types Λ_1 and Λ_2, respectively. That the angular distribution of the mesons is uniform in the center of mass system, independent of any particular assumption about the V's is evident, and follows directly from our approximation of neglecting $\frac{1}{2}k$ compared with $p \pm p_0$.

The method of calculation which has just been used for symmetric pseudoscalar mesons with pseudovector coupling can easily be extended to the case of symmetric scalar mesons with scalar coupling. The total Hamiltonian (5) is replaced by

$$H = \int dx \Big\{ \frac{1}{2M} (\nabla\psi_\rho{}^*) \cdot (\nabla\psi_\rho) + \tfrac{1}{2}[\pi_i{}^2 + (\nabla\phi_i)^2 + \mu^2\phi_i{}^2] + (4\pi)^{\frac{1}{2}} f \psi_\rho{}^* (\tau_{\rho\mu})_i \phi_i \psi_\rho + \int dx' \psi_\rho{}^*(x) \psi_\sigma{}^*(x') \times (J_{\rho\mu}{}^{\alpha} \cdot J_{\sigma\nu}{}^{\alpha}) V^{(\alpha)}(x-x') \psi_\mu(x) \psi_\nu(x') \Big\}. \tag{24}$$

Proceeding in exactly the same fashion as for the pseudoscalar case, we obtain an expression for $|H'|^2$ identical with (19) except that f^2 replaces $(g/\mu)^2$ and $K^{\alpha *}$ is defined by $[J^{\alpha}(\tau_i v_i{}^{s*}) - (\tau_i v_i{}^{s*}) J^{\alpha}]$ instead of (14a). Evaluation of the spurs then leads

TABLE I. Values of $A^s(\Lambda_1, \Lambda_2)$.

	p-p collision ($\Lambda_1=\Lambda_2=1$)	n-n collision ($\Lambda_1=\Lambda_2=-1$)	n-p collision ($\Lambda_1=\pm 1$, $\Lambda_2=\mp 1$)
pos. meson ($s=1$)	4	0	2
neg. meson ($s=2$)	0	4	2
neutral meson ($s=3$)	0	0	4

TABLE II. Values of total cross section for Yukawa potential.

(E_0/E_t)	1.25	1.50	2.00
$\frac{\sigma_T{}^s(\Lambda_1,\Lambda_2)}{(g^2/\hbar c)A^s(\Lambda_1,\Lambda_2)}$	$4.4\cdot10^{-31}$ cm²	$4.4\cdot10^{-30}$ cm²	$2.0\cdot10^{-29}$ cm²

to the following values for the P's and Q's (see Eqs. (20a)–(21d)):

$$P_1{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=-16A^s(\Lambda_1,\Lambda_2)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix},\qquad(25a)$$

$$P_2{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=P_1{}^{\alpha\beta}(\Lambda_2,\Lambda_1,s),\qquad(25b)$$

$$P_3{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=-P_1{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s),\qquad(25c)$$

$$P_4{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=P_3{}^{\alpha\beta}(\Lambda_2,\Lambda_1,s).\qquad(25d)$$

$$Q_1{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=-8B^s(\Lambda_1,\Lambda_2)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&3\\0&0&3&-3\end{pmatrix},\qquad(26a)$$

$$Q_2{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=-8B^s(\Lambda_1,\Lambda_2)\times\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&-3\\0&0&-3&3\end{pmatrix},\qquad(26b)$$

$$Q_3{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=Q_2{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s),\qquad(26c)$$

$$Q_4{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s)=Q_1{}^{\alpha\beta}(\Lambda_1,\Lambda_2,s),\qquad(26d)$$

where $A^s(\Lambda_1,\Lambda_2)$ is defined by (22a) and

$$B^s(\Lambda_1,\Lambda_2)=2\delta_1\delta_2(-1+(v_3{}^s)^2)-\epsilon^s(\delta_1+\delta_2).$$

