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On the Theory of Diffraction by an Aperture in an Infinite Plane Screen. II.

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The diffraction of a scalar plane wave by an aperture in an infinite plane screen is examined theoretically. The wave function at an arbitrary point of space is expressed in terms of the discontinuity in its normal derivative at the screen, where the boundary condition is that of vanishing wave function. An integral equation for the discontinuity in normal derivative (or the residual function which measures its deviation from the simple distribution appropriate to a completely infinite screen) is the result of applying the boundary condition to the space wave function. Utilizing the integral equation (whose solution is generally unobtainable), the diffracted spherical wave amplitude at large distances from the aperture is cast into a form which is stationary with respect to small variations (relative to the correct values) of the residual functions arising from a pair of incident waves. An homogeneous expression for the amplitude is exhibited wherein the part independent of the residual functions defines a Kirchoff approximation. The connection with another stationary form of the amplitude, involving a pair of aperture wave functions, is examined. A variational expression for the plane wave transmission cross section of the aperture is based on the amplitude observed in the direction of incidence. The variational formulation is applied for a wave incident normally on a circular aperture. By comparison with the exact results available for this problem, it appears that use of simple residual functions in the variational formulation yields approximate, yet accurate expressions for the diffracted amplitude and transmission cross section over a wide range of frequencies.

I. INTRODUCTION

In a previous paper of the same title,[1] the diffraction of a scalar plane wave by an aperture in an infinite plane screen was described in terms of a variational principle. The viewpoint adopted in this formulation regards the aperture as a coupling surface of the half spaces on opposite sides of the screen. For a boundary condition of vanishing wave function on the screen, the amplitude of the diffracted spherical wave at large distances from the aperture is exhibited in a form which is stationary with respect to small variations (relative to the correct values) of the aperture fields arising from a pair of incident waves.

The purpose of this paper is to develop a second variational principle for the diffracted amplitude by considering the screen as an obstacle to the propagation of the incident wave through free space. With the same boundary condition as previously, a pair of discontinuities in normal derivative of the fields at the screen appear in the role of functions which render the variational expression stationary. (Such functions are analogous to the surface currents excited on a perfectly conducting screen by incident electromagnetic waves.)

For explicit construction of the second variational principle, it proves convenient to deal with the residual functions which measure the deviation of the discontinuities in normal derivative from the simple distributions appropriate to a completely infinite screen. This procedure leads directly to the desired spherical wave amplitude in the radiation field of the screen, as distinct from the plane wave part of the radiation field that exists even in the absence of the aperture.

[1] H. Levine and J. Schwinger, Phys. Rev. **74**, 958 (1948); hereafter referred to as I.

The diffracted spherical wave amplitude is shown to be invariant with respect to reversal in the sense of excitation and observation along a pair of directions in space. With the assistance of this reciprocity relation, the amplitude is exhibited in a form which is stationary for small independent variations (relative to the correct values) of a pair of residual functions arising from excitation along the foregoing directions. Following a scale transformation of the residual functions, this relation is converted to a homogeneous form. A part of the diffracted amplitude, independent of the residual functions, is identified as the Kirchoff approximation for it is the result obtained by ignoring the influence of the aperture on the discontinuity in normal derivative of the wave function at the screen.

The plane wave transmission cross section of the aperture is related to the imaginary part of the spherical wave amplitude observed in the direction of incidence, and hence retains the stationary property. In the limit of vanishingly small wave-length, λ, compared to the aperture dimensions, the Kirchoff part alone contributes and the geometrical optics form of the cross section emerges. At long wave-lengths the cross section varies with λ^{-2} except for the correct residual functions, unlike the behavior of the earlier variational form of the cross section, where the λ^{-4} variation is obtained with any real aperture field (subject to the boundary condition at the rim of the screen). Thus the two variational principles are individually adapted to opposite extremes of wave-length, and their over-all compatibility indicates the departure from a correct solution.

The variational formulation developed below is applied in detail for a plane wave normally incident on a circular aperture. Numerical values of the

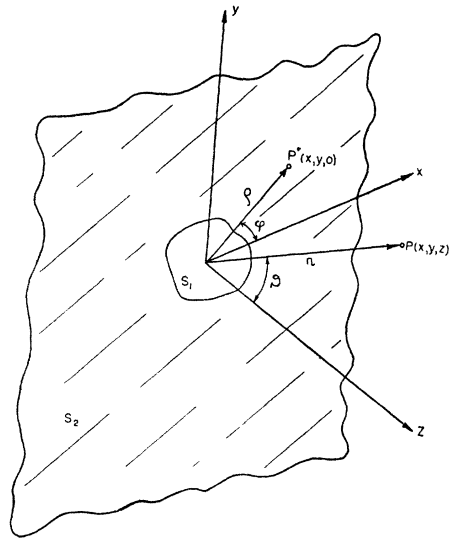

FIG. 1. Diffracting aperture in a plane screen.

transmission coefficient (transmission cross section/area of aperture) based on a relatively simple form of residual function are found to exhibit a high order of accuracy; it may also be noted that the Kirchoff procedure affords a reasonably satisfactory approximation.

II. INTEGRAL EQUATION FORMULATION FOR AN APERTURE OF ARBITRARY OPENING

We consider an infinitesimally thin plane screen S_2, of infinite extent, which is perforated by an aperture S_1. A rectangular coordinate system is chosen with origin at some point of the aperture, and oriented so that the screen lies in the x,y plane (Fig. 1).

A plane wave is incident on the screen in the half-space $z<0$, and it is desired to investigate the diffracted field. The incident wave, propagating in the direction ϑ', φ' (ϑ' measured from the positive direction of the z axis, and φ' from the positive direction of the x axis, in the x,y plane) is described by the scalar wave function

$$\phi^{\text{inc}}(\mathbf{r})=\exp(ik\mathbf{n}'\cdot\mathbf{r})=\exp[ik(x\sin\vartheta'\cos\varphi' +y\sin\vartheta'\sin\varphi'+z\cos\vartheta')], \quad (2.1)$$

where $\mathbf{n}'$ is a unit vector in the direction of propagation, $k=2\pi/\lambda$ is the free space propagation constant, and λ the corresponding wave-length. The harmonic time dependence $\exp(-i\omega t)$, $\omega=kc$, with c the velocity of wave propagation, is omitted throughout.

