known[1] that for no value of the real parameter λ will the differential equation

$$\varphi''+(f(s)+\lambda)\varphi=0 \qquad (1)$$

be such as to possess only solutions satisfying

$$\int_0^\infty |\varphi(s)|^2 ds<\infty. \qquad (2)$$

In other words, if $\varphi=\varphi_1(s)$ and $\varphi=\varphi_2(s)$ are two linearly independent solutions of Eq. (1) for some λ, then at least one of the functions φ_1, φ_2 will violate condition (2). This means that Eq. (1) and any homogeneous, linear boundary condition at $s=0$ (such as $\varphi(0)=0$ or $\varphi'(0)=0$) will *determine* an eigenvalue problem. This will, in particular, be the case if the potential $f(s)$ tends to a finite limit when $s\rightarrow\infty$.

The latter particular case of the restriction $|f(s)|<\text{const.}$ can be assumed in the form

$$\lim_{s\rightarrow\infty} f(s)=0. \qquad (3)$$

In this case, it can be expected, and recently it was proved,[2] that the spectrum contains every point of the half-line $0\leqq\lambda<\infty$.

In order to simplify the manner of speaking, let the "essential spectrum" of a differential Eq. (1) be defined as follows: A λ-value is in the essential spectrum if it is in the spectrum of *every* boundary condition assigned at $s=0$. In this terminology, the fact mentioned before can be expressed by saying that every point of the half-line $0\leqq\lambda<\infty$ is in the essential spectrum of Eq. (1) if the potential satisfies condition (3).

It was emphasized in a previous note[3] that an exact definition of the notion of a spectrum cannot be reduced to heuristic ideas. It will be shown below that intuitive notions on the spectrum are capable of leading to results which prove to be wrong.

First, an intuitive approach to the notion of a spectrum leads to the following definition:[4] A λ-value is in the spectrum (for *some* boundary condition assigned at $s=0$) if and only if the corresponding Eq. (1) has a solution $\varphi=\varphi(s)$ which remains bounded

$$|\varphi(s)|<\text{const. when } s\rightarrow\infty \qquad (4)$$

(but is not the trivial solution, $\varphi(s)\equiv 0$). This definition clearly implies that a given λ cannot be in the essential spectrum unless the corresponding Eq. (1) has a solution $\varphi(s)$ satisfying condition (4). It is assumed, of course, that $f(s)$ is subject to some reasonable restriction when $s\rightarrow\infty$, such as the restriction mentioned before Eq. (1).

This restriction is surely satisfied if $f(s)$ is such as to satisfy condition (3). Since the latter condition assures that every point of the half-line $0\leqq\lambda<\infty$ is in the essential spectrum, there results the following assertion: If $f(s)$ is subject to condition (3), then there cannot exist a positive λ-value corresponding to which Eq. (1) fails to possess *some* solution satisfying condition (4). The truth of this assertion was deduced from the customary intuitive definition of a spectrum. Hence, if the assertion deduced will turn out to be false, it will follow that the intuitive definition must in general be discarded.

Accordingly, it is sufficient to ascertain that, if $f(s)$ is a real-valued, continuous function satisfying assumption (3), and if λ has a positive value, say $\lambda=1$, then *all* solutions of the corresponding differential Eq. (1), that is, of

$$\varphi''+(f(s)+1)\varphi=0, \qquad (5)$$

can be such as to violate condition (4). But the existence of such an $f(s)$ is assured by the method of construction applied in an earlier paper.[5] In fact, the detailed construction given in that paper implies that such an $f(s)$ results by placing

$$f(s)=3\chi(s)\sin s-\chi^2(s)\cos^2 s-\chi'(s)\cos s,$$

if $\chi(s)$ is defined by

$$\chi(s)=s^{-\frac{1}{2}}(\cos s)\cos(\log s)$$

for $1\leqq s<\infty$ (and arbitrarily for $0\leqq s<1$, with the only restriction that $\chi'(s)=d\chi(s)/ds$ exists and is continuous for $0\leqq s\leqq 1$ also).

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[1] H. Weyl, Math. Annalen **68**, 238 (1910).

[2] P. Hartman, Am. J. Math. **71**, 71 (1949).

[3] A. Wintner, Phys. Rev. **72**, 81 (1947).

[4] See, for instance, L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1935), p. 58, or H. A. Kramers, *Die Grundlagen der Quantentheorie* (1938), end of first paragraph on p. 72.

[5] A. Wintner, Am. J. Math. **68**, 385–397 (1946). An $f(s)$ of the type desired above results by choosing $\lambda=0$ and $\mu=\infty$ on p. 396 and then proceeding as on the bottom of p. 396 and the top of p. 397. For details, see Theorem (i) in a paper of the authors, to appear shortly in the Am. J. Math.

