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The Magnetic Internal Conversion Coefficient*

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The magnetic internal conversion coefficient for K-electrons is rederived with the Dirac current for the electron introduced into the electron-nucleus interaction matrix. Beyond resolving a discrepancy found in the literature, the formula obtained indicates magnetic conversion to be an effect of importance comparable with electric conversion. Its influence on the total conversion coefficient, on the K- to L-shell conversion ratio, and on the lifetimes of isomeric levels is discussed.

I. INTRODUCTION

CONVERSION of magnetic multipole radiation by K-electrons is a relativistic effect depending upon the spin of the electrons. This was first pointed out by Dancoff and Morrison,[1] who gave a relativistic calculation of the magnetic internal conversion coefficient neglecting binding (Born approximation). Recently, Goertzel and Lowen,[2] and Berestetzky[3] have independently calculated this effect on the basis of the Pauli two-component theory of the spinning electron. In the limit of weak binding and soft β-rays ($v_{el} \ll c$, $Ze^2/hv_{el} \ll 1$), their result disagrees with the formula of Dancoff and Morrison by a factor $l/(2l+1)$, where the final state of the ejected conversion electron possesses l units of orbital angular momentum. This discrepancy is here shown to be due to the extremely singular character of the magnetic multipole potential. It is resolved by introducing the four-component Dirac current for the electron into the electron-nucleus interaction matrix.

An important consequence of these calculations is that magnetic conversion contributes more significantly than previously appreciated. This is particularly evidenced for soft γ-rays in the neighborhood of the K-electron threshold, since magnetic conversion approaches a large finite value, whereas electric K-conversion vanishes at threshold. Large magnetic conversion can be expected to have an appreciable effect on the total conversion coefficient and on isomeric lifetimes. Of principal concern to the experimentalist will be its influence on the K- to L-shell conversion ratio.

II. CALCULATION

The number of electronic transitions per second from a state o in the K-shell to a continuum state f resulting from magnetic multipole radiation is given for two electrons in the K-shell:[4]

$$N_e d\Omega = (2\pi e^2/\hbar)|\text{M.E.}|^2 d\Omega,$$

where

$$\text{M.E.} = \sum_{\text{spin}} \int dv(\psi_f^* \boldsymbol{\alpha} \psi_o)\cdot\mathbf{A} = \frac{1}{ec}\sum_{\text{spin}} \int dv \mathbf{j}_{fo}\cdot\mathbf{A}. \quad (1)$$

ψ_f is normalized to unit energy, the $(\alpha_1, \alpha_2, \alpha_3)$ are the four-component Dirac spin matrices, and the spin sum extends over final and initial spin states. With Dirac's equation we make the familiar[5] decomposition of the current density,

$$\mathbf{j}_{fo} = (e\hbar/2im)(\psi_f^*\beta\, \mathbf{grad}\psi_o - \psi_o^*\beta\, \mathbf{grad}\psi_f) + (e\hbar/2m)\, \mathbf{curl}(\psi_f^*\boldsymbol{\sigma}\beta\psi_o) - (ie\hbar/2mc)(\partial/\partial t)(\psi_f^*\boldsymbol{\alpha}\beta\psi_o). \quad (2)$$

The first term of (2) is the Schrödinger current. It does not contribute to flux of s-electrons into the continuum because of magnetic multipole radiation, as is seen from the following symmetry considerations. An electron will make a transition

* Assisted by the joint program of the Office of Naval Research and Atomic Energy Commission.

[1] S. M. Dancoff and P. Morrison, Phys. Rev. 55, 128 (1939).

[2] G. Goertzel and I. S. Lowen, Phys. Rev. 67, 203 (1945).

[3] V. Berestetzky, J. Phys. USSR. 10, 137 (1946).

[4] Cf. reference 1. The notation of Dancoff and Morrison is used. Berestetzky has given (J. Phys., USSR. 11, 85 (1947)) a group theoretical argument showing the gauge in which the scalar potential vanishes to be appropriate for magnetic multipole radiation.