The wave function describing the complete (incident+diffracted) field satisfies the wave equation

$$(\nabla^2+k^2)\phi(\mathbf{r})=0 \quad (2.2)$$

at all points of space, and is subject to the prescribed boundary condition

$$\phi(\mathbf{r})=0, \quad \mathbf{r} \text{ on } S_2; \quad (2.3)$$

in addition, the wave function and its normal (i.e., z) derivative vary continuously on passing through the aperture.

Pursuant to a description of the screen as an obstacle imbedded in free space, we apply Green's second scalar identity,

$$\int[G(\mathbf{r}',\mathbf{r})(\nabla'^2+k^2)\phi(\mathbf{r}') -\phi(\mathbf{r}')(\nabla'^2+k^2)G(\mathbf{r}',\mathbf{r})]d\tau' =\int[G(\mathbf{r}',\mathbf{r})(\partial/\partial n')\phi(\mathbf{r}') -\phi(\mathbf{r}')(\partial/\partial n')G(\mathbf{r}',\mathbf{r})]dS', \quad (2.4)$$

to the wave function $\phi(\mathbf{r})$ and free space scalar Green's function,

$$G(\mathbf{r},\mathbf{r}')=\exp(ik|\mathbf{r}-\mathbf{r}'|)/4\pi|\mathbf{r}-\mathbf{r}'|=G(\mathbf{r}',\mathbf{r}), \quad (2.5)$$

within the domain bounded by the surfaces S_-, S_+ and parts of the respective faces of the screen (Fig. 2); the derivative is taken along the outward normal at each point of the bounding surface.

By virtue of (2) and the wave equation for the Green's function,

$$(\nabla^2+k^2)G(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}'), \quad (2.6)$$

where $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function, the volume integral in (4) yields simply $\phi(\mathbf{r})$.

If the surface S_+ on the far side of the screen ($z>0$) is sufficiently remote, the wave and Green's functions there exhibit the behavior of spherical waves diverging from the aperture and field point $\mathbf{r}$, respectively, and the surface integral in (4) vanishes. Moreover, for a similarly disposed surface S_- on the near side of the screen ($z<0$), where the wave function describes incident and specularly reflected plane waves appropriate to a completely infinite screen (i.e., with no aperture), *viz.*:

$$\phi^{(0)}(\mathbf{r})=\phi^{\text{inc}}(x,y,z)-\phi^{\text{inc}}(x,y,-z), \quad \phi^{(0)}(x,y,0)=0, \quad (2.7)$$

in addition to a spherical wave diverging from the aperture, it is only $\phi^{(0)}(\mathbf{r})$ which contributes to the surface integral in (2.4). Thus, utilizing the boundary condition (2.3) and taking account of the oppositely directed normal derivatives at the two faces of the screen, we obtain for the wave function at an arbitrary point in the enclosed region,

$$\phi(\mathbf{r})=\int_{S_-}[G(\mathbf{r},\mathbf{r}')(\partial/\partial n')\phi^{(0)}(\mathbf{r}')-\phi^{(0)}(\mathbf{r}')(\partial/\partial n')G(\mathbf{r},\mathbf{r}')]dS'+\int_{[S_2]}\Psi(\varrho')G(\mathbf{r};x',y',0)dS', \quad (2.8)$$

with

$$\Psi(\varrho)=(\partial/\partial z)\phi(x,y,-0)-(\partial/\partial z)\phi(x,y,+0). \quad (2.9)$$

Here ϱ denotes a position vector in the x,y plane, and $[S_2]$ that portion of the screen encompassed by the integration contour (ultimately all inclusive).

In the absence of an aperture, the result of applying Green's second scalar identity to the functions $\phi^{(0)}(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$ within the region bounded by S_- and the screen, is

$$\int_{S_-}[G(\mathbf{r},\mathbf{r}')(\partial/\partial n')\phi^{(0)}(\mathbf{r}')-\phi^{(0)}(\mathbf{r}')(\partial/\partial n')G(\mathbf{r},\mathbf{r}')]dS'+\int_{[S_2]}\Psi^{(0)}(\varrho')G(\mathbf{r};x',y',0)dS'+\int_{S_1}\Psi^{(0)}(\varrho')G(\mathbf{r};x',y',0)dS'$$
$$=\phi^{(0)}(\mathbf{r}),\quad z<0,$$
$$=0,\quad z>0, \quad (2.10)$$

where, by (2.1), (2.7)

$$\phi^{(0)}(\mathbf{r})=2i\sin(kz\cos\vartheta')\exp(ik\mathbf{n}'\cdot\varrho),$$
$$\Psi^{(0)}(\varrho)=(\partial/\partial z)\phi^{(0)}(x,y,-0)=2ik\cos\vartheta'\exp(ik\mathbf{n}'\cdot\varrho). \quad (2.11)$$

Hence, on subtracting (2.10) from (2.8), we obtain

$$\phi(\mathbf{r})=\begin{Bmatrix}\phi^{(+)}(\mathbf{r}),\ z>0\\ \phi^{(-)}(\mathbf{r}),\ z<0\end{Bmatrix}=\begin{Bmatrix}0\\ \phi^{(0)}(\mathbf{r})\end{Bmatrix}+\int_{S_2}[\Psi(\varrho')-\Psi^{(0)}(\varrho')]G(\mathbf{r};x',y',0)dS'-\int_{S_1}\Psi^{(0)}(\varrho')G(\mathbf{r};x',y',0)dS', \quad (2.12)$$

where integration over the entire screen is permissible, since $\Psi-\Psi^{(0)}$ vanishes at a point infinitely removed from the aperture. Introducing

$$\Psi(\varrho)-\Psi^{(0)}(\varrho)=2ik\cos\vartheta'\psi_{\mathbf{n}'}(\varrho) \quad (2.13)$$