Nuclear Quadrupole Effects in the Microwave Spectrum of the Asymmetric Top Molecule, Vinyl Chloride*

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The study of nuclear quadrupole effects in molecular spectra has heretofore been confined to linear and symmetric top molecules. The theory of these effects has recently been extended to asymmetric tops by Bragg.[1] This communication is a preliminary report of the analysis of the observed hyperfine structure in the microwave spectrum of the asymmetric top vinyl chloride (C_2H_3Cl) with the aid of the first-order theory.

Table I gives the principal lines in the *K*-band spectrum of vinyl chloride and the frequencies corrected for quadrupole shift, together with the values of the rotational constants and the asymmetry parameter κ derived from them. The transitions are designated in the notation of King, Hainer, and Cross,[2] and the listed frequencies are those of the principal component of each multiplet.

The transitions were identified on the basis of an assumed planar model for the molecule, and were verified by measurement of Stark effect, relative intensities, temperature coefficient of intensity, and isotopic shift. The rotational constants were calculated from the corrected $1_{10}\rightarrow 2_{11}$ and $1_{11}\rightarrow 2_{12}$ frequencies using the Mecke sum rules,[3] and κ was evaluated on the assumption of planarity. From these results, the predicted frequencies for the $1_{01}\rightarrow 2_{02}$ transitions are 22,946.3 Mc and 22,485.3 Mc for $C_2H_3Cl^{35}$ and $C_2H_3Cl^{37}$, respectively. The close agreement of these values with the observed frequencies indicates that the deviation from planarity is very small.

The first-order quadrupole interaction energy was calculated from the expression:

$$E_Q=\frac{2}{2J+1}\left[\frac{\frac{3}{4}C(C+1)-I(I+1)J(J+1)}{2I(2I-1)(2J-1)(2J+3)}\right]\sum_{\tau'}\{\chi_{aa}(\lambda^a{}_{J_\tau;J_{\tau'}}-\lambda^c{}_{J_\tau;J_{\tau'}})+\chi_{bb}(\lambda^b{}_{J_\tau;J_{\tau'}}-\lambda^c{}_{J_\tau;J_{\tau'}})\}, \qquad (1)$$

where $C=F(F+1)-I(I+1)-J(J+1)$, χ_{aa} and χ_{bb} are the two parameters of the problem, defined by $\chi_{aa}=eQ(\partial^2 V/\partial a^2)$ and $\chi_{bb}=eQ(\partial^2 V/\partial b^2)$, *a* and *b* being the principal axes of least and intermediate moment of inertia, respectively; *Q* is the nuclear quadrupole moment; and $\partial^2 V/\partial_a{}^2$ and $\partial^2 V/\partial_b{}^2$ are components of the molecular electric field gradient at the chlorine nucleus. The quantities $\lambda^a{}_{J_\tau;J_{\tau'}}$, etc., are the line strengths, tabulated by Cross, Hainer, and King[2] for the transition $J_\tau \rightarrow J_{\tau'}$ appropriate to the given value of κ, and for a component of dipole moment parallel to the axis indicated by the superscript. The values of χ_{aa} and χ_{bb}, determined by a least square fit of two of the well resolved transitions of $C_2H_3Cl^{35}$ are given in Table II.

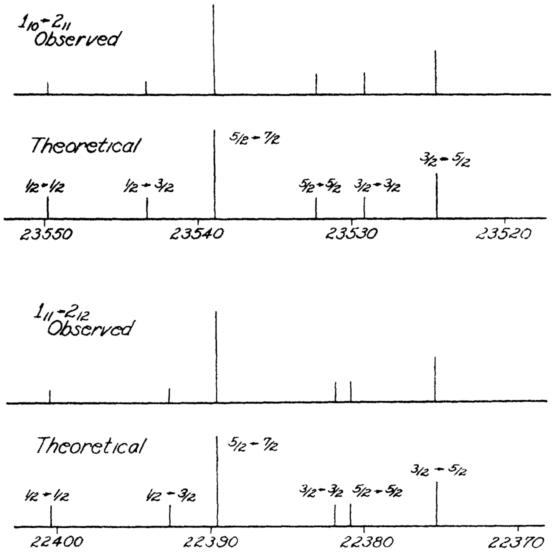

FIG. 1. Comparison of observed and calculated hyperfine structures.

A comparison of the observed hyperfine structures with those calculated from the values of χ_{aa} and χ_{bb} given in columns I and II of Table II is shown in Fig. 1. The indicated intensities are only approximate. The two weak components corresponding to $\Delta F=-1$ were not observed. Analysis of the structure of the $1_{01}\rightarrow 2_{02}$ transition gave a value of -57 ± 1 Mc for χ_{aa}, using for χ_{bb} the average given in Table II.

TABLE I. K-band spectrum and rotational constants of vinyl chloride.