If we now sum the P's and Q's, we find $S_1{}^{\alpha\beta}\equiv0$ and $S_2{}^{\alpha\beta}\equiv0$. Hence, in the present approximation, the symmetric scalar theory yields zero cross section. This result was discovered independently by W. S. MacAfee.[8] This means, of course, that one must take account of the recoil of the nucleons in order to get a non-vanishing cross section on the symmetrical scalar theory.[5] However, if this is done, it is no longer self-consistent to neglect the "exchange" terms compared to the "ordinary" terms (see Introduction). We have, therefore, not calculated the meson production cross section on the basis of the symmetric scalar theory.[9]

3. RESULTS AND DISCUSSION

We apply formula (23) to calculate the π-meson production cross section on the basis of two different assumptions regarding the spatial behavior of the four V's: a Yukawa potential for all four V's and a square-well potential for all four V's. These two assumptions are taken from the work of Serber and Christian[7] who have obtained the best fit of the experimental angular distribution for neutron-proton scattering at 90 Mev. Serber and Christian find for the best Yukawa potential:

$$V(r)=\tfrac{1}{2}(1+P_M)({}^\alpha g)^2(e^{-Kr}/r),\qquad(27)$$

where $K=0.87\mu$, $({}^1g)^2=0.280$, $({}^3g)^2=0.404$ and P_M is the Majorana operator. Translated into our notation, (27) becomes:

$$V(r)=\{3(c_1+c_3)+(-3c_1+c_3)\sigma_1\cdot\sigma_2+(c_1-3c_3)\tau_1\cdot\tau_2+(-c_1-c_3)\sigma_1\cdot\sigma_2\tau_1\cdot\tau_2\}e^{-Kr}/16r,\qquad(28)$$

where $c_1=({}^1g)^2$, $c_3=({}^3g)^2$. Substitution of (28) into (23) yields the differential cross section for the production of pseudoscalar mesons:

$$d\sigma^s(\Lambda_1,\Lambda_2)=\frac{(c_1-c_3)^2}{2\pi}\left(\frac{gMk}{\mu\omega_k}\right)^2\left(\frac{kp}{p_0}\right)A^s(\Lambda_1,\Lambda_2)\times\left\{\frac{1}{(K^2+p^2+p_0{}^2)^2-4p_0{}^2p^2}-\frac{\tanh^{-1}[2p_0p/(K^2+p^2+p_0{}^2)]}{2p_0p(K^2+p^2+p_0{}^2)}\right\}d\Omega_kd\omega_k,\qquad(29)$$

where $k=(\omega_k{}^2-\mu^2)^{\frac{1}{2}}$ and $p=(p_0{}^2-M\omega_k)^{\frac{1}{2}}$. The total cross section, $\sigma_T{}^s(\Lambda_1,\Lambda_2)$, can easily be evaluated; we find the results given in Table II for three values of the incident energy E_0. In Table II the incident energy is measured in units of the threshold energy E_t, chosen as 290 Mev[8], i.e., the mass of the π-meson is taken as 145 Mev. The quantity $A^s(\Lambda_1,\Lambda_2)$ is given by Table I while $(g^2/\hbar c)$ can be taken from Pauli;[10] the product $(g^2/\hbar c)A^s$ is of order unity. The rapid increase of cross section is characteristic of the pseudoscalar theory.

A corresponding calculation has been performed for a square-well potential using the constants determined by Serber and Christian,[7] namely:

$$V(r)=\tfrac{1}{2}(1+P_M)^\alpha J(r),\qquad(30)$$

where

$$\left.\begin{array}{l}{}^1J(r)=0.166\mu\\{}^3J(r)=0.252\mu\end{array}\right\}\text{ for } r\leq1.5/\mu$$

TABLE III. Values of total cross section for square well potential.

(E_0/E_t)	1.25	1.50	2.00
$\frac{\sigma_T{}^s(\Lambda_1,\Lambda_2)}{(g^2/\hbar c)A^s(\Lambda_1,\Lambda_2)}$	$8.4\cdot10^{-31}$ cm²	$9.6\cdot10^{-30}$ cm²	$1.7\cdot10^{-29}$ cm²

[8] Cornell doctoral dissertation under H. A. Bethe; we are indebted to Professor Bethe for informing us of Mr. MacAfee's result.