[5] E. L. Hill and R. Landshoff, Rev. Mod. Phys. 10, 87 (1938).

from the K-shell with zero units to a state with l units of orbital angular momentum upon absorbing a 2^l-pole quantum.[6] The potential of a magnetic 2^l-pole is proportional to Y_l^m, and thus transforms under rotation as a $(2l+1)$-dimensional representation of the rotation group. On the other hand, the Schrödinger current density for this transition transforms as $Y_{l+1}^{m'}$. The angular integral of the product of these two factors vanishes. In terms of parity conservation, the electron parity change accompanying this transition is $(-)^l$, whereas the field of a magnetic 2^l-pole has parity $(-)^{l+1}$. The second term may be identified as the magnetic dipole current associated with the electron spin. The third term is the electric dipole current. It may be neglected in consistency with the accuracy of these calculations. The $\boldsymbol{\alpha}$-matrices mix large and small components of the wave functions and the time derivative is proportional to the energy difference ($\ll$mc^2) between initial bound and final continuum states of the electron.

Introducing (2) into (1), carrying out one partial integration, and employing Gauss's theorem, we obtain for the perturbing matrix element,

$$\text{M.E.} = (\hbar/2mc) \sum_{\text{spin}} \left[\int dv (\psi_f^* \boldsymbol{\sigma}\beta\psi_o) \cdot \mathbf{H} + \int_\Sigma d\boldsymbol{\Sigma} \cdot (\psi_f^* \boldsymbol{\sigma}\beta\psi_o \times \mathbf{A}) \right]. \quad (3)$$

$\mathbf{H} = \text{curl}\mathbf{A}$ is the magnetic field strength, and the bounding surface Σ consists of two spheres whose radii approach infinity and zero. Laplace's equation is valid in the annular region between the two spheres which is appropriate for the internal conversion problem.[1]

The first term of (3) gives just the contribution to the magnetic conversion coefficient at low energies as calculated with the Pauli theory. There will be no contribution from the second term because of the surface at infinity. Because of the extremely singular character of the potential $\mathbf{A}$ there will be one from the small sphere about the origin. A magnetic 2^l-pole has parity $(-)^{l+1}$. It induces transitions of s-electrons with $\frac{1}{2}$ unit of total angular momentum to final continuum states of type (a), with $(l-1)$ units of orbital angular momentum and $(l-\frac{1}{2})$ units of total angular momentum, and of type (b), with $(l+1)$ units of orbital angular momentum and $(l+\frac{1}{2})$ units of total angular momentum. Investigation of the behavior of the wave functions and of the magnetic multipole potential for small arguments shows that the surface term in (3) contributes to transitions to continuum states of type (a) from a small sphere about the origin. A regular solution of the coulomb field problem, representing a state of $(l-1)$ units of orbital angular momentum, behaves as r^{l-1} near the origin. Since the singular potential caused by a magnetic 2^l-pole diverges[6] as $r^{-(l+1)}$, the integrand approaches a finite limit as the inner boundary shrinks to zero radius.

To calculate matrix element (3) we limit ourselves to the two large components of the electronic wave functions. This is permitted since the $(\boldsymbol{\sigma}\beta)$ matrices mix large with large and small with small components. The result will be accurate for $v_{\text{el}}/c \ll 1$ and will thus serve as a first-order relativistic approximation. The wave functions are for the K-shell:

$$\psi_o = (a^3/\pi)^{\frac{1}{2}} e^{-ar} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and for the continuum state with l-units of orbital angular momentum:

$$\psi_{l+\frac{1}{2},\, l,\, \frac{1}{2}} = R_l \begin{pmatrix} \sqrt{(l+1)/(2l+1)}\, Y_l^o \\ -\sqrt{(l)/(2l+1)}\, Y_l^1 \end{pmatrix}, \quad \text{(Type a)},$$