Transition	$C_2H_3Cl^{35}$ Observed	$C_2H_3Cl^{35}$ Corr. for quad. effect	$C_2H_3Cl^{37}$ Observed	$C_2H_3Cl^{37}$ Corr. for quad. effect
$1_{01}\rightarrow 2_{02}$	22,946.9 Mc	22,945.7 Mc	22,485.9 Mc	22,485.0 Mc
$1_{10}\rightarrow 2_{11}$	23,538.9	23,535.6	23,055.0	23,052.4
$1_{11}\rightarrow 2_{12}$	22,369.6	22,366.2	21,930.2	21,927.5
B, Mc	6030.5		5903.7	
C, Mc	5445.2		5341.3	
κ	−0.9769		−0.9779	

TABLE II. Values of the quadrupole coupling parameters χ_{aa} and χ_{bb}.

	I $1_{10}\rightarrow 2_{11}$	II $1_{11}\rightarrow 2_{12}$	III Average
χ_{aa} Mc	−57.6	−57.2	−57.4
χ_{bb} Mc	26.1	26.3	26.2

The value of χ_{zz}, the quadrupole coupling constant along the C−Cl bond calculated from the above data and assuming an axially symmetric bond is −62 Mc. However, the assumption of axial symmetry leads to an unreasonable angle of orientation of the bond in the principal axis system, and would, moreover, not be expected if appreciable double bond character is present. Alternately, the assumption that the C−Cl bond is a principal axis of the χ-dyadic, at an inclination of about 20° to the a axis, as determined from the moments of inertia, leads to a value of −67 Mc for χ_{zz}. The calculations indicate that the value of χ_{zz} probably lies between −62 and −70 Mc.

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On the Decay of Isotropic Turbulence

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IN a recent paper Heisenberg[1] has discussed the decay of isotropic turbulence on the basis of the equation

$$-\frac{\partial}{\partial t}\int_0^k F(k,t)dk=2\left\{\frac{\mu}{\rho}+\kappa\int_k^\infty\left(\frac{F(k,t)}{k^3}\right)^{\frac{1}{2}}dk\right\}\int_0^k F(k,t)k^2dk, \tag{1}$$

where $F(k, t)$ represents the spectrum of turbulence at time t, μ the viscosity, ρ the density and κ a pure number of order unity. Heisenberg has pointed out that Eq. (1) admits solutions of the form

$$F(k,t)=t^{-\frac{1}{2}}f(k\sqrt{t}), \tag{2}$$

where $f(x)$ satisfies the integral equation

$$\int_0^x f(x)dx-\tfrac{1}{2}xf(x)=2\left\{\frac{\mu}{\rho}+\kappa\int_x^\infty\left(\frac{f(x)}{x^3}\right)^{\frac{1}{2}}dx\right\}\int_0^x f(x)x^2dx. \tag{3}$$

Heisenberg justifies the similarity hypothesis (2) on dimensional grounds.

Now from Eq. (3) it readily follows that

$$f(x)\rightarrow\text{constant}\times x \quad\text{as}\quad x\rightarrow 0. \tag{4}$$

Heisenberg, further, states that for $\mu=0$, $f(x)\sim$constant $\times x^{-5/3}$ (i.e., the stationary Kolmogoroff spectrum) for $x\rightarrow\infty$, and that for $\mu\rightarrow\infty$ (i.e., for small Reynolds numbers) $f(x)$ $\rightarrow$constant$\times x^{-7}$ for $x\rightarrow\infty$. It does not seem that the latter statement concerning the solution of Eq. (3) is correct. For letting

$$g=\kappa^2x^3f(x)\quad\text{and}\quad y=\kappa^2\int_0^x f(x)x^2dx, \tag{5}$$

we can show that g as a function of y satisfies the second-order differential equation

$$g^{\frac{3}{2}}g''+2y(4+g')+2g^{\frac{1}{2}}(4-g')-8g=0, \tag{6}$$

where primes denote differentiation with respect to y.

The requirement that $f(x)\rightarrow$constant$\times x$ as $x\rightarrow 0$ now implies that $g(y)$ is tangential to the line $g=4y$ at $y=0$. From Eq. (6) it now follows that there is a one-parametric family of solutions which have this property: indeed, the asymptotic behavior of these solutions at the origin is given by

$$g=4y+y^{\frac{3}{2}}\left(a+\frac{4}{3}\log y\right)+O(y^2\log^2 y), \tag{7}$$

where a is an arbitrary constant. The constant a is related to μ/ρ of Eq. (3) by

$$\mu/c\rho=\tfrac{1}{6}-0.375a-\underset{\substack{x\rightarrow 0\\ y\rightarrow 0}}{\text{limit}}\left(\int_x^\infty (g)^{\frac{1}{2}}\frac{dx}{x^3}+\tfrac{1}{2}\log y\right), \tag{8}$$

where c is a constant; also, in evaluating the integral on the