(with dependence of $\psi_{\mathbf{n}'}$ on the direction of the incident plane wave indicated explicitly) and employing (2.11), Eq. (12) takes the form

$$\begin{Bmatrix}\phi^{(+)}(\mathbf{r})\\ \phi^{(-)}(\mathbf{r})\end{Bmatrix}=\begin{Bmatrix}0\\ \phi^{(0)}(\mathbf{r})\end{Bmatrix}+2ik\cos\vartheta'\left[\int_{S_2}\psi_{\mathbf{n}'}(\varrho')G(\mathbf{r};x',y',0)dS'-\int_{S_1}\exp(ik\mathbf{n}'\cdot\varrho')G(\mathbf{r};x',y',0)dS'\right]. \quad (2.14)$$

The wave functions $\phi^{(+)}$, $\phi^{(-)}$ in (14) are equal at any point of the aperture, since $\phi^{(0)}$ is an odd function of z and vanishes in the plane of the screen. Furthermore, with the Fourier integral representation of the Green's function (I, 2.5), it follows that

$$(\partial/\partial z)G(x,y,0;x',y',0)=\mp\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\exp\{i[k_x(x-x')+k_y(y-y')]\}dk_xdk_y=\mp\tfrac{1}{2}\delta(x-x')\delta(y-y'),$$

where the upper and lower signs apply for $z\to+0$, $z\to-0$, respectively; thus, from (14)

$$\begin{Bmatrix}(\partial/\partial z)\phi^{(+)}(x,y,0)\\ (\partial/\partial z)\phi^{(-)}(x,y,0)\end{Bmatrix}=\begin{Bmatrix}0\\ 2ik\cos\vartheta'\exp(ik\mathbf{n}'\cdot\varrho)\end{Bmatrix}\mp ik\cos\vartheta'\left[\int_{S_2}\psi_{\mathbf{n}'}(\varrho')\delta(x-x')\delta(y-y')dS'-\int_{S_1}\exp(ik\mathbf{n}'\cdot\varrho')\delta(x-x')\delta(y-y')dS'\right].$$

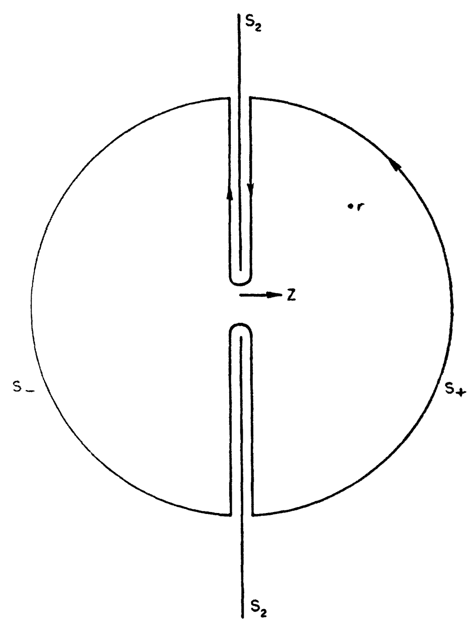

FIG. 2. Integration domain for Eq. (2.4).

Using familiar properties of the delta-function, we obtain

$$(\partial/\partial z)\phi^{(+)}(x,y,0)=(\partial/\partial z)\phi^{(-)}(x,y,0)=(\partial/\partial z)\phi^{\mathrm{inc}}(x,y,0),\quad \varrho \text{ in } S_1$$

for a point in the aperture (compare I, 2.14), and (after subtraction) the relation (2.13) for a point on the screen. Finally, by imposing the boundary condition (3) on either of the wave functions (14), we arrive at the integral equation

$$\int_{S_2}\psi_{\mathbf{n}'}(\varrho')G(\varrho,\varrho')dS'=\int_{S_1}\exp(ik\mathbf{n}'\cdot\varrho')G(\varrho,\varrho')dS',\quad \varrho \text{ on } S_2,\qquad G(\varrho,\varrho')=G(x,y,0;x',y',0), \tag{2.15}$$

to determine $\psi_{\mathbf{n}'}(\varrho)$ and thus, from (14), the wave function at any point of space.

According to (2.15), the projection of $\mathbf{n}'$ on the x,y plane distinguishes each function $\psi_{\mathbf{n}'}$ corresponding to a given direction of incidence. In particular, with excitation restricted to the half-space $z<0$, a reversal in the sign of $\mathbf{n}'$ implies an increase by π in the azimuthal angle φ'. By invoking the reflection symmetry with respect to the plane of the screen, this sign reversal can be ascribed to excitation in the direction opposite to $\mathbf{n}'$.

As in I, we shall confine our attention to the properties of the diffracted field at distances from the aperture large compared to its dimensions and the wave-length. Since a rigorous and explicit solution of the integral Eq. (2.15) is not generally feasible, the far field amplitude will be cast into a form which is insensitive to small deviations of the assumed solutions from the correct ones.

Inserting the asymptotic form of the Green's function

$$G(\mathbf{r},\mathbf{r}')\simeq\exp(ik\{r-\mathbf{n}\cdot\mathbf{r}'\})/4\pi r,\quad r\rightarrow\infty \tag{2.16}$$

in (14), we obtain the transmitted field in the form of a diverging spherical wave,

$$\phi^{(+)}(\mathbf{r})\simeq A(\mathbf{n},\mathbf{n}')(e^{ikr}/r),\quad r\rightarrow\infty, \tag{2.17}$$

with the directionality factor

$$A(\mathbf{n},\mathbf{n}')=\frac{ik\cos\vartheta'}{2\pi}\left[\int_{S_2}\psi_{\mathbf{n}'}(\varrho)\exp(-ik\mathbf{n}\cdot\varrho)dS-\int_{S_1}\exp(ik(\mathbf{n}'-\mathbf{n})\cdot\varrho)dS\right]. \tag{2.18}$$

It is clear from (2.14), (2.18) that an identical spherical wave appears along the direction obtained by reflecting $\mathbf{n}$ in the plane of the screen.