[9] See L. W. Nordheim and G. Nordheim, Phys. Rev. 54, 254 (1938) and reference 5.

[10] W. Pauli, *Meson Theory of Nuclear Forces* (Interscience Publishers, Inc., New York, 1946).

and

$$^{a}J(r)=0 \text{ for } r>1.5/\mu.$$

Table III gives the results[11] for the same range of values for the incident energy of the nucleon as Table II.

Comparison of Tables II and III makes manifest the rather expected insensitivity of the meson production cross section to the shape of the nuclear potential as long as the nucleon-nucleon scattering experiments are fitted at high energies. For both the square-well and Yukawa potentials, the meson production cross section turns out to be surprisingly small. This result is in striking contrast to the result obtained by Morette and Peng[2] on the basis of a third-order field-theoretic perturbation calculation. Morette and Peng obtain a meson production cross section of the order of 10^{-27} cm^2 at an incident nucleon energy of twice the threshold energy.

The large discrepancy between the two results has been discussed by one of the authors (R.E.M.) with Dr. Morette. Dr. Morette has reexamined the details of her calculation and arrived at the conclusion that the "exchange" terms are responsible for the large cross section predicted by the field-theoretic treatment. This is a consequence of the fact that whereas the δ-interaction terms between the two nucleons are effectively subtracted out from the "ordinary" terms, they are not subtracted out from the "exchange" terms so that the latter make the largest contributions to the cross section. This is contrary to our statement in the introduction and is due to the singularities which arise in present-day meson field theories of nuclear forces. In a correct convergent field theory of nuclear forces, the "exchange" terms should not be large compared to the "ordinary" terms, except for accidental cancellations as in the symmetric scalar theory.

[11] The fluctuations due to the sine and cosine functions (arising from the Fourier transform of the square well) were averaged out.

We believe that the small cross sections—not necessarily those predicted by the symmetric pseudoscalar theory—will correspond to experiment. We feel that the value of our method lies precisely in its correlation of the meson production cross section with the nucleon-nucleon scattering cross section at high energies and its bypassing of the existing singular field theories of nuclear forces. The quantitative cross section and the energy spectrum of the mesons just above the threshold should therefore contribute to our knowledge of the meson-nucleon coupling. We do not believe, however, that our results and calculations along similar lines (for other types of meson-nucleon coupling and more complicated (tensor) forms of nuclear interaction) will shed light on the correct meson theory of nuclear forces.

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Note added in proof: One of the authors (L.L.F.) has investigated the effect of tensor forces on the angular distribution of the mesons produced in nucleon-nucleon collisions. Toward this end, a calculation was performed with the neutral pseudoscalar theory (pseudovector coupling) with the same approximations as were made above with a phenomenological interaction of the form:

$$(\sigma_1\cdot\text{grad})(\sigma_2\cdot\text{grad})U(r)=\tfrac{1}{3}S_{12}\left[\frac{d^2U}{dr^2}-\frac{1}{r}\frac{dU}{dr}\right]+\tfrac{1}{3}\sigma_1\cdot\sigma_2\left[\frac{d^2U}{dr^2}+\frac{2}{r}\frac{dU}{dr}\right],$$

where S_{12} is the tensor force operator. The results indicate that in contrast to the isotropic angular distribution predicted for central forces, a force of the above type yields a $(\sin^2\theta)$ angular distribution where θ is the angle between the direction of emission of the meson and the direction of the incident nucleon. By adding to the above interaction a central force which cancels out the central part, the resulting angular distribution becomes $(\frac{4}{3}-\sin^2\theta)$. Hence, in the first instance, the emission of mesons perpendicular to the direction of the nucleons is favored while for a pure tensor force, emission parallel to this direction is favored.