Periods of Motion in Periodic Orbits in the Equatorial Plane of a Magnetic Dipole

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The differential equations of motion of an electrically charged particle in the equatorial plane of a magnetic dipole are solved in a way that time appears directly. The period of motion for periodic orbits is then found, and an application to cosmic radiation is made. For electrons moving in the equatorial plane of the magnetic field of the earth the periods in seconds have been determined.

I. INTRODUCTION

A STUDY of the periodic orbits of a charged particle in the equatorial plane of a magnetic dipole has already been made.[1] The work cited is concerned principally with the geometry of these orbits. Nevertheless, since these authors did not compute the actual periods, it seems worth while to do so. In the other works[2] there is introduced a function σ which defines a conformal transformation of the meridian plane. The introduction of the function σ is necessary when we are treating the general problem,[3] in which case there is at present no other method of solution than numerical or mechanical integration. The problem treated in this paper can be solved in terms of elliptic functions, and the use of σ, whose knowledge as a function of time implies one more integration, is not required.

II. THE EQUATIONS OF MOTION

From the general equations of motion[4] one easily finds the following equations for the equatorial plane:

$$m/Mq[(d^2\rho/dt^2)-\rho(d\varphi/dt)^2] = -(1/\rho^2)(d\varphi/dt), \quad (1)$$

$$(m/Mq)(1/\rho)(d/dt)[\rho^2(d\varphi/dt)] = (1/\rho^3)(d\rho/dt), \quad (2)$$

where ρ and φ are the polar coordinates. From the principle of the conservation of energy one finds, furthermore:

$$(d\rho/dt)^2+\rho^2(d\varphi/dt)^2=v^2. \quad (v=\text{constant}). \quad (3)$$

Let us take for the unit of length the length c_1 defined by Störmer and given by equation

$$c_1=(M|q|/mv)^{\frac{1}{2}}\ \text{cm}, \quad (4)$$

in which M is the magnetic moment of the dipole in electromagnetic units, q the charge of the particle in the same units, v its velocity, and m its transverse mass. If one writes

$$\rho=c_1r, \quad (5)$$

$$v=c_1(ds/dt), \quad (6)$$

then the equations of motion become

$$(d^2r/ds^2)-r(d\varphi/ds)^2=-(1/r^2)(d\varphi/ds), \quad (7)$$

$$(1/r)(d/ds)[r^2(d\varphi/ds)]=(1/r^3)(dr/ds), \quad (8)$$

$$(dr/ds)^2+r^2(d\varphi/ds)^2=1. \quad (9)$$

It is seen from the last equation that s is the length of arc measured along the orbit.

III. INTEGRATION OF THE EQUATIONS OF MOTION

A first integral of Eq. (8) is

$$r^2(d\varphi/ds)+(1/r)=2\gamma_1, \quad (10)$$

where γ_1 is a constant.

From (9) and (10) one obtains

$$s=\pm\int(r^4-4\gamma_1^2r^2+4\gamma_1r-1)^{-\frac{1}{2}}r^2dr. \quad (11)$$

The right-hand side of this is an elliptic integral which can be expressed in terms of Legendre canonical forms. For this purpose we use

$$y^2=1+(2\gamma_1/r)-(1/r^2). \quad (12)$$

Without loss of generality one can choose the

[1] C. Graef and S. Kusaka, J. Math. and Phys. **17**, 43 (1938).
[2] J. Lifshitz, J. Math. and Phys. **21**, 94 (1942).
[3] G. Lemaître and M. S. Vallarta, Phys. Rev. **49**, 719 (1936).
[4] G. Lemaître and M. S. Vallarta, Phys. Rev. **43**, 87 (1933).

TABLE I. Values of the period in seconds t for some values of the number l of loops and the number n of turns, for electrons moving in the equatorial plane of the magnetic field in the neighborhood of the earth. If the greatest common divisor of l and n is d, then the orbit has d identical orbits.

$n \backslash l$	1	2	3	4	5	6	7	8	9	10
1	0.3399									
2	0.6607	0.6797								
3	0.9829	0.9982	1.0196							
4	1.3053	1.3215	1.3413	1.3595						
5	1.6278	1.6444	1.6593	1.6771	1.6993					
6	1.9506	1.9658	1.9822	1.9964	2.0144	2.0392				
7	2.2754	2.2879	2.3082	2.3215	2.3377	2.3534	2.3791			
8	2.5950	2.6107	2.6283	2.6429	2.6532	2.6827	2.6916	2.7189		
9	2.9169	2.9331	2.9488	2.9654	2.9838	2.9947	3.0080	3.0286	3.0588	
10	3.2396	3.2557	3.2705	3.2888	3.3037	3.3185	3.3347	3.3542	3.3679	3.3987

plus sign in Eq. (11), in which case one finds

$$s=\pm\int[(1+\gamma_1^2-y^2)(2-y^2)]^{-\frac{1}{2}}[r(y)]^2dy. \quad (13)$$

The double sign reappears because of the freedom of choice of the sign in

$$1/r(y)=\gamma_1\pm(\gamma_1^2-y^2+1)^{\frac{1}{2}} \quad (14)$$

obtained from (12).