$$\psi_{l-\frac{1}{2},\, l,\, \frac{1}{2}} = R_l \begin{pmatrix} \sqrt{(l)/(2l+1)}\, Y_l^o \\ \sqrt{(l+1)/(2l+1)}\, Y_l^1 \end{pmatrix}, \quad \text{(Type b)},$$

where

$$R_l = (2mp/\pi\hbar^2)^{\frac{1}{2}} \{ |\Gamma(l+1+in)| / (2l+1)! \} e^{\frac{1}{2}n\pi} \times (2pr)^l e^{ipr} F(l+1-in,\, 2l+2,\, -2ipr),$$

the Y_l^m are normalized spherical harmonics, F is the confluent hypergeometric function, $a = Z\alpha mc/\hbar$, $\alpha = e^2/\hbar c$, $n = a/p = Z\alpha/[2\nu - (Z\alpha)^2]^{\frac{1}{2}}$, where ν is the γ-ray energy in units of mc^2, and $p = 1/\hbar$ times the momentum of the ejected

[6] Cf. W. Heitler, Proc. Camb. Phil. Soc. **32**, 112 (1936) for derivation of the multipole fields which appear in reference (1). He shows the quantized electric or magnetic 2^l-pole field to represent a light quantum with angular momentum $\mathbf{L}\hbar$. Berestetzky has given an alternate derivation in (4).

electron. A simplification may be achieved, as Berestetzky[4] pointed out, by immediately setting to zero, m, the z axis projection of the angular momentum of the emitted multipole radiation. The flux of electrons from the K-shell is independent of the axis orientation. The potential for a 2^l magnetic multipole reads then:[1]

$$A_z = 0, \quad A_\pm = A_x \pm iA_y = -b_l Y_l^{\pm 1} f_l(kr).$$

b_l measures the magnetic multipole moment, k is $1/\hbar$ times the momentum of the quantum, and radial function $f_l(kr)$ is $H_{l+\frac{1}{2}}^{(1)}(kr)/(kr)^{\frac{1}{2}}$, where $H_{l+\frac{1}{2}}^{(1)}(kr)$ is the Hankel function of the first kind. In the radial integrations, $f_l(kr)$ is replaced by its most singular term:[7]

$$(\Gamma(l+\tfrac{1}{2})/i\pi)(2^{l+\frac{1}{2}})/(kr)^{l+1}.$$

The magnetic conversion coefficient for two K-electrons is obtained by dividing the electronic flux N_e, calculated from (1) for magnetic 2^l-pole radiation, by the corresponding flux of quanta from the atom,[1]

$$|b_l|^2/\pi^2\hbar k \text{ quanta/sec.}$$

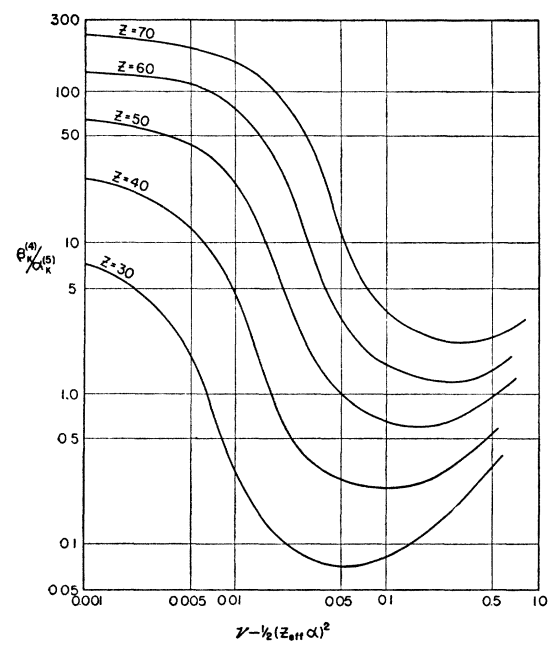

FIG. 1. Graph of the ratio of the K-shell conversion coefficient for 2^4-pole magnetic radiation to the coefficient for 2^5-pole electric radiation *vs.* the kinetic energy of the ejected electron in units of mc^2.