The transmission cross section of the aperture for a plane wave is calculated from the diffracted amplitude in the direction of incidence by the relation developed in I, namely,

$$\sigma(\mathbf{n}')=-(2\pi/k)ImA(\mathbf{n}',\mathbf{n}'). \tag{2.19}$$

III. VARIATIONAL PRINCIPLE FOR DIFFRACTED WAVE AMPLITUDE

Multiplying both sides of (2.15) by $\psi_{\mathbf{n}''}(\varrho)$ and integrating over the screen, we find

$$\int_{S_2}\psi_{\mathbf{n}'}(\varrho)G(\varrho,\varrho')\psi_{\mathbf{n}''}(\varrho')dSdS'=\int_{S_2}\psi_{\mathbf{n}''}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(ik\mathbf{n}'\cdot\varrho')dS'=\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS' \tag{3.1}$$

with recognition of the symmetry exhibited by the first term in $\mathbf{n}'$, $\mathbf{n}''$ (or the angular coordinates ϑ', φ' and ϑ'', φ''). Further, by invoking the additive nature of integration in the domains S_1, S_2, and with recourse to (2.15) once again, it follows that

$$\begin{aligned}\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS'&=\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1+S_2}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS'\\&\quad-\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_2}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS'\\&=\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1+S_2}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS'\\&\quad-\int_{S_1}\exp(ik\mathbf{n}'\cdot\varrho)dS\int_{S_2}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS'\\&=\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1+S_2}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS'\\&\quad-\int_{S_1}\exp(ik\mathbf{n}'\cdot\varrho)dS\int_{S_1+S_2}G(\varrho,\varrho')\times\exp(ik\mathbf{n}''\cdot\varrho')dS'\\&\quad+\int_{S_1}\exp(ik(\mathbf{n}'\cdot\varrho+\mathbf{n}''\cdot\varrho'))G(\varrho,\varrho')dSdS'.\end{aligned} \tag{3.2}$$

The integration in S_1+S_2 is conveniently performed by use of the Fourier integral representation

$$G(\varrho,\varrho')=\frac{\exp(ik|\varrho-\varrho'|)}{4\pi|\varrho-\varrho'|}=\frac{i}{8\pi^2}\int_{-\infty}^{\infty}\frac{\exp[i\{k_x(x-x')+k_y(y-y')\}]}{(k^2-k_x^2-k_y^2)^{\frac{1}{2}}}dk_xdk_y,\qquad \arg(k^2-k_x^2-k_y^2)^{\frac{1}{2}}\geq 0; \tag{3.3}$$

thus

$$\int_{S_1+S_2} G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS' = \frac{i}{2}\int_{-\infty}^{\infty}\delta(k_x-kn_x'')\delta(k_y-kn_y'') \times\frac{\exp[i\{k_x x+k_y y\}]}{(k^2-k_x^2-k_y^2)^{\frac{1}{2}}}dk_x dk_y=\frac{i\exp(ik\mathbf{n}''\cdot\varrho)}{2k|\cos\vartheta''|} \quad (3.4)$$

(where n_x'', n_y'' denote components of the vector $\mathbf{n}''$ along the x,y directions, respectively). Hence the relation (2) becomes

$$\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(ik\mathbf{n}''\cdot\varrho')dS'$$
$$=[i/2k|\cos\vartheta''|]\left[\int_{S_2}\psi_{\mathbf{n}'}(\varrho)\exp(ik\mathbf{n}''\cdot\varrho)dS - \int_{S_1}\exp(ik(\mathbf{n}'+\mathbf{n}'')\cdot\varrho)dS\right]$$
$$+\int_{S_1}\exp(ik(\mathbf{n}'\cdot\varrho+\mathbf{n}''\cdot\varrho'))G(\varrho,\varrho')dSdS'$$

and from (3.1), (2.18),

$$\int_{S_2}\psi_{\mathbf{n}'}(\varrho)G(\varrho,\varrho')\psi_{-\mathbf{n}''}(\varrho')dSdS' - \int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(-ik\mathbf{n}''\cdot\varrho')dS'$$
$$-\int_{S_2}\psi_{-\mathbf{n}''}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(ik\mathbf{n}'\cdot\varrho')dS'$$
$$=-[\pi/k^2\cos\vartheta'\cos\vartheta'']\times[A(\mathbf{n}'',\mathbf{n}')-A_K(\mathbf{n}'',\mathbf{n}')]$$
$$=-[\pi/k^2\cos\vartheta'\cos\vartheta'']\times[A(-\mathbf{n}',-\mathbf{n}'')-A_K(-\mathbf{n}',-\mathbf{n}'')] \quad (3.5)$$

with

$$A_K(\mathbf{n}'',\mathbf{n}')=A_K(-\mathbf{n}',-\mathbf{n}'') = -[k^2\cos\vartheta'\cos\vartheta''/\pi] \times\int_{S_1}\exp(ik(\mathbf{n}'\cdot\varrho-\mathbf{n}''\cdot\varrho'))G(\varrho,\varrho')dSdS'. \quad (3.6)$$

The expressions (3.5), (3.6) apply for ϑ', $\vartheta''<\pi/2$, since the absolute value signs have been discarded; the remaining amplitudes are readily obtained by means of the reflection symmetry in the plane of the screen. With omission of $\psi_{\mathbf{n}'}$, $\psi_{-\mathbf{n}''}$, or equivalently, on employing the wave functions appropriate to a completely infinite screen, the amplitudes A, A_K are identical. A_K may thus be described as a Kirchoff approximation for the diffracted amplitude, since it disregards the modification of the fields occasioned by the presence of an aperture. This Kirchoff approximation is to be distinguished from the more familiar type, in which aperture and incident fields are identified, with disregard for the presence of the screen.

Equality of the amplitudes $A(\mathbf{n}'',\mathbf{n}')$ and $A(-\mathbf{n}',-\mathbf{n}'')$ (or the corresponding Kirchoff amplitudes) describes a reciprocity condition for incidence and observation along a pair of directions in space.

On performing independent variations of $\psi_{\mathbf{n}'}$, $\psi_{-\mathbf{n}''}$ in (3.5), it is found that a stationary value of $A(\mathbf{n}'',\mathbf{n}')$ obtains, provided these functions obey integral equations of the form (2.15). If we introduce the scale transformations

$$\psi_{\mathbf{n}'}(\varrho)\rightarrow\alpha\psi_{\mathbf{n}'}(\varrho),\quad \psi_{-\mathbf{n}''}(\varrho)\rightarrow\beta\psi_{-\mathbf{n}''}(\varrho)$$

($\alpha=\beta=0$ for the Kirchoff approximation) and apply the stationary requirement to the determination of α,β, we arrive at a variational expression for $A(\mathbf{n}'',\mathbf{n}')$,

$$A(\mathbf{n}'',\mathbf{n}')=A_K(\mathbf{n}'',\mathbf{n}')+[k^2\cos\vartheta'\cos\vartheta''/\pi] \times\frac{\left(\int_{S_2}\psi_{\mathbf{n}'}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(-ik\mathbf{n}''\cdot\varrho')dS'\right)\left(\int_{S_2}\psi_{-\mathbf{n}''}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(ik\mathbf{n}'\cdot\varrho')dS'\right)}{\int_{S_2}\psi_{\mathbf{n}'}(\varrho)G(\varrho,\varrho')\psi_{-\mathbf{n}''}(\varrho')dSdS'}, \quad (3.7)$$

which is homogeneous in $\psi_{\mathbf{n}'}$, $\psi_{-\mathbf{n}''}$.