In the case of periodic orbits the values of r lie between $r_{\min}$ and $r_{\max}$, where

$$1/r_{\min}=\gamma_1+(\gamma_1^2+1)^{\frac{1}{2}}, \quad (15)$$

$$1/r_{\max}=\gamma_1+(\gamma_1^2-1)^{\frac{1}{2}}; \quad (16)$$

we must now use the plus sign in (14) and we obtain

$$s=2\gamma_1^2\int[(1+\gamma_1^2-y^2)(2-y^2)]^{-\frac{1}{2}}(y^2-1)^{-2}dy$$
$$-2\gamma_1\int(2-y^2)^{-\frac{1}{2}}(y^2-1)^{-2}dy$$
$$-\int[(1+\gamma_1^2-y^2)(2-y^2)]^{-\frac{1}{2}}(y^2-1)^{-1}dy. \quad (17)$$

The second integral represents an elementary function. If we measure s from a point where $r=r_{\min}$, the lower limit in each of the definite integrals corresponding to (17) is zero.

The third integral in (17) is an elliptic integral of the third kind which is reduced to

$$\int_0^u\frac{du}{sn^2u-\alpha}=(2sna\ cna\ dna)^{-1}\times\left[ln\frac{H(a-u)}{H(a+u)}+2uZ(a)\right], \quad (18)$$

where

$$y=2^{\frac{1}{2}}sn\ u, \quad (19)$$

$$sn^2\ a=\alpha, \quad (20)$$

if $\alpha=\frac{1}{2}$.

Finally, we can evaluate the first integral in (17) by differentiating Eq. (18) with respect to α before setting $\alpha=\frac{1}{2}$; we obtain in this manner:

$$s=\tfrac{1}{2}(1+\gamma_1^2)^{\frac{1}{2}}\left[\frac{H'(a-u)}{H(a-u)}-\frac{H'(a+u)}{H(a+u)}+2uZ'(a)\right]+\frac{\gamma_1 snu\ cnu}{sn^2u-sn^2a}. \quad (21)$$

All the above functions of Jacobi are constructed with the modulus

$$k=[2/(1+\gamma_1^2)]^{\frac{1}{2}}. \quad (22)$$

We must note that the three integrals in (17) are improper and do not converge for the value $y=1$. Nevertheless, the sum of these integrals must be a continuous function in the neighborhood of $y=1$, since the integral (13) converges for this value of the upper limit of the corresponding definite integral. One can see that this is actually the case by transforming the right-hand side of Eq. (21) using the relation[5]

$$\frac{2sna\ cna\ dna}{sn^2u-sn^2a}=\frac{H'(u-a)}{H(u-a)}-\frac{H'(u+a)}{H(u+a)}+2Z(a), \quad (23)$$

[5] E. B. Wilson, *Advanced Calculus* (Ginn and Company, Boston, 1912), p. 513.

obtaining thus:

$$s=\gamma_1\frac{2snu\,cnu-1}{2sn^2u-1}-(1+\gamma_1^2)^{\frac{1}{2}}\times\left[\frac{H'(u+a)}{H(u+a)}-Z(a)-uZ'(a)\right]. \quad (24)$$

From this equation it follows also that when $u=0$, which corresponds to $r=r_{\min}$, then $s=0$, as we were expecting.

One can also write (24) in the form

$$s=\gamma_1\frac{2snu\,cnu-1}{2sn^2u-1}-(1+\gamma_1^2)^{\frac{1}{2}}\left[\frac{cn(u+a)\,dn(u+a)}{sn(u+a)}+Z(u+a)-Z(a)-uZ'(a)\right]. \quad (25)$$

Eliminating s from (9) and (10) one obtains the differential equation of the orbit, whose integral is

$$\varphi=am\,u+[\gamma_1u/(1+\gamma_1^2)^{\frac{1}{2}}]. \quad (26)$$

The last two equations, together with

$$(r^2+2\gamma_1r-1)^{\frac{1}{2}}(1/r)=2^{\frac{1}{2}}sn\,u, \quad (27)$$

obtained from (12) and (19), determine the motion completely.

IV. PERIOD

From (27) it follows that r is periodic in u with period $2K$. The increase in s when u increases from u to $u+2K$ is, according to Eq. (25),

$$s_1=2(1+\gamma_1^2)^{\frac{1}{2}}KZ'(a). \quad (28)$$

If the orbit has l loops, then its total length is

$$L=2l(1+\gamma_1^2)^{\frac{1}{2}}KZ'(a), \quad (29)$$

and the period in t is therefore

$$T=(c_1/v)L. \quad (30)$$

For numerical applications it is convenient to transform (29) by means of the formula[6]

$$dn^2\,u=Z'(u)+(E/K), \quad (31)$$

and we obtain in this way:

$$L=2l(1+\gamma_1^2)^{\frac{1}{2}}[(K\gamma_1/(1+\gamma_1^2))-E], \quad (32)$$

where K and E are the complete elliptic integrals of the first and second kinds, respectively.

V. NUMERICAL RESULTS AND APPLICATION TO COSMIC RADIATION

Using values of γ_1 corresponding to periodic orbits,[1] we have calculated L, using Eq. (32), for several values of l and for various values of the number n of turns. Furthermore, these results have been applied to the case of electrons moving in the equatorial plane of the magnetic field in the neighborhood of the earth, and by means of Eq. (30) the periods of motion in t have been calculated. The results are given in the Table I.

[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Teddington, 1940), fourth edition, p. 518.