That this ratio is the experimentally determined quantity is demonstrated by Taylor and Mott.[8] They show that, because of direct interaction between the electrons and nucleus, the flux of "observable" quanta which appear external to the atom is equal, with an error of the order of magnitude of the fine structure constant, to the flux, which a bare nucleus would emit upon undergoing the same multipole transition.

The result for the magnetic conversion coefficient for two K-electrons, due to 2^l-pole magnetic radiation, is

$$\beta_K^{(l)} = \frac{l(l+2)}{(2l+1)(l+1)}\left(\frac{\nu^2}{4}\right)\alpha_K^{(l+1)} + 2Z^3\alpha^4\left(\frac{2}{\nu}\right)^{l+\frac{3}{2}} \times \frac{l+1}{2l+1}\cdot\frac{\pi n}{(1+n^2)^{l-\frac{1}{2}}}\cdot\frac{\prod_{i=1}^{l-1}(i^2+n^2)}{[\Gamma(l)]^2(1-e^{-2\pi n})}$$

$$= \frac{l(l+2)}{(2l+1)(l+1)}\left(\frac{\nu^2}{4}\right)\alpha_K^{(l+1)} + \frac{2^{2l+4}\pi\alpha(l+1)}{(Z\alpha)^{2l}(2l+1)[\Gamma(l)]^2} \times \frac{n^{2l+4}\prod_{i=1}^{l-1}(i^2+n^2)}{(1+n^2)^{2l+1}(1-e^{-2\pi n})}, \quad (4)$$

where ν is the γ-ray energy in units of mc^2, and $\alpha_K^{(l+1)}$ is the electric conversion coefficient for two K-electrons due to $2^{(l+1)}$-pole electric radiation.[9] For $l=1$,

$$\prod_{i=1}^{l-1}(i^2+n^2) = 1.$$

The first term in (4) is the principal contribution given by the Pauli theory.[2,3] For low Z it approaches[10] the limiting expression

$$\{l/2l+1\}Z^3\alpha^4(2/\nu)^{l+\frac{3}{2}}.$$

The second term in (4) is just the contribution

[7] Dancoff and Morrison point out that this is accurate to $(v_{el}/c)^2$.

[8] H. M. Taylor and N. F. Mott, Proc. Roy. Soc. **A142**, 215 (1933).

[9] The author has learned in a private communication that R. J. Bessey has performed this calculation using four-component wave functions. His result is similar to ours.

[10] Cf. formula (15), p. 128 of reference 1.

due to the second term in (3) from a small sphere about the origin. In the limit of low $Z(n\rightarrow o)$ this surface term approaches

$$\{l+1/2l+1\}Z^3\alpha^4(2/\nu)^{l+\frac{3}{2}}.$$

Formula (4) thus agrees with the result of Dancoff and Morrison (Eq. (20), p. 129 of reference 1) for low Z. Condition $v_{\text{el}}/c \ll 1$ limits the validity of this result.

An alternate derivation of (4) indicates clearly the failure of the Pauli theory. By decomposition of the Dirac equation into two two component equations,[5] one can express the small components, X_i, of the wave function in terms of the large ones Φ_i,

$$X_i=(c\boldsymbol{\sigma}\cdot\mathbf{p})/(2mc^2+Ze^2/r)\Phi_i, \quad (5)$$

where $\mathbf{p}$ is the momentum operator and $\boldsymbol{\sigma}$ the 2×2 Pauli spin matrices. Introduction of (5) into (1) gives, in place of (3),

$$\text{M.E.}=\hbar/2mc\sum_{\text{spin}}\left[\int dv\Phi_f^*\sigma\cdot\mathbf{H}\Phi_o + 2emc^2\int dv\Phi_f^*(\boldsymbol{\sigma}\cdot((Ze/r^3)\mathbf{r}\times\mathbf{A}))/((2mc^2+Ze^2/r)^2)\Phi_o\right]. \quad (6)$$