A comparison of the two variational principles for the spherical wave amplitude is instructive; to this end, we recall the equations (I, 2.11, 2.12)

$$\phi^{(+)}(\mathbf{r})=\int_{S_1}\phi(\varrho')(\partial/\partial z')G(\mathbf{r};x',y',0)dS', \quad (3.8)$$

$$\phi^{(-)}(\mathbf{r})=\phi^{(0)}(\mathbf{r})-\int_{S_1}\phi(\varrho')(\partial/\partial z')G(\mathbf{r};x',y',0)dS', \quad (3.9)$$

for the wave function at any point of space in terms of its values in the aperture. In accord with (2.9), (2.13), we obtain on differentiating (3.8), (3.9),

$$\Psi(\varrho)-\Psi^{(0)}(\varrho)=2ik\cos\vartheta\psi_{\mathbf{n}}(\varrho)$$
$$=-2\int_{S_1}\phi_{\mathbf{n}}(\varrho')K(\varrho,\varrho')dS',$$

and hence

$$\psi_{\mathbf{n}}(\varrho)=(i/k\cos\vartheta)\int_{S_1}\phi_{\mathbf{n}}(\varrho')K(\varrho,\varrho')dS',\quad(3.10)$$

where

$$K(\varrho,\varrho')=(\partial/\partial z)(\partial/\partial z')G(x,y,0;x',y',0)=K(\varrho',\varrho).\quad(3.11)$$

The relation (10) may be adopted as a general form of the residual function, to be evaluated with an assumed aperture field. If this be inserted in (3.5) and the resulting expression transformed after the fashion of (3.2)–(3.4), it turns out that

$$A(\mathbf{n}'',\mathbf{n}')=\frac{1}{4\pi}\bigg[\int_{S_1}\phi_{\mathbf{n}'}(\varrho)K(\varrho,\varrho')\phi_{-\mathbf{n}''}(\varrho')dSdS'+ik\cos\vartheta'\int_{S_1}\phi_{-\mathbf{n}''}(\varrho)\exp(ik\mathbf{n}'\cdot\varrho)dS$$
$$+ik\cos\vartheta''\int_{S_1}\phi_{\mathbf{n}'}(\varrho)\exp(-ik\mathbf{n}''\cdot\varrho)dS\bigg]-\frac{1}{\pi}\int_{S_1}\bigg[\int_{S_1}\phi_{\mathbf{n}'}(\varrho'')K(\varrho'',\varrho)dS''$$
$$-ik\cos\vartheta'\exp(ik\mathbf{n}'\cdot\varrho)\bigg]G(\varrho,\varrho')\bigg[\int_{S_1}K(\varrho',\varrho''')\phi_{-\mathbf{n}''}(\varrho''')dS'''-ik\cos\vartheta''\exp(-ik\mathbf{n}''\cdot\varrho')\bigg]dSdS'.$$

We observe that with aperture wave functions which satisfy the exact integral equation

$$\int_{S_1}\phi_{\mathbf{n}}(\varrho')K(\varrho,\varrho')dS'=ik\cos\vartheta\exp(ik\mathbf{n}\cdot\varrho),\quad\varrho\text{ on }S_1\quad(\text{I},2.9)$$

(obtained by equating the z derivatives of (3.8), (3.9) in the aperture), only the first three terms in the preceding equation remain; more generally, on retaining just these terms, the amplitude $A(\mathbf{n}'',\mathbf{n}')$ is stationary with respect to variations relative to the exact aperture wave functions. Following a scale transformation of the trial wave functions, the resulting amplitude

$$\frac{1}{A(\mathbf{n}'',\mathbf{n}')}=\frac{(4\pi/k^2)\int_{S_1}\phi_{\mathbf{n}'}(\varrho)K(\varrho,\varrho')\phi_{-\mathbf{n}''}(\varrho')dSdS'}{\cos\vartheta'\cos\vartheta''\int_{S_1}\phi_{\mathbf{n}'}(\rho)\exp(-ik\mathbf{n}''\cdot\varrho)dS\int_{1}\phi_{-\mathbf{n}''}(\varrho)\exp(ik\mathbf{n}'\cdot\varrho)dS},\quad(\text{I},3.2)$$

is identical with that derived from the variational principle in I. Thus, a practical measure of proximity to the correct solution is obtained from the over-all discrepancy in the two variational principles, following the use of assumed trial functions.

Combining (2.19) with (3.6), (3.7), the transmission cross section of the aperture for a plane wave incident in the direction $\mathbf{n}$ assumes the form

$$\sigma(\mathbf{n})=\sigma_K(\mathbf{n})-2k\cos^2\vartheta Im$$
$$\times\frac{\left(\int_{S_2}\psi_{\mathbf{n}}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(-ik\mathbf{n}\cdot\varrho')dS'\right)\left(\int_{S_2}\psi_{-\mathbf{n}}(\varrho)dS\int_{S_1}G(\varrho,\varrho')\exp(ik\mathbf{n}\cdot\varrho')dS'\right)}{\int_{S_2}\psi_{\mathbf{n}}(\varrho)G(\varrho,\varrho')\psi_{-\mathbf{n}}(\varrho')dSdS'},\quad(3.12)$$

where

$$\sigma_K(\mathbf{n})=2k\cos^2\vartheta Im\int_{S_1}\exp\{ik\mathbf{n}\cdot(\varrho-\varrho')\}G(\varrho,\varrho')dSdS'\quad(3.13)$$

is the Kirchoff contribution.

To examine the behavior of the cross section at high frequencies, a convenient point of departure is the Fourier integral representation (3.3) of $G(\varrho,\varrho')$.

Thus

$$\int_{S_1} \exp\{ik\mathbf{n}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')\}G(\boldsymbol{\rho},\boldsymbol{\rho}')dSdS'$$

$$=(i/8\pi^2)\int_{-\infty}^{\infty}\int_{S_1}\exp[i\{\alpha(x-x')+\beta(y-y')\}]$$

$$\times[dk_xdk_y/(k^2-k_x^2-k_y^2)^{\frac{1}{2}}]dSdS',$$

$$\alpha=k_x+kn_x,\quad \beta=k_y+kn_y,$$

and, by virtue of the increasing oscillations of the exponential factor as $k\to\infty$, the integration variables x, y (or x', y') can be extended throughout the plane $z=0$, with the result

$$\int_{S_1} \exp\{ik\mathbf{n}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')\}G(\boldsymbol{\rho},\boldsymbol{\rho}')dSdS'$$

$$\simeq\frac{i}{2}\int_{-\infty}^{\infty}\int_{S_1}\exp[-i\{\alpha x'+\beta y'\}]\delta(\alpha)\delta(\beta)$$

$$\times[dk_xdk_y/(k^2-k_x^2-k_y^2)^{\frac{1}{2}}]dS'$$

$$=iS_1/2k\cos\vartheta,\quad k\to\infty,\qquad(3.14)$$

where S_1 is the area of the aperture.

Owing to the delta-function character of $G(\boldsymbol{\rho},\boldsymbol{\rho}')$ in the limit of infinite frequency (cf. (3.3)), the second term in (12) tends to zero since the integration domains for the factors in its numerator are mutually exclusive. Thus, from (3.12)–(3.14),*

$$\sigma(\mathbf{n})\simeq\sigma_K(\mathbf{n})\simeq S_1\cos\vartheta,\quad k\to\infty,\qquad(3.15)$$

in which the last term contains the geometrical optics cross section, namely, the projected area of the aperture on a plane normal to the direction of the incident wave.

Although the cross section in the limit of infinite frequency is rigorously obtained independently of the residual functions, the previous considerations imply that deviations in this quantity are inherent for all other frequencies unless the correct residual functions are employed. In particular, with a choice of real, frequency independent residual functions $\psi(\boldsymbol{\rho})$, unrelated to the direction of excitation, and use of the expansions

$$G(\boldsymbol{\rho},\boldsymbol{\rho}')\doteqdot[1/4\pi|\boldsymbol{\rho}-\boldsymbol{\rho}'|]+[ik/4\pi]-\cdots,$$
$$\exp(\pm ik\mathbf{n}\cdot\boldsymbol{\rho})\doteqdot1\pm ik\mathbf{n}\cdot\boldsymbol{\rho}-\cdots,\quad k\to0,$$

we find as the leading term in the frequency expansion of the cross section (12),

$$\sigma(\mathbf{n})=(k^2/2\pi)\cos^2\vartheta\left\{S_1-\frac{\left(\int_{S_2}\psi(\boldsymbol{\rho})dS\right)\left(\int_{S_2}\psi(\boldsymbol{\rho})dS\int_{S_1}(dS'/|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\right)}{\int_{S_2}(\psi(\boldsymbol{\rho})\psi(\boldsymbol{\rho}')/|\boldsymbol{\rho}-\boldsymbol{\rho}'|)dSdS'}\right\}^2,\quad k\to0.\qquad(3.16)$$

The cross section (3.16) departs from the k^4 behavior characteristic of low frequencies, although it embodies the correct angular dependence (compare I, 3.7). Nevertheless, as shown in a later example, these low frequency deviations may prove insignificant when viewed on a scale appropriate to the over-all frequency variation of the cross section.

IV. DIFFRACTION BY A CIRCULAR APERTURE

To illustrate the utility of the variational formulation, we describe an application for the case of normal incidence of a plane wave on a circular aperture. We calculate, in particular, the transmission coefficient of the aperture as a function of the characteristic parameter, $ka=2\pi$(radius of aperture/wave-length).

From (3.12), (3.13) the transmission coefficient for normal incidence on an arbitrary aperture becomes

$$t=\sigma(0)/S_1=t_K-[2k/S_1]Im\times\frac{\left(\int_{S_2}\psi(\boldsymbol{\rho})dS\int_{S_1}G(\boldsymbol{\rho},\boldsymbol{\rho}')dS'\right)^2}{\int_{S_2}\psi(\boldsymbol{\rho})G(\boldsymbol{\rho},\boldsymbol{\rho}')\psi(\boldsymbol{\rho}')dSdS'},\qquad(4.1)$$

$$t_K=(2k/S_1)Im\int_{S_1}G(\boldsymbol{\rho},\boldsymbol{\rho}')dSdS',\qquad(4.2)$$

involving the single residual function $\psi(\boldsymbol{\rho})$ (omitting subscript) which arises from plane wave excitation on opposite sides of the screen.

For a circular domain S_1, it is convenient to introduce polar coordinates ρ, φ in the plane of the aperture, with origin at its center, whence $G(\boldsymbol{\rho},\boldsymbol{\rho}')=G(\rho,\varphi;\rho',\varphi')$ and $\psi(\boldsymbol{\rho})=\psi(\rho)$ is a func-

* Observe that this result follows from (2.18), (2.19) with omission of the residual function.

tion only of the radial coordinate. The Kirchoff transmission coefficient t_K is readily evaluated in this case with the aid of the integral representation (see I, 4.12),

$$G(\rho,\varphi;\rho',\varphi')=\frac{1}{4\pi}\int_0^\infty J_0(\zeta R)\frac{\zeta d\zeta}{(\zeta^2-k^2)^{\frac{1}{2}}},$$
$$R=(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi'))^{\frac{1}{2}}, \quad (4.3)$$

where J_0 denotes the zero-order Bessel function. The integration contour in (4.3) avoids a singularity at $\zeta=k$ by an indentation below the singular point, and $\arg(\zeta^2-k^2)^{\frac{1}{2}}=0$, $\zeta>k$; $=-\pi/2$, $\zeta<k$. Thus, employing the Bessel function addition theorem

$$J_0(\zeta R)=\sum_{n=0}^{\infty}(2-\delta_{0n})J_n(\zeta\rho)J_n(\zeta\rho')\cos n(\varphi-\varphi'),$$
$$\delta_{pq}=\begin{matrix}0, & p\neq q\\ 1, & p=q\end{matrix} \quad (4.4)$$

it follows that

$$t_K=(2k/\pi a^2)Im\int_{S_1}G(\rho,\varphi;\rho',\varphi')\rho d\rho d\varphi\rho' d\rho' d\varphi'$$
$$=(2k/a^2)Im\int_0^\infty\zeta(\zeta^2-k^2)^{-\frac{1}{2}}d\zeta \times\left(\int_0^a\rho J_0(\zeta\rho)d\rho\right)^2.$$

Extracting the imaginary part of the latter expression and performing the ρ-integration, we find after a change of variable,

$$t_K=2\int_0^1(1-v^2)^{-\frac{1}{2}}J_1^2(kav)(dv/v), \quad (4.5)$$

whence (see appendix)

$$\begin{aligned} t_K&=1-(1/ka)J_1(2ka)\\ &\simeq\tfrac{1}{2}(ka)^2, \quad ka\ll 1,\\ &\simeq 1, \quad ka\gg 1. \end{aligned} \quad (4.6)$$

Numerical values of t_K are given in Fig. 3, for the interval $0<ka<10$, together with exact values of the transmission coefficient calculated by Bouwkamp (I, reference 5). A comparison of this Kirchoff approximation with one based on a constant aperture field (see I, Fig. 2) reveals that both approach the value unity at high frequencies and possess a k^2 behavior at low frequencies. The latter approximation, however, is distinctly less accurate for intermediate frequencies, failing to take values in excess of unity and exhibit characteristic oscillations of the diffraction curve.

We consider next a more refined approximation to the transmission coefficient, deduced from the variational expression (1). A simple choice of frequency independent residual function is

$$\psi(\rho)=(a/\rho)^3(1-(a/\rho)^2)^{-\frac{1}{2}}, \quad (4.7)$$

featuring a characteristic (low frequency) singularity at the rim of the screen and decreasing with sufficient rapidity therefrom to ensure integrability over the entire screen. The integrals in the second term of (4.1) are again simplified with the aid of (4.3), (4.4). Thus

$$\int_{S_2}\psi(\rho)\rho d\rho d\varphi\int_{S_1}G(\rho,\varphi;\rho',\varphi')\rho' d\rho' d\varphi'$$
$$=\pi\int_0^\infty\zeta(\zeta^2-k^2)^{-\frac{1}{2}}d\zeta\left(\int_a^\infty(a/\rho)^3(1-(a/\rho)^2)^{-\frac{1}{2}}\times\rho J_0(\zeta\rho)d\rho\right)\left(\int_0^a\rho' J_0(\zeta\rho')d\rho'\right)$$
$$=\pi a^3\int_0^\infty(\zeta^2-k^2)^{-\frac{1}{2}}J_1(\zeta a)d\zeta \times\int_1^\infty(x^2-1)^{-\frac{1}{2}}J_0(\zeta ax)(dx/x).$$

To perform the x integration, we consider

$$F(\xi)=\int_1^\infty J_0(\xi x)(x^2-1)^{-\frac{1}{2}}(dx/x), \quad F(0)=\pi/2$$

and observe that[2]

$$dF/d\xi=-\int_1^\infty J_1(\xi x)(x^2-1)^{-\frac{1}{2}}dx=-\sin\xi/\xi,$$

whence

$$F(\xi)=(\pi/2)-\int_0^\xi(\sin x/x)dx=(\pi/2)-Si\,\xi=si\,\xi,$$

where Si denotes the sine integral.

Consequently,

$$\int_{S_2}\psi(\rho)\rho d\rho d\varphi\int_{S_1}G(\rho,\varphi;\rho',\varphi')\rho' d\rho' d\varphi'$$
$$=\pi a^3(R_1(ka)+iI_1(ka)), \quad (4.8)$$

with

$$\begin{aligned} I_1(\alpha)&=\int_0^\alpha J_1(x)(\alpha^2-x^2)^{-\frac{1}{2}}si\,x\,dx,\\ R_1(\alpha)&=\int_\alpha^\infty J_1(x)(x^2-\alpha^2)^{-\frac{1}{2}}si\,x\,dx. \end{aligned} \quad (4.9)$$

[2] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, London, 1945), p. 417.

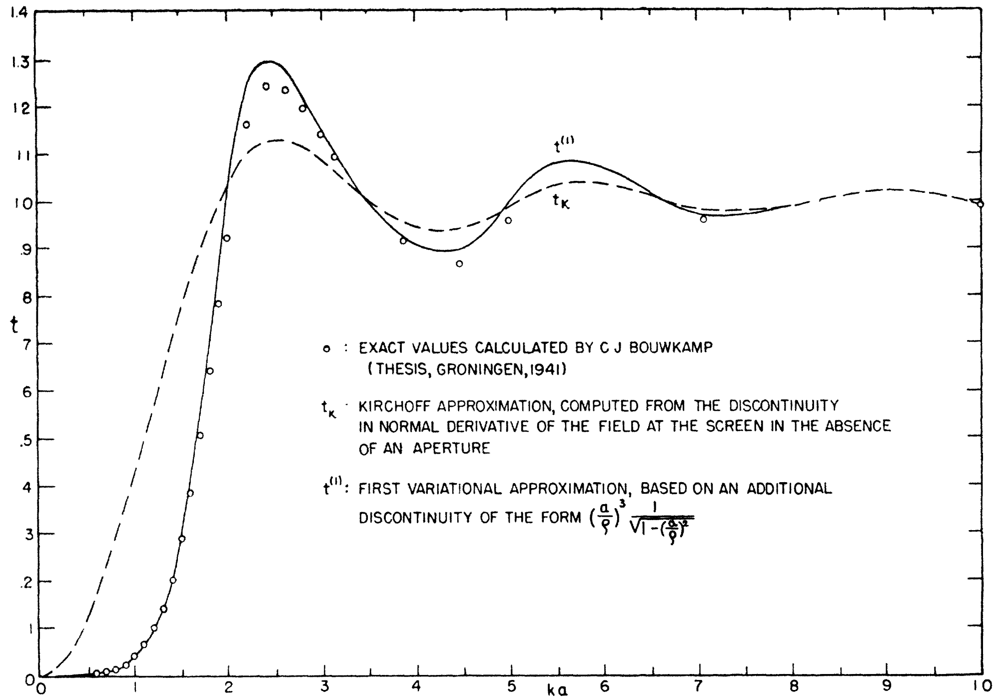

FIG. 3. Transmission coefficient of circular aperture for normally incident plane waves, as a function of the parameter $ka = 2\pi$ (radius of aperture/wave-length). Scalar wave function assumed to vanish on the screen.

In a similar fashion, we obtain

$$\int_{S_2} \psi(\rho) G(\rho, \varphi; \rho', \varphi') \psi(\rho') \rho d\rho d\varphi \rho' d\rho' d\varphi' = \pi a^3 (R_2(ka) + iI_2(ka)), \quad (4.10)$$

with

$$I_2(\alpha) = \int_0^\alpha x(\alpha^2 - x^2)^{-\frac{1}{2}} si^2 x dx,$$
$$R_2(\alpha) = \int_\alpha^\infty x(x^2 - \alpha^2)^{-\frac{1}{2}} si^2 x dx. \quad (4.11)$$

The result of inserting (4.6), (4.8), (4.10) in (4.1) yields as a first variational approximation to the transmission coefficient,

$$t^{(1)} = 1 - [J_1(2ka)/ka] - 2ka[(2I_1R_1R_2 - I_2(R_1{}^2 - I_1{}^2))/(R_2{}^2 + I_2{}^2)], \quad (4.12)$$

in which the argument, ka, of the functions I_1, I_2, R_1, R_2 is omitted. The values of $t^{(1)}$ presented in Fig. 3 are based on numerical integrations of (4.9), (4.11), with appropriate modifications for the singular point. A very accurate low frequency behavior is exhibited therein, although considerations of the previous section imply some residual k^2 dependence in $t^{(1)}$. We have, in fact,

$$\left.\begin{aligned} &I_1(\alpha) \doteq \pi\alpha/4, \quad I_2(\alpha) \doteq \pi^2\alpha/4 \\ &R_1(\alpha) \doteq \int_0^\infty J_1(x) six(dx/x) = (\pi/2) - 1 \\ &R_2(\alpha) \doteq \int_0^\infty si^2 x dx = \pi/2 \end{aligned}\right\} \alpha \rightarrow 0,$$

whence, from (4.12),

$$t^{(1)} \doteq (1/2)(\pi - 3)^2 (ka)^2 = 0.010(ka)^2, \quad ka \rightarrow 0. \quad (4.13)$$

(The same result is obtained from (3.16) for the case of normal incidence.) We may note that if the radical is omitted in (4.7), the coefficient of (4.13) is replaced by 0.041. From the over-all agreement of the end results in the two variational formulations, we may infer their proximity to that of an exact solution, quite apart from Bouwkamp's work.

The construction of analogous variational principles for the diffraction of electromagnetic waves by an aperture in a perfectly conducting screen will appear in a forthcoming paper.

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APPENDIX

To evaluate the integral

$$I=\int_0^1 (1-v^2)^{-\frac{1}{2}}J_1{}^2(\alpha v)(dv/v),$$

we introduce the representation

$$J_1{}^2(z)=(1/\pi)\int_0^\pi J_2(2z\sin\vartheta)d\vartheta$$

and obtain, with the change of variable $v=\sin\varphi$,

$$I=(1/2\pi)\int_0^\pi \sin^{-1}\varphi d\varphi\int_0^\pi J_2(2\alpha\sin\vartheta\sin\varphi)d\vartheta.$$

Regarding ϑ, φ as the polar angles of a point on a unit sphere, we may write

$$I=(1/2\pi)\int_{x\geq 0} x^{-1}J_2(2\alpha x)d\omega,$$

where $x=\sin\vartheta\sin\varphi$, $d\omega=\sin\vartheta d\vartheta d\varphi$, and the integration extends over the surface of the hemisphere on which x is positive. The replacement of x by $z(=\cos\vartheta)$ corresponds to a rotation of the coordinate system and does not affect the value of the integral. Hence

$$I=(1/2\pi)\int_{z\geq 0} z^{-1}J_2(2\alpha z)d\omega$$

$$=\int_0^1 \mu^{-1}J_2(2\alpha\mu)d\mu$$

$$=-\int_0^1 d(J_1(2\alpha\mu)/2\alpha\mu)=\tfrac{1}{2}[1-(J_1(2\alpha)/\alpha)],$$

which allows the verification of Eq. (4.6) in the text.

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The Energy Spectrum of the Decay Particles and the Mass and Spin of the Mesotron*

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Energy values determined from curvature measurements of 75 cloud-chamber tracks of decay particles of cosmic-ray mesotrons at sea level, in a magnetic field of 7250 gauss, are here reported. The observed spectrum extends from 9 Mev to 55 Mev with an apparently continuous distribution of intermediate energy values and a mean energy of 34 Mev. The shape of the spectrum and the value of its upper limit are strong evidence that the mesotron disintegrates into an electron and two neutrinos. It is concluded that the mesotron has half-integral spin. The value of the observed upper limit of the energy spectrum corresponds to a mass value of the mesotron equal to 217±4 electron masses.

I. INTRODUCTION

MEASUREMENTS of the energy of the particles resulting from the decay of mesotrons have previously been made in three ways. In a very few cases, the energies have been determined directly by measurement of the curvature of cloud-chamber tracks of the decay particles in a magnetic field.[1] In other experiments, the energies of the decay particles have been inferred from measurement of their absorption in various materials,[2] and recently, from measurement of their scattering in photographic emulsions.[3] The very small number of cases available for measurement, and the difficulty in making precise energy measurements, have made it impossible so far to distinguish between a continuous spectrum of energies and two or three discrete energies for the decay particles. Thus, the basic nature of the spectrum has so far not been established.

The measurements of the energies of decay particles that are here reported were made on 75 cases obtained in a cloud chamber operated in a magnetic field. The precision of the measurements and the number of cases provide strong evidence for a continuous decay spectrum, and yield some information as to its shape.

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