Direct evaluation of (6) yields the identical result (4). In the reduction of the Dirac to the Pauli theory one usually neglects the second term of (6) relative to the first term as a second-order correction reduced in the ratio of v_{el}/c. This is an erroneous procedure to adopt in this case because of the extremely singular nature of the magnetic multipole potential, which necessitates preservation of the singular term in the energy denominator.

III. DISCUSSION

The importance of the second term in (4) extends beyond its purely academic significance in removing the discrepancy from the literature. It dominates in formula (4), particularly for elements of medium Z, and indicates magnetic K-conversion to be an effect fully as important as electric K-conversion.

This is exemplified in Figs. 1 and 2. The electric coefficients were obtained from the calculations of Hebb and Nelson.[11] Their shielding correction for the K-shell was also employed.[12]

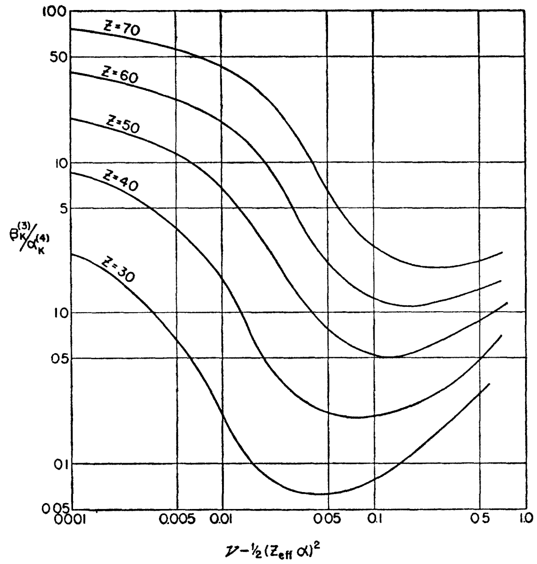

FIG. 2. Graph of the ratio of the K-shell conversion coefficient for 2^3-pole magnetic radiation to the coefficient for 2^4-pole electric radiation *vs.* the kinetic energy of the ejected electron units of mc^2.

Whereas electric K-conversion vanishes as we approach the K-shell threshold $(n\rightarrow\infty)$, the magnetic coefficient approaches the large finite value

$$\beta_K^{(l)}=2^{2l+4}\pi\alpha(l+1)/((Z\alpha)^{2l}(2l+1)[\Gamma(l)]^2).(n\gg l)$$

Since the L-shell threshold is one-fourth the K-shell binding energy for the same value of nuclear charge Z, electric L-shell conversion is predominant[11] in this energy region. However, the large magnetic conversion will influence the experimentally observed K to L-shell conversion ratio quite appreciably, and will contribute significantly to the total conversion coefficient.

This contribution will be reflected in the lifetimes of isomeric levels. One may write for the rate of decay, R, of a nucleus from a metastable to ground state[13]

$$R\propto\sum_{l=|J-J'|}^{J+J'}|\sigma_l|^2(1+\alpha^{(l)}) + \sum_{l'=|J-J'|}^{J+J'}|b_{l'}|^2(1+\beta^{(l')}), \quad (7)$$

[11] M. H. Hebb and E. Nelson, Phys. Rev. **58**, 486 (1940).

[12] From the calculations of J. C. Slater, Phys. Rev. **36**, 57 (1930), we take for the K-shell, $Z_{\text{eff}}=Z-0.30$.

[13] M. H. Hebb and G. E. Uhlenbeck, Physica **5**, 605 (1938).

where σ_l is a measure of the electric 2^l-multipole, $b_{l'}$ is a measure of the magnetic $2^{l'}$-multipole, $\alpha^{(l)}$ is the electric conversion coefficient for 2^l-pole electric radiation, and $\beta^{(l')}$ the magnetic coefficient for the $2^{l'}$-pole magnetic radiation. **J** and **J**′ are the quantum numbers of total angular momentum of the initial and final states of the nucleus. Barring particular nuclear symmetries, one is justified in keeping only the lowest permitted multipole order.[14] It is seen from (4), and from Eqs. (14) and (20) of reference 1, that the magnetic and electric conversion coefficients increase with l, roughly as $(1/\nu)^l$. Since the γ-ray intensity falls off with increasing l as $(k\delta)^{2l}$, δ being the nuclear radius, the total number of ejected electrons decreases as $(k\delta/\nu^{\frac{1}{2}})^{2l}\approx(p\delta)^{2l}$. For electron energies <1 Mev, $p\delta\approx 1/100$.

By the parity selection rule there corresponds to a given parity change and l value a minimum electric and magnetic multipole.[1] For a "parity allowed" type of transition the lowest allowed multipoles are electric 2^l and magnetic 2^{l+1}, so that (7) reduces to

$$R_{\mathrm{P.A.}}=R_o(1+\alpha^{(l)}), \tag{8}$$

where R_o is the decay rate due to pure γ-emission alone. We have neglected in (8) $|b_{l+1}|^2$ as reduced relative to $|\sigma_l|^2$ by $(k\delta)^4$, and[15]

$$|b_{l+1}|^2\beta^{(l+1)}/|\sigma_l|^2\alpha^{(l)}$$
$$\approx(k\delta)^4\cdot(\beta^{(l+1)}/\alpha^{(l+2)}\quad(\alpha^{(l+2)}/\alpha^{(l)})$$
$$\approx(k\delta)^4/\nu^2\approx(p\delta)^4.$$

[14] A detailed argument appears in reference 1, pp. 123–124.

[15] One sees from the graphs that the deviation of the ratio $\beta^{(l+1)}/\alpha^{(l+2)}$ from one does not destroy the validity of this argument.

For "parity forbidden" transitions, $l'=l-1$ in (7). Electric 2^l-pole amplitude, $|\sigma_l|^2$, and magnetic $2^{(l-1)}$-pole amplitude, $|b_{l-1}|^2$, are of the same order of magnitude. Their relative numerical values will be determined by a nuclear form factor, f, which is a function of the charge-current distribution in the nucleus. f tells what percent of the nuclear transitions to ground level give rise to electric radiation. We may write $|\sigma_l|^2=fR_o$; $|b_{l-1}|^2=(1-f)R_o$. In these terms, (7) become

$$R_{\mathrm{P.F.}}=R_o(1+f\alpha^{(l)}+(1-f)\beta^{(l-1)}).$$

One would expect f to be usually in the neighborhood of $\frac{1}{2}$.[16]

This result makes more ambiguous the interpretation of internal conversion data. First of all, one cannot know in advance whether the transition is parity allowed or forbidden. Secondly, if parity forbidden, a precise formula to compare with an internal conversion coefficient would appear hard to come by without further knowledge of f.

The relation between isomeric lifetimes and internal conversion coefficients has been investigated in detail by P. Axel and S. M. Dancoff.[17]

It is a pleasure to acknowledge my indebtedness to Professor S. M. Dancoff for suggesting this problem and for his friendly guidance and constructive criticisms during its progress. I also wish to thank Professor A. Nordsieck for helpful comments. Thanks are due Mr. Gerald P. Beck, who performed the computations for the graphs.

[16] However, recent unpublished experiments by Professors Hill and Scharff-Goldhaber on Te^{125} indicate $f=.05$ in order to account for the large observed K- to L-shell conversion ratio with a parity forbidden $l=4$ transition. The calculated ratio is too small by a factor of ten if only electric conversion is considered.

[17] To